Hence either

$$x-x_j<x-\frac{x_j+x_{j+1}}{2}<x-x_{j+1} \quad \text{or} \quad x-x_j>x-\frac{x_j+x_{j+1}}{2}>x-x_{j+1}$$

then either

$$(x-x_j)^2<\left(x-\frac{x_j+x_{j+1}}{2}\right)^2 \text{ or } (x-x_{j+1})^2<\left(x-\frac{x_j+x_{j+1}}{2}\right)^2.$$

Consequently it is

$$\text{(19)} \quad \text{either } \left(\frac{x-x_j}{2}\right)^2<\left(x-\frac{x_j+x_{j+1}}{2}\right)^2 \text{ or } \left(\frac{x-x_{j+1}}{2}\right)^2<\left(x-\frac{x_j+x_{j+1}}{2}\right)^2.$$

The inequalities (18) and (19) imply (12). Therefore the inequality (12) is true in all cases.

Thus we have shown that no point $\left(\frac{x_j+x_{j+1}}{2},0\right)$ belongs to $K_{y_0}(A_0,E_2)$.

Formula (11) implies that between two points

$$\left(\frac{x_{j_1}+x_{j_1+1}}{2},0\right) \quad \text{and} \quad \left(\frac{x_{j_2}+x_{j_2+1}}{2},0\right)$$

for $j_1 \neq j_2$ lies at least one point $x_{j'}$ belonging to $K_{y_0}(A_0,E_2)$ since according to (6) $f(x_{j'})<f(x_0)$. Consequently the points $\left(\frac{x_{j_1}+x_{j_1+1}}{2},0\right)$ and $\left(\frac{x_{j_2}+x_{j_2+1}}{2},0\right)$ for $j_1 \neq j_2$ lie in the different components of $E_2-K_{y_0}(A_0,E_2)$.

Consequently $E_2-K_{y_0}(A_0,E_2)$ contains an infinite number of components. Hence $K_{y_0}(A_0,E_2)$ is not locally contractible.

10. Problem. *Let A_0 be a compact subset of the n-dimensional Euclidean space E_n. Let R denote the set of all positive numbers r, such that $K_r(A_0,E_n)$ is not homeomorphic to a polytope. The problem is, whether the set R is necessarily of first category (in the sense of Baire) and of measure zero (in the sense of Lebesgue)?*

Simply connected spaces.

By

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1. There are two ways of defining simple connectedness for topological spaces.

The first way is based on closed paths and their deformation:

An arcwise and locally arcwise connected topological space is termed *simply connected* whenever each of its closed paths is homotopic to a point ([9], p. 310; [10], p. 221) [1]). Such spaces will be referred to hereafter as *pathwise simply connected*.

Another way of defining simple connectedness makes use of the idea of a covering space:

A connected and locally connected topological space is termed *simply connected* whenever it admits only a trivial covering space ([5], p. 44). These will be referred to merely as *simply connected* spaces.

The first definition requires arcwise connectedness, while the second has a meaning even for Hausdorff-Lennes connected and locally connected spaces.

Similarly, the fundamental group of a space may be defined either as the group of paths, or as the group of covering homeomorphisms of the simply connected covering space (*Deckbewegungsgruppe*).

2. It is the purpose of this paper to state some theorems on simply connected spaces, which do not hold true for pathwise simple connectedness. As a consequence, it will be shown that, without further local assumptions, the two definitions are not equivalent [2]).

Our main goal is the proof of two kinds of approximation theorems: one related to the so-called ε-mappings, the other concerning convergent families of sets.

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[1]) Numbers in brackets refer to the bibliography at the end of the paper.

[2]) Pathwise simple connectedness implies simple connectedness, but less than that is needed.

There is a striking analogy between our theorems concerning the fundamental group, defined in terms of covering spaces, and some known results regarding the behaviour of the homology groups of a space (compare namely our § 6 with [7], our § 7 with [12], finally our § 8 with [13]). This analogy ceases to exist when the fundamental group is defined in terms of paths, as will be seen with the help of an example described in [3].

3. The terminology adopted is generally that of [9] (p. 1-40), and for the parts concerning the theory of covering spaces, we follow [5] (p. 40-60). The few group-theoretic concepts used, are in accordance with [10] (p. 3-25), and for limit-groups we refer again to [9] (p. 54-56).

We recall here the basic definitions. A *topological space* is a set of points, in which open subsets have been selected, satisfying the usual axioms ([9], p. 5). We shall always assume in addition, that the Hausdorff *separation axiom* T_2 holds ([9], p. 24). A *neighbourhood* of a subset is an open set, containing that subset. No distinction is made between a point and the set consisting of that single point. A point is *adherent* to a set if each of its neighbourhoods meets that set. *Subspaces* are always understood with the *relative topology* ([9], p. 10). $A\times B$ denotes the *cartesian product* of the two sets A and B. Let $Y=\varphi(X)$, where X is a space, Y a set and φ a transformation; the *strongest topology in* Y, for which φ is continuous, is obtained by specifying $V\subset Y$ to be open if and only if $\varphi^{-1}(V)$ is open in X (cf. [9], p. 12). If $\{A_\alpha\}$ and $\{B_\beta\}$ denote *collections of sets*, $\{A_\alpha\}\succ\{B_\beta\}$ means: each set A_α is contained in some set B_β; $\mathfrak{S}tA_\lambda$ is the *union* of all the A_α meeting A_λ ([9], p. 13, 324). Sometimes, when considering collections of sets, we shall omit the subscripts, and write merely $\{A\}$ instead of $\{A_\alpha\}$; but then A_1, A_2 will always mean sets of the given collection. If $\{V\}$ is a collection of subsets of $Y=\varphi(X)$, $\{\varphi^{-1}(V)\}$ is the aggregate of all the $\varphi^{-1}(V)\subset X$ with $V\in\{V\}$. A space is *connected* if it is not the union of two non-void disjoint open sets. It is *locally connected* when each neighbourhood of any point contains a connected neighbourhood of that point ([5], p. 40). *Compact* is used here in the sense of bicompact ([9], p. 17; [1], p. 86).

Let $\varphi:X\to Y$, $\psi:Y\to Z$ be transformations. $\psi\varphi:X\to Z$ is the transformation defined by $\psi\varphi(x)=\psi[\varphi(x)]$ for $x\in X$. For $A\subset X$ we denote by $\omega:A\to A\subset X$ the *inclusion map*, which is defined by $\omega(a)=a\in X$ for $a\in A$. $\Phi=\varphi\omega$ is termed the *contraction* of φ to A; it is defined on A and satisfies $\Phi(a)=\varphi(a)\in Y$ for $a\in A\subset X$. We shall denote a map and its contraction to a subset by the same symbol, i. e. we write $\varphi:X\to Y$ and $\varphi:A\to Y$, for $A\subset X$, when no misunderstanding is to be feared.

Let $f:\widetilde{E}\to E$ be continuous, into. A subset $A\subset E$ is termed *evenly covered* by $(\widetilde{E},f)$, if $f^{-1}(A)$ is not empty and each component of $f^{-1}(A)$ is topologically mapped by f onto A ([5], p. 40, D. 2). A *covering space* [3]) of the topological space E is a pair $(\widetilde{E},f)$ formed by a connected and locally connected space $\widetilde{E}$, and a continuous map f of $\widetilde{E}$ onto E, such that each point of E has a neighbourhood evenly covered by $(\widetilde{E},f)$. If the space E admits a covering space, since f is a local homeomorphism, E is connected and locally connected. *If E is a Hausdorff space, $\widetilde{E}$ has the same property.* Let $(\widetilde{E},f)$ be a covering space of E, A a connected, locally connected subset of E and $\widetilde{A}$ a component of $f^{-1}(A)$; then $(\widetilde{A},f)$, with f denoting the contraction to $\widetilde{A}$ of $f:\widetilde{E}\to E$, is a covering space of A ([5], p. 42, L. 5). Two covering spaces $(\widetilde{E}_1,f_1)$ and $(\widetilde{E}_2,f_2)$ of the space E are *isomorphic*, if there exists an onto-homeomorphism $f:\widetilde{E}_1\to\widetilde{E}_2$ such that $f_1=f_2f$ ([5], p. 43). The space E is *simply connected* if it is connected, locally connected and each of its covering spaces is isomorphic to (E,θ), where $\theta:E\to E$ is the identity map ([5], p. 44, D. 1). If $(\widetilde{E},f)$ is a covering space of E, the cardinal number of $f^{-1}(x)$ is independent of $x\in E$, and is known as the *number of leaves* of $(\widetilde{E},f)$. The group $\mathcal{F}$ of those homeomorphisms ξ of $\widetilde{E}$ onto itself, satisfying $f\xi=f$ on $\widetilde{E}$, is termed the *group of automorphisms* of the covering space $(\widetilde{E},f)$. When $\widetilde{E}$ is simply connected, $\mathcal{F}$ is termed the *Poincaré*, or *fundamental group* of the space E, ([5], p. 52, D. 1), and is denoted by $\pi_1(E)$. If $\widetilde{E}$ is a Hausdorff space and $\widetilde{x}_1,\widetilde{x}_2\in\widetilde{E}$ are any two points with $f(\widetilde{x}_1)=f(\widetilde{x}_2)$, there exists at most one element $\xi\in\mathcal{F}$ such that $\xi(\widetilde{x}_1)=\widetilde{x}_2$ ([5], p. 51, L. 1). The covering space $(\widetilde{E},f)$ is termed *regular*, whenever for each pair $\widetilde{x}_1,\widetilde{x}_2\in\widetilde{E}$ with $f(\widetilde{x}_1)=f(\widetilde{x}_2)$ there is a $\xi\in\mathcal{F}$ with $\xi(\widetilde{x}_1)=\widetilde{x}_2$. The simply connected covering space, if it exists, is always regular ([5], p. 52, P. 3). Finally we recall a fundamental property of simply connected spaces, which will often be used in the sequel:

[3]) This definition is due to C. Chevalley ([5], p. 40, D. 3). The idea of defining a covering space as a pair formed by a topological space and an interior map has already been exploited by S. Stoïlow in his work on Riemann surfaces; see for instance C. R. Congrès Int. Math. Oslo, II (1936), p. 143-144.

Let be: X a simply connected space, $(\widetilde{Y},g)$ a covering space of Y, $\varphi: X\to Y$ continuous, into, $x\in X$ and $\widetilde{y}\in\widetilde{Y}$ two points with $\varphi(x)=g(\widetilde{y})$. There exists then a uniquely determined continuous, into, $\widetilde{\varphi}: X\to\widetilde{Y}$ with $\widetilde{\varphi}(x)=\widetilde{y}$ and $g\widetilde{\varphi}=\varphi$ on the whole of X ([5], p. 50, P. 1).

4. We state first some elementary lemmas which we shall need later.

4.1. **Lemma.** *Let $\varphi: X\to Y$ be continuous, into, where X and Y are both connected, locally connected Hausdorff spaces. Let $(\widetilde{X},f)$ and $(\widetilde{Y},g)$ be regular covering spaces of X and Y, with $\mathcal{F}$ and $\mathcal{G}$ as their groups of automorphisms. Each into, continuous $\widetilde{\varphi}:\widetilde{X}\to\widetilde{Y}$ with $g\widetilde{\varphi}=\varphi f$ on the whole of $\widetilde{X}$, is called paired to φ. Then:*

(i) *If $\widetilde{X}$ is simply connected, for each pair $\widetilde{x}\in\widetilde{X}$, $\widetilde{y}\in\widetilde{Y}$ with $\varphi f(\widetilde{x})=g(\widetilde{y})$, there exists a single $\widetilde{\varphi}$ paired to φ, with $\widetilde{\varphi}(\widetilde{x})=\widetilde{y}$.*

(ii) *If $\widetilde{\varphi}$ is paired to φ, for each $\xi\in\mathcal{F}$ there exists a single $\eta\in\mathcal{G}$ such that $\eta\widetilde{\varphi}=\widetilde{\varphi}\xi$ holds on the whole of $\widetilde{X}$. By assigning to each $\xi\in F$ that element $\eta\in\mathcal{G}$ satisfying $\eta\widetilde{\varphi}=\widetilde{\varphi}\xi$ on $\widetilde{X}$, we produce an into-homomorphism $\Phi:\mathcal{F}\to\mathcal{G}$, called paired to φ.*

(iii) *The superposition of an inner automorphism of $\mathcal{G}$ and a homomorphism paired to φ, is again a homomorphism of $\mathcal{F}$ into $\mathcal{G}$, paired to φ.*

(iv) *For any two homomorphisms paired to φ, there exists an inner automorphism of $\mathcal{G}$, such that its superposition with the first homomorphism produces the second.*

(v) *If one of the homomorphisms paired to φ is onto, or an isomorphism into, so are the others.*

Proof. (i) Follows directly from [5] (p. 50, P. 1).

(ii) $g\widetilde{\varphi}\xi(\widetilde{x})=\varphi f\xi(\widetilde{x})=\varphi f(\widetilde{x})=g\widetilde{\varphi}(\widetilde{x})$ and since $(\widetilde{Y},g)$ is regular, there exists a single $\eta\in\mathcal{G}$ with $\eta\widetilde{\varphi}(\widetilde{x})=\widetilde{\varphi}\xi(\widetilde{x})$. $\eta\widetilde{\varphi}$ and $\widetilde{\varphi}\xi$ also satisfy $g\eta\widetilde{\varphi}=g\widetilde{\varphi}=\varphi f=\varphi f\xi=g\widetilde{\varphi}\xi$ on the whole of $\widetilde{X}$, so that $\eta\widetilde{\varphi}=\widetilde{\varphi}\xi$ is a consequence of [5] (p. 51, L. 1). If $\eta_1,\eta_2\in\mathcal{G}$ are images of $\xi_1,\xi_2\in\mathcal{F}$ under Φ, from $\eta_1\widetilde{\varphi}=\widetilde{\varphi}\xi_1$, $\eta_2\widetilde{\varphi}=\widetilde{\varphi}\xi_2$ it follows that $\eta_1\eta_2\widetilde{\varphi}=\eta_1\widetilde{\varphi}\xi_2=\widetilde{\varphi}\xi_1\xi_2$, and thus Φ is a homomorphism.

(iii) With $\zeta\in\mathcal{G}$, let $\Phi':\mathcal{F}\to\mathcal{G}$ be the homomorphism assigning to each $\xi\in\mathcal{F}$ the element $\eta'=\zeta\eta\zeta^{-1}\in\mathcal{G}$, where $\eta\in\mathcal{G}$ is the image of ξ under Φ, that is $\eta\widetilde{\varphi}=\widetilde{\varphi}\xi$. $\widetilde{\varphi}'=\zeta\widetilde{\varphi}$ maps $\widetilde{X}$ into $\widetilde{Y}$ and $g\widetilde{\varphi}'=g\zeta\widetilde{\varphi}=g\widetilde{\varphi}=\varphi f$ on $\widetilde{X}$, i. e. $\widetilde{\varphi}'$ is also paired to φ. From

$$\eta'\widetilde{\varphi}'=\zeta\eta\zeta^{-1}\zeta\widetilde{\varphi}=\zeta\eta\widetilde{\varphi}=\zeta\widetilde{\varphi}\xi=\widetilde{\varphi}'\xi$$

it follows now that the homomorphism Φ' is also paired to φ.

(iv) For $i=1,2$, let $\Phi_i: F\to G$ both be paired to φ, produced by $\eta_i\widetilde{\varphi}_i=\widetilde{\varphi}_i\xi$ with $g\widetilde{\varphi}_i=\varphi f$, $\xi\in F$, $\eta_i\in G$. Choose $\widetilde{x}\in\widetilde{X}$. Then $g\widetilde{\varphi}_1(\widetilde{x})=\varphi f(\widetilde{x})=g\varphi_2(\widetilde{x})$ and there exists $\zeta\in G$ with $\varphi_2(\widetilde{x})=\zeta\widetilde{\varphi}_1(\widetilde{x})$. Since $g\widetilde{\varphi}_2=\varphi f=g\widetilde{\varphi}_1=g\zeta\widetilde{\varphi}_1$, from [5] (p. 51, L. 1) it follows that $\widetilde{\varphi}_2=\zeta\widetilde{\varphi}_1$, hence $\eta_2\zeta\widetilde{\varphi}_1=\eta_2\widetilde{\varphi}_2=\widetilde{\varphi}_2\xi=\zeta\widetilde{\varphi}_1\xi=\zeta\eta_1\widetilde{\varphi}_1$, i. e. $\eta_2\zeta=\zeta\eta_1$. Then $\eta_2=\zeta\eta_1\zeta^{-1}$, where $\zeta\in G$ is independent of $\xi\in F$, and thus Φ_2 is the superposition of Φ_1 with an inner automorphism of G.

(v) Is an easy consequence of (iii) and (iv).

4.2. **Lemma.** *Let $\varphi: X\to Y$ be continuous, into, where X and Y are both connected, locally connected Hausdorff spaces, admitting simply connected covering spaces $(\widetilde{X},f)$ and $(\widetilde{Y},g)$. $\pi_1(X)$ is isomorphic to a retract*[4] *of $\pi_1(Y)$ if there exists a regular covering space $(\hat{Y},h)$ of Y, whose group $\mathcal{G}$ of automorphisms is the isomorphic image of $\pi_1(X)$, under an isomorphism of $\pi_1(X)$ onto $\mathcal{G}$, paired to φ.*

Proof. There exists $\hat{\varphi}:\widetilde{X}\to\hat{Y}$ with $h\psi=\varphi f$ on $\widetilde{X}$, (4.1. i), and, by assumption, the relation $\gamma\hat{\varphi}=\hat{\varphi}\xi$, where $\xi\in\pi_1(X)$, $\gamma\in\mathcal{G}$, defines an onto-isomorphism $\Phi:\pi_1(X)\to\mathcal{G}$, (4.1. v).

There exists a map $\psi:\widetilde{Y}\to\hat{Y}$ with $h\psi=g$ on $\widetilde{Y}$, and as can readily be seen, ψ is onto, $(\widetilde{Y},\psi)$ is a covering space of $\hat{Y}$ and the homomorphism $\Psi:\pi_1(Y)\to\mathcal{G}$ produced by $\gamma\psi=\psi\eta$, with $\eta\in\pi_1(Y)$, $\gamma\in\mathcal{G}$, is onto (4.1. i, ii).

There exists $\widetilde{\varphi}:\widetilde{X}\to\widetilde{Y}$ such that $\psi\widetilde{\varphi}=\hat{\varphi}$, hence

$$g\widetilde{\varphi}=h\psi\widetilde{\varphi}=h\hat{\varphi}=\varphi f$$

on $\widetilde{X}$, and the relation $\eta\widetilde{\varphi}=\widetilde{\varphi}\xi$, where $\xi\in\pi_1(X)$, $\eta\in\pi_1(Y)$, defines an into-homomorphism $\widetilde{\Phi}:\pi_1(X)\to\pi_1(Y)$, (4.1. ii).

From $\xi\in\pi_1(X)$, $\eta\in\pi_1(Y)$ with $\eta\widetilde{\varphi}=\widetilde{\varphi}\xi$, and $\gamma\in\mathcal{G}$ with $\gamma\psi=\psi\eta$, it follows that $\hat{\varphi}\xi=\psi\widetilde{\varphi}\xi=\psi\eta\widetilde{\varphi}=\gamma\psi\widetilde{\varphi}=\gamma\hat{\varphi}$, hence $\Phi(\xi)=\Psi\widetilde{\Phi}(\xi)$ for any $\xi\in\pi_1(X)$.

$\widetilde{\Phi}(\xi_1)=\widetilde{\Phi}(\xi_2)$ implies $\Phi(\xi_1)=\Psi\widetilde{\Phi}(\xi_1)=\Psi\widetilde{\Phi}(\xi_2)=\Phi(\xi_2)$, hence $\xi_1=\xi_2$, since Φ is an isomorphism. Thus

$$\widetilde{\Phi}:\pi_1(X)\to\pi_1(Y)$$

is an isomorphism into. $\mathcal{R}=\widetilde{\Phi}[\pi_1(X)]\subset\pi_1(Y)$ is a subgroup of $\pi_1(Y)$, isomorphic under $\widetilde{\Phi}$ to $\pi_1(X)$.

Let $P=\widetilde{\Phi}\Phi^{-1}\Psi:\pi_1(Y)\to\mathcal{R}\subset\pi_1(Y)$. If $\eta\in\mathcal{R}$, then $\eta=\widetilde{\Phi}(\xi)$ with $\xi\in\pi_1(X)$, and $P(\eta)=\widetilde{\Phi}\Phi^{-1}\Psi\widetilde{\Phi}(\xi)=\widetilde{\Phi}\Phi^{-1}\Phi(\xi)=\widetilde{\Phi}(\xi)=\eta$, that is: P is a retraction of $\pi_1(Y)$ onto $\mathcal{R}\subset\pi_1(Y)$.

[4]) The subgroup $R\subset G$ is a *retract* of the group G, if there is an endomorphism $\varrho: G\to R\subset G$, termed *retraction*, satisfying $\varrho(r)=r$ for each $r\in R$.

4.3. **Lemma.** *Let $(\tilde{E},f)$ be a regular covering space of the connected, locally connected Hausdorff space E, with $\mathcal{G}$ as its group of automorphisms. Let $A\subset E$ be a connected, locally connected subspace and $\tilde{A}$ a component of $f^{-1}(A)\subset\tilde{E}$. Then $(\tilde{A},f)$ is a regular covering space of A*[5]*; its group $\mathcal{F}$ of automorphisms is isomorphic to a subgroup $\mathcal{S}\subset\mathcal{G}$, under an into-isomorphism $\Omega:\mathcal{F}\to\mathcal{G}$, paired to the inclusion map $\omega: A\to A\subset E$. Necessary and sufficient for $\mathcal{S}=\mathcal{G}$ is the connectedness of $f^{-1}(A)$.*

Proof. That $(\tilde A,f)$ is a covering space of A follows from [5] (p. 42, L. 5). Let $\mathcal S$ be the set of all the $\sigma\in\mathcal G$ satisfying $\sigma(\tilde A)=\tilde A$. It is clear that $\mathcal S$ is a subgroup of $\mathcal G$. Let $\tilde\omega:\tilde A\to\tilde A\subset\tilde E$ be the inclusion map: it satisfies $f\tilde\omega=\omega f$ on the whole of $\tilde A$. For $\sigma\in\mathcal S$, $\xi=\tilde\omega^{-1}\sigma\tilde\omega$ is a homeomorphism of $\tilde A$ onto itself, satisfying $f\xi=f$ on $\tilde A$, hence $\xi\in\mathcal F$. Since $(\tilde E,f)$ is regular, for $\tilde a_1,\tilde a_2\in\tilde A$ with $f(\tilde a_1)=f(\tilde a_2)$, there exists $\sigma\in\mathcal G$ with $\sigma(\tilde a_1)=\tilde a_2$; thus $\sigma\in\mathcal S$ and the regularity of $(\tilde A,f)$ follows from $\xi(\tilde a_1)=\tilde a_2$, with $\xi=\tilde\omega^{-1}\sigma\tilde\omega\in\mathcal F$.

The relation $\eta\tilde\omega=\tilde\omega\xi$ on $\tilde A$, with $\xi\in\mathcal F$, $\eta\in\mathcal G$, defines an into-homomorphism $\Omega:\mathcal F\to\mathcal G$, paired to ω, (4.1. ii). From $\eta\tilde\omega=\tilde\omega\xi$ it follows that $\eta(\tilde A)=\tilde A$, hence $\Omega(\mathcal F)\subset\mathcal S\subset\mathcal G$. If $\sigma\in\mathcal S$, $\xi=\tilde\omega^{-1}\sigma\tilde\omega\in\mathcal F$ and $\tilde\omega\xi=\sigma\tilde\omega$ holds on $\tilde A$; in addition, from $\tilde\omega\xi_1=\sigma\tilde\omega=\tilde\omega\xi_2$, with $\xi_1,\xi_2\in\mathcal F$ and $\sigma\in\mathcal S$, follows $\xi_1=\xi_2$. Thus Ω is an isomorphism of $\mathcal F$ onto the subgroup $\mathcal S\subset\mathcal G$.

If $f^{-1}(A)$ is connected, $f^{-1}(A)=\tilde A$, $\eta f^{-1}(A)=f^{-1}(A)$ for any $\eta\in\mathcal G$, and so $\mathcal S=\mathcal G$. Conversely, if: $\mathcal S=\mathcal G$, $\tilde B$ is any component of $f^{-1}(A)$, $\tilde a\in\tilde A$, $a=f(\tilde a)$ and $\tilde b\in\tilde B\cap f^{-1}(a)$, there is an $\eta\in\mathcal G=\mathcal S$ with $\eta(\tilde a)=\tilde b$, hence $\tilde A=\tilde B$ and $f^{-1}(A)$ is connected.

5. Essential for our purpose, is the possibility of extending a covering space of a subset $A\subset E$ to a suitable neighbourhood of A in E.

Denoting by E a normal ([9], p. 26), locally connected Hausdorff space, and by $A\subset E$ a compact, connected, locally connected subspace, we shall prove the following two propositions:

5.1. **Proposition.** *Let $(\tilde U,g)$ be a covering space of U, where $A\subset U\subset E$ and U is open in E and connected. There exists then a set G, open in E and connected, satisfying $A\subset G\subset U$, and such that:*

(i) *$\tilde G\cap g^{-1}(A)=\tilde A$ holds for any component $\tilde A$ of $g^{-1}(A)$, if $\tilde G$ denotes the component of $g^{-1}(G)$ that contains $\tilde A$;*

(ii) *Under the same assumptions, the covering space $(\tilde G,g)$ of G is regular, if so is the covering space $(\tilde A,g)$ of A.*

5) Where f in $(\tilde A,f)$ means the contraction to $\tilde A$ of $f:\tilde E\to E$.

5.2. **Proposition.** *For each covering space $(\tilde A,f)$ of A, there exists a set $U\supset A$, open in E and connected, possessing a covering space $(\tilde U,g)$ such that:*

(i) *$g^{-1}(A)$ is connected;*

(ii) *The covering spaces $(\tilde A,f)$ and $(g^{-1}(A),g)$ of the space A are isomorphic, in the sense of* [5] (p. 43, § VII).

Proof of (5.1). Since A is compact and locally connected, and then by a suitable star-refinement ([9], p. 234) in the normal and locally connected space E, one can find:

a) a finite collection of n sets $V_j\subset U$, open in E and evenly covered by $(\tilde U,g)$, with each $A\cap V_j$ non-empty and connected, satisfying $A\subset\bigcup_{j=1}^n V_j$;

b) a finite collection of m sets G_i, open in E and connected, with each $A\cap G_i\neq 0$, satisfying $A\subset\bigcup_{i=1}^m G_i$ and $\{\mathfrak{St}\,G_i\}\succ\{V_j\}$.

$G=\bigcup_{i=1}^m G_i$ is then connected, open in E, hence locally connected and $A\subset G\subset U$.

Notice first that if $\tilde V_j$ is any component of a $g^{-1}(V_j)$, since V_j is evenly covered by $(\tilde U,g)$ and thus $g^{-1}(A)\cap\tilde V_j$ is topologically mapped by g onto $A\cap V_j$, each $g^{-1}(A)\cap\tilde V_j$ is itself connected. Thus, if it meets a component $\tilde A$ of $g^{-1}(A)$, it is entirely contained in $\tilde A$.

Now let $\tilde A$ be an arbitrary component of $g^{-1}(A)$ and let $\tilde G$ denote the union of all the components meeting $\tilde A$, of the different sets $g^{-1}(G_i)$, $i=1,\dots,m$. It is clear that: $\tilde G\supset\tilde A$, $\tilde G$ is connected, $\tilde G\subset g^{-1}(G)$ and $\tilde G\cap g^{-1}(A)\supset\tilde A$. Any $\tilde x\in\tilde G\cap g^{-1}(A)$ is contained in a component $\tilde G_h$, meeting $\tilde A$, of some $g^{-1}(G_h)$, $1\leqslant h\leqslant m$. There is now a $V_k\supset G_h$, $1\leqslant k\leqslant n$, and let $\tilde V_k$ denote the component of $g^{-1}(V_k)$ containing $\tilde G_h$. Since $0\neq\tilde A\cap\tilde G_h\subset g^{-1}(A)\cap\tilde V_k$, it follows that $g^{-1}(A)\cap\tilde V_k\subset\tilde A$, hence $\tilde x\in\tilde A$ and so $\tilde G\cap g^{-1}(A)=\tilde A$.

To prove (i) it only remains to show that $\tilde G$ is a component of $g^{-1}(G)$, and since $\tilde G$ is connected, this will be accomplished by proving $\tilde G$ to be open and closed in $g^{-1}(G)$. For this purpose, let $\tilde p\in g^{-1}(G)$ be adherent to $\tilde G$, and let $p=g(\tilde p)\in G$. There is a $G_l\ni p$, $1\leqslant l\leqslant m$; let $\tilde G_l$ be the component of $g^{-1}(G_l)$ containing $\tilde p$. The local connectedness of $\tilde U$ implies that $\tilde G_l$ is a neighbourhood of $\tilde p$; thus there is a $\tilde q\in\tilde G\cap\tilde G_l$, hence $\tilde q\in\tilde G_r$, where $\tilde G_r$ is a component of $g^{-1}(G_r)$, meeting $\tilde A$, $1\leqslant r\leqslant m$. From $g(\tilde q)\in G_l\cap G_r$ it follows that $G_l\cup G_r\subset V_s$ for some s, $1\leqslant s\leqslant n$, and if $\tilde V_s$ is the component of $\tilde q$ in $g^{-1}(V_s)$, then $\tilde G_l\cup\tilde G_r\subset\tilde V_s$, $0\neq\tilde A\cap\tilde G_r\subset g^{-1}(A)\cap\tilde V_s$, hence $g^{-1}(A)\cap\tilde V_s\subset\tilde A$. From $A\cap G_l\neq 0$ it follows that $0\neq g^{-1}(A)\cap\tilde G_l\subset g^{-1}(A)\cap\tilde V_s\subset\tilde A$, hence $\tilde A\cap\tilde G_l\neq 0$, and finally $\tilde p\in\tilde G_l\subset\tilde G$. Thus $\tilde p$ is also interior to $\tilde G$, and this proves (i).

Before considering (ii), note that, since it is connected and contained in some V_j, each G_i is also evenly covered ([5], p. 41, L. 2) by $(\tilde{U}, g)$. Then, for any $x \in G_i$ and component $\tilde{G}_i$ of $g^{-1}(G_i)$, $\tilde{G}_i \cap g^{-1}(x)$ contains exactly one point. Since $G_i \subset G$ and $\tilde{G}$ is a component of $g^{-1}(G)$, it is clear that if $\tilde{G}_i$ meets $\tilde{G}$, it is entirely contained in $\tilde{G}$, hence $\tilde{G}_i \cap g^{-1}(A) \subset \tilde{G} \cap g^{-1}(A) = \tilde{A}$.

Let now $\mathcal{F}$ and $\mathcal{G}$ be the groups of automorphisms of the covering spaces $(\tilde{A}, g)$ and $(\tilde{G}, g)$. Choose $a_i \in A \cap G_i$ for $i = 1, \ldots, m$ and put $H(x) = \{h \mid G_h \ni x\}$ for each $x \in G$. We extend now to the whole of $\tilde{G}$ any given $\xi \in \mathcal{F}$.

For $\tilde{x} \in \tilde{G}$, hence $x = g(\tilde{x}) \in G$ and any $k \in H(x)$ let be: $\tilde{G}_k$ the component of $g^{-1}(G_k)$ containing $\tilde{x}$, $\tilde{a}_k = \tilde{G}_k \cap g^{-1}(a_k) \subset \tilde{A}$, $\tilde{a}'_k = \xi(\tilde{a}_k)$, $\tilde{G}'_k$ the component of $\tilde{a}'_k$ in $g^{-1}(G_k)$ and $\eta_k(\tilde{x}) = \tilde{G}'_k \cap g^{-1}(x)$. First we prove, that for given ξ, $\eta_k(\tilde{x})$ is independent of the subscript $k \in H(x)$ which served for its computation. In fact, for any other $l \in H(x)$, $x \in G_k \cap G_l$ implies $x, a_k, a_l \in G_k \cup G_l \subset V_s$ for some s, $1 \leqslant s \leqslant n$. If $\tilde{V}_s$ denotes the component of $\tilde{x}$ in $g^{-1}(V_s)$, from [5] (p. 41, L. 2) it follows that $\tilde{G}_h = \tilde{V}_s \cap g^{-1}(G_h)$ and $\tilde{a}_h = \tilde{V}_s \cap g^{-1}(a_h)$ for $h = k, l$. Since it is connected and meets $\tilde{A}$, $g^{-1}(A) \cap \tilde{V}_s \subset \tilde{A}$ and for $h = k, l$, $\tilde{a}'_h \in \xi[g^{-1}(A) \cap \tilde{V}_s]$ which is still connected. If $\tilde{V}'_s$ is the component of $g^{-1}(V_s)$ containing $\xi[g^{-1}(A) \cap \tilde{V}_s]$, from $\tilde{a}'_h \in \tilde{G}'_h \cap \tilde{V}'_s$ and $G_h \subset V_s$ it follows that $\tilde{G}'_h \subset \tilde{V}'_s$, hence, since g is univalent on $\tilde{V}'_s$:

$$\eta_h(\tilde{x}) = \tilde{V}'_s \cap g^{-1}(x) \quad \text{for} \quad h = k, l.$$

Denoting this common value by $\eta(\tilde{x})$, we have a single valued function $\eta: \tilde{G} \to \tilde{G}$, satisfying $g\eta = g$, and a straightforward calculation shows that η is a homeomorphism of $\tilde{G}$ onto itself: $\eta \in \mathcal{G}$.

Suppose now that $(\tilde{A}, g)$ is regular and let $\tilde{x}^1, \tilde{x}^2 \in \tilde{G}$ with $g(\tilde{x}^1) = g(\tilde{x}^2) = x$, hence $x \in G_h$ for some $h \in H(x)$. Let $\tilde{G}^t_h$ be the component of $g^{-1}(G_h)$ containing $\tilde{x}^t$, and $\tilde{a}^t_h = \tilde{G}^t_h \cap g^{-1}(a_h) \subset \tilde{A}$ for $t = 1, 2$. Since $(\tilde{A}, g)$ is regular, there exists $\xi \in \mathcal{F}$ with $\xi(\tilde{a}^1_h) = \tilde{a}^2_h$ and the corresponding $\eta \in \mathcal{G}$ satisfies $\eta(\tilde{x}^1) = \tilde{x}^2$, hence $(\tilde{G}, g)$ is regular.

P r o o f of (5.2). Since A is compact and locally connected, then by suitable star-refinements ([9], p. 324) in the normal and locally connected space E, one can find:

a) a finite collection of r sets W_s, open in E, with each $A \cap W_s$ evenly covered by $(\tilde{A}, f)$, such that $A \subset \bigcup_{s=1}^{r} W_s$;

b) a finite collection of n sets V_j, open in E and connected, with each $A \cap V_j$ non-empty and connected, such that $A \subset \bigcup_{j=1}^{n} V_j$ and $\{\mathfrak{St} V_j\} \succ \{W_s\}$;

c) a finite collection of m sets U_i, open in E and connected, with each $A \cap U_i \neq 0$, such that $A \subset \bigcup_{i=1}^{m} U_i$ and $\{\mathfrak{St} U_i\} \succ \{V_j\}$.

$U = \bigcup_{i=1}^{m} U_i$ is connected, open in E, hence locally connected and $A \subset U$.

Choose $a_i \in A \cap U_i$ for $i = 1, \ldots, m$, and for each $x \in U$ put:

$$H(x) = \{h \mid U_h \ni x\}, \quad S(x) = \bigcup_{h \in H(x)} U_h \quad \text{and} \quad \alpha(x) = \{a_h \mid h \in H(x)\};$$

thus $\alpha(x) \subset S(x)$. $f: \tilde{A} \to A \subset E$ maps $\tilde{A}$ into E and for any subset $B \subset E$, $f^{-1}(B) = f^{-1}(A \cap B)$. This remark will be extensively used to shorten the notation in the sequel.

Let $\Sigma = \bigcup_{i=1}^{m} U_i \times f^{-1}(a_i)$. It is readily seen that taking as a base for Σ the family of all the sets $N_i \times \tilde{a}_i$, with $N_i \subset U_i$ connected and open in U, and $\tilde{a}_i \in f^{-1}(a_i)$, $1 \leqslant i \leqslant m$, turns Σ into a topological space which is locally connected.

Now let $\tilde{U}$ be the collection of all the sets $x \times \tilde{\alpha}(x)$ such that: $x \in U$ and $\tilde{\alpha}(x) = \tilde{V}_l \cap f^{-1}[\alpha(x)]$, where $\tilde{V}_l$ is a component of $f^{-1}(V_l)$ with $V_l \supset S(x) \supset \alpha(x)$; such V_l exist according to c).

Each $x \times \tilde{\alpha}(x)$ is a subset of Σ and each point of Σ is in a set of $\tilde{U}$. Suppose $x \times \tilde{a} \in [y \times \tilde{\alpha}(y)] \cap [z \times \tilde{\alpha}(z)]$. Then

$$y = x = z, \quad \tilde{\alpha}(y) = \tilde{V}_l \cap f^{-1}[\alpha(x)], \quad \tilde{\alpha}(z) = \tilde{V}_k \cap f^{-1}[\alpha(x)],$$

where $\tilde{V}_l$, $\tilde{V}_k$ are components of $f^{-1}(V_l)$, $f^{-1}(V_k)$, satisfying $V_l \cap V_k \supset S(x) \supset \alpha(x)$. From b) follows $V_l \cup V_k \subset W_s$ for some s, $1 \leqslant s \leqslant r$ hence, with $\tilde{W}_s$ denoting the component of $\tilde{a}$ in $f^{-1}(W_s)$, thus $\tilde{V}_l \cup \tilde{V}_k \subset \tilde{W}_s$, since f is univalent on $\tilde{W}_s$, it follows that

$$\tilde{\alpha}(y) = \tilde{V}_l \cap f^{-1}[\alpha(x)] = \tilde{W}_s \cap f^{-1}[\alpha(x)] = \tilde{V}_k \cap f^{-1}[\alpha(x)] = \tilde{\alpha}(z).$$

Thus any two sets of $\tilde{U}$ are disjoint or identical.

Let $\varphi: \Sigma \to \tilde{U}$ assign to each point of Σ that set of $\tilde{U}$, in which the given point is contained: $\varphi(\Sigma) = \tilde{U}$. Topologize $\tilde{U}$ with the strongest topology for which φ is continuous.

From $x \times \tilde{a} \in \varphi^{-1}\varphi(N_i \times \tilde{a}_i)$, where $N_i \subset U_i$ is connected, open in U, and $\tilde{a}_i \in f^{-1}(a_i)$, it follows that $\tilde{a} = \tilde{a}_j \in f^{-1}(a_j)$ and $x \in N_i \cap U$ for some j, $1 \leqslant j \leqslant m$. If N_j is now the component of $N_i \cap U_j$ containing x, it is readily seen on account of b) that

$$x \times \tilde{a} \in N_j \times \tilde{a}_j \subset \varphi^{-1}\varphi(N_i \times \tilde{a}_i);$$

thus $\varphi^{-1}\varphi(N_i \times \tilde{a}_i)$ is open in Σ, and since $\{N_i \times \tilde{a}_i\}$ is a base for Σ, φ is an open map.

As a consequence, $\widetilde{U}=\varphi(\Sigma)$ is locally connected.

For $x\times\widetilde{a}\in\Sigma$, also let $pr(x\times\widetilde{a})=x$. $pr(\Sigma)=U$ and pr is continuous and open. Finally, let

$$g=pr\varphi^{-1}.$$

Notice that $g:\widetilde{U}\to U$ is single-valued, onto, and since pr and φ are continuous and open, the same holds for g.

It is readily seen that $\varphi pr^{-1}(U_l)=\varphi[U_l\times f^{-1}(a_l)]$ and it follows that:

$$g^{-1}(U_l)=\bigcup_{\widetilde{a}_l\in f^{-1}(a_l)}\varphi(U_l\times\widetilde{a}_l).$$

The sets $\varphi(U_l\times\widetilde{a}_l)$, for $\widetilde{a}_l\in f^{-1}(a_l)$, are open in $\widetilde{U}$, connected and no two of them meet. Thus they are the components of $g^{-1}(U_l)$, and since each of them is univalently mapped onto U_l by g, which is open and continuous, each U_l is evenly covered by $(\widetilde{U},g)$.

Now, for $\widetilde{a}\in\widetilde{A}$ let: $a=f(\widetilde{a})\in A$, $V_l\supset S(a)\ni a\cup\alpha(a)$; with $1\leqslant l\leqslant n$, $\widetilde{V}_l$ be the component of $\widetilde{a}$ in $f^{-1}(V_l)$, $\widetilde{\alpha}(a)=\widetilde{V}_l\cap f^{-1}[\alpha(a)]$ and $\psi(\widetilde{a})=a\times\widetilde{\alpha}(a)\in\widetilde{U}$. On account of b) $\psi:\widetilde{A}\to\widetilde{U}$ is single valued and $g\psi(\widetilde{a})=f(\widetilde{a})$ for each $\widetilde{a}\in\widetilde{A}$, thus $g\psi(\widetilde{A})=A$ and $\psi(\widetilde{A})\subset g^{-1}(A)$. Let $a^*\in g^{-1}(A)\subset\varphi(\Sigma)$, hence $a^*=\varphi(a\times\widetilde{a}_l)$ with $a\in U_l$, $a_l\in\alpha(a)$ and $g(a^*)=a\in A\subset U$. Let: $V_l\supset S(a)\supset a\cup\alpha(a)$, $\widetilde{V}_l$ be the component of $f^{-1}(V_l)$ containing $\widetilde{a}_l$, $\widetilde{\alpha}(a)=\widetilde{V}_l\cap f^{-1}[\alpha(a)]\ni\widetilde{a}_l$ and $\widetilde{a}=\widetilde{V}_l\cap f^{-1}(a)$. It follows that $\psi(\widetilde{a})=a\times\widetilde{\alpha}(a)=\varphi(a\times\widetilde{a}_l)=a^*$, hence $\psi(\widetilde{A})=g^{-1}(A)$.

From $\psi(\widetilde{a}')=\psi(\widetilde{a}'')$ it follows that $\widetilde{a}'$ and $\widetilde{a}''$ are in the same component of some $f^{-1}(V_l)$ and $f(\widetilde{a}')=f(\widetilde{a}'')$, hence $\widetilde{a}'=\widetilde{a}''$ and thus $\psi:\widetilde{A}\to g^{-1}(A)$ is one-one, onto.

Let $\widetilde{a}\in\widetilde{A}$, $a^*=\psi(\widetilde{a})\in\varphi(N_l\times\widetilde{a}_l)$ with $N_l\subset U_l$, connected, open in U and $\widetilde{a}_l\in f^{-1}(a_l)$. It follows that $\psi(\widetilde{a})=\varphi(a\times\widetilde{a}_l)$ with $a\in N_l\subset U_l$ and $f(\widetilde{a})=a\in A$. Let G' be open in U, with $a\in G'\subset N_l$ and $G=A\cap G'$ evenly covered by $(\widetilde{A},f)$. The component $\widetilde{G}$ of $f^{-1}(G)$ containing $\widetilde{a}$ is open in $\widetilde{A}$, and on account of b) $a^*\in\psi(\widetilde{G})\subset\varphi(N_l\times\widetilde{a}_l)$. Since φ is open and continuous, $\{\varphi(N_l\times\widetilde{a}_l)\}$ is a base for $\widetilde{U}$, hence $\psi:\widetilde{A}\to g^{-1}(A)\subset\widetilde{U}$ is continuous.

As a consequence, $g^{-1}(A)=\psi(\widetilde{A})$ is connected, and so is $\widetilde{U}$. $(\widetilde{U},g)$ and $(g^{-1}(A),g)$ are thus covering spaces of U and A. Since $f:\widetilde{A}\to A$ is continuous and open, while $\psi:\widetilde{A}\to g^{-1}(A)$ is continuous and satisfies $f=g\psi$ on $\widetilde{A}$, as can easily be seen, ψ is also an open map, thus a homeomorphism.

The covering spaces $(\widetilde{A},f)$ and $(g^{-1}(A),g)$ of A are thus isomorphic under ψ and (5.2) is proved.

An immediate consequence of (5.1) is the

5.3. **Corollary.** *If A is evenly covered by $(\widetilde{U},g)$, there exists a set G, open in E, satisfying $A\subset G\subset U$, and evenly covered by $(\widetilde{U},g)$.*

5.4. ***Remark.*** Throughout this section, the assumed compactness of the set A is used only to derive the statements a), b) and a), b), c) in the proofs of (5.1) and (5.2). Similar statements hold, however, in some other cases too. For instance, they hold when E is a metric, even non-separable, connected and locally connected space, while A is an arbitrary connected and locally connected subspace. The open coverings of A, there considered, need not be finite any longer, but this requires no change whatever in the remainder of the proofs. Thus (5.1) and (5.2) may be restated with A denoting an arbitrary connected and locally connected subspace of the metric, connected and locally connected space E.

6. We pass now to the investigation of the so called ε-mappings, but first define:

6.1. *ε denoting an open covering* ([9], p. 13) *of the space X, $\varphi:X\to Y$ is an ε-mapping when it is onto, continuous, and each point $y\in Y$ has a neighbourhood whose inverse-image is entirely contained in a member of ε. The space Y is an ε-image of X if there is an ε-mapping $\varphi:X\to Y$.*

The two spaces X and Y are termed quasi-homeomorphic ([8], p. 252) *if, for each open covering ε of X, Y is an ε-image of X, and conversely.*

For metric compact spaces, these definitions become the usual special cases ([1], p. 103), with ε replaced by a positive number.

We shall also need:

6.2. *X denoting a connected and locally connected space, $\varphi:X\to Y$ is termed quasi-monotone if it is onto, continuous, and for each domain (open connected subset) $V\subset Y$ and component U of $\varphi^{-1}(V)$, $\varphi(U)=V$ holds. It is monotone if it is onto, continuous, and for each point $y\in Y$ and domain $V\subset Y$, $\varphi^{-1}(y)$ and $\varphi^{-1}(V)$ are connected* (cf. [14], p. 151, 152, 127).

We are now in position to state and prove:

6.3. **Lemma.** *Let $(\widetilde{X},f)$ be a covering space of X, with $\mathcal{F}$ as group of automorphisms, and let $Y=\varphi(X)$, φ continuous, where X and Y are connected, locally connected Hausdorff spaces.*

Consider the following two assumptions:

a) There exists an open connected[6] *base $\{V\}$ for Y and an open connected covering*[6] *$\{U\}$ of X, such that: $\{\varphi^{-1}(V)\}\succ\{U\}$ and $U_1\cup U_2$ is evenly covered by $(\widetilde{X},f)$ for any two $U_1,U_2\in\{U\}$ with $U_1\cap U_2\neq 0$;*

b) $\varphi:X\to Y$ is monotone and there exists: an open connected base $\{V\}$ for Y, and also an open connected covering $\{U\}$ of X, such that each U is evenly covered by $(\widetilde{X},f)$ and $\{\varphi^{-1}(V)\}\succ\{U\}$.

Each assumption implies that Y possesses a covering space $(\hat{Y},h)$, with $\mathcal{G}$ as its group of automorphisms, such that:

(i) $(\widetilde{X},f)$ and $(\hat{Y},h)$ have the same number of leaves;

(ii) $\mathcal{G}$ is the isomorphic image of $\mathcal{F}$, under an onto-isomorphism $\Phi:\mathcal{F}\to\mathcal{G}$, paired to φ;

(iii) The covering spaces $(\widetilde{X},f)$ and $(\hat{Y},h)$ are both regular or not;

(iv) Assumption b) implies the existence of a monotone $\hat{\varphi}:\widetilde{X}\to\hat{Y}$, paired to φ.

Proof. Let $\hat{Y}$ denote the family of all the sets $\widetilde{U}\cap f^{-1}\varphi^{-1}(y)$, such that:

$y\in Y$, $\varphi^{-1}(y)\subset\varphi^{-1}(V)\subset U$ for some U and V, $\widetilde{U}$ is a component of $f^{-1}(U)$.

Each $\widetilde{x}\in\widetilde{X}$ is in a set of $\hat{Y}$ and each set of $\hat{Y}$ is a subset of $\widetilde{X}$. Suppose

$$\widetilde{x}\in[\widetilde{U}_1\cap f^{-1}\varphi^{-1}(V)]\cap[\widetilde{U}_2\cap f^{-1}\varphi^{-1}(V)],$$

where $f(\widetilde{x})\in\varphi^{-1}(V)\subset U_1\cap U_2$ and $\widetilde{U}_i$ is the component of $f^{-1}(U_i)$ containing $\widetilde{x}$, for $i=1,2$.

If a), from $f(\widetilde{x})\in U_1\cap U_2$ it follows that $W=U_1\cup U_2$ is evenly covered by $(\widetilde{X},f)$, and if $\widetilde{W}$ denotes the component of $f^{-1}(W)$ that contains $\widetilde{x}$, since f is univalent on $\widetilde{W}$, it follows that:

$$\widetilde{U}_i\cap f^{-1}\varphi^{-1}(V)=\widetilde{W}\cap f^{-1}\varphi^{-1}(V),\quad i=1,2.$$

If b), $\widetilde{U}_i\cap f^{-1}\varphi^{-1}(V)$ is the component of $f^{-1}\varphi^{-1}(V)$ containing $\widetilde{x}$, $i=1,2$.

Hence in both cases $\widetilde{U}_1\cap f^{-1}\varphi^{-1}(V)=\widetilde{U}_2\cap f^{-1}\varphi^{-1}(V)$.

As a consequence, any two sets of $\hat{Y}$ are either disjoint or identical.

[6]) i. e. each of its members is connected.

Let $\hat{\varphi}(\widetilde{x})\in\hat{Y}$ denote the set of $\hat{Y}$ containing $\widetilde{x}\in\widetilde{X}$. $\hat{\varphi}:\widetilde{X}\to\hat{Y}$ is single valued, onto, and the same holds for

$$h=\varphi f\hat{\varphi}^{-1}:\hat{Y}\to Y.$$

In addition, $h\hat{\varphi}=\varphi f$ holds on the whole of $\widetilde{X}$, and from $f(\widetilde{x}_1)=f(\widetilde{x}_2)$, $\hat{\varphi}(\widetilde{x}_1)=\hat{\varphi}(\widetilde{x}_2)$ it follows easily that $\widetilde{x}_1=\widetilde{x}_2$.

If we take as a base for $\hat{Y}$, the family of all the sets $\hat{\varphi}[\widetilde{U}\cap f^{-1}\varphi^{-1}(V)]$, with $\varphi^{-1}(V)\subset U$, $V\in\{V\}$, $U\in\{U\}$, and $\widetilde{U}$ a component of $f^{-1}(U)$, then the set $\hat{Y}$ becomes a topological space.

Choose now V and $U\supset\varphi^{-1}(V)$. Let $\widetilde{U}_\alpha$ be the components of $f^{-1}(U)$.

The definition of $\hat{\varphi}$ implies

$$\hat{\varphi}^{-1}\hat{\varphi}[\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]=\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V),$$

which is open in $\widetilde{X}$, hence, since $\hat{\varphi}[\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]$ is basic for $\hat{Y}$, $\hat{\varphi}$ is continuous. As a consequence, $\hat{Y}=\hat{\varphi}(\widetilde{X})$ is connected.

Under the same assumptions, from

$$h^{-1}(V)=\cup_\alpha\hat{\varphi}[\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]$$

and

$$h\hat{\varphi}[\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]=\varphi f[\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]=V$$

it follows that h is continuous and open. In addition, from

$$\widetilde{x}_i\in\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V),\quad i=1,2,\quad\text{and}\quad h\hat{\varphi}(\widetilde{x}_1)=h\hat{\varphi}(\widetilde{x}_2)$$

it follows that

$$\varphi f(\widetilde{x}_1)=h\hat{\varphi}(\widetilde{x}_1)=h\hat{\varphi}(\widetilde{x}_2)=\varphi f(\widetilde{x}_2)=y\in V,$$

hence $f(\widetilde{x}_i)\in\varphi^{-1}(y)\subset\varphi^{-1}(V)\subset U$ and $\widetilde{x}_i\in\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(y)$, $i=1,2$, thus finally $\hat{\varphi}(\widetilde{x}_1)=\hat{\varphi}(\widetilde{x}_2)$, which shows that h is univalent on each set $\hat{\varphi}[\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]$. Since it is continuous and open, h maps each such set topologically onto V. It follows that each $\hat{\varphi}[\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]$ is connected.

One consequence is, that $\hat{Y}$ is locally connected.

On the other hand, as they are connected, open and no two of them meet, the sets $\hat{\varphi}[\widetilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]$, for fixed V and $U\supset\varphi^{-1}(V)$, are the components of $h^{-1}(V)$, and so each V is evenly covered by $(\hat{Y},h)$. Since $\{V\}$ is a base for Y, $(\hat{Y},h)$ is a covering space of Y.

For $x\in X$ and $y=\varphi(x)\in Y$, $\hat{\varphi}$ is a one-one transformation of $f^{-1}(x)$ to $h^{-1}(y)$, and this proves (i).

If $\xi\in\mathcal{F}$, $\gamma=\Phi(\xi)=\hat{\varphi}\xi\hat{\varphi}^{-1}$ is easily seen to be a homeomorphism of $\hat{Y}$ onto itself, satisfying on the whole of $\hat{Y}$:

$$h\gamma=h\hat{\varphi}\xi\hat{\varphi}^{-1}=\varphi f\xi\hat{\varphi}^{-1}=\varphi f\hat{\varphi}^{-1}=h.$$

Hence $\gamma\in\mathcal{G}$ and in addition $\gamma\hat{\varphi}=\hat{\varphi}\xi\hat{\varphi}^{-1}\hat{\varphi}=\hat{\varphi}\xi$ holds on $\tilde{X}$.

If $\gamma\in G$, let $\xi(\tilde{x})=f^{-1}f(\tilde{x})\cap\hat{\varphi}^{-1}\gamma\hat{\varphi}(\tilde{x})$ for any $\tilde{x}\in\tilde{X}$. ξ is a single-valued transformation of $\tilde{X}$ into itself, satisfying $f\xi=f$. To prove its continuity, choose U, V, and W open in X, evenly covered by $(\tilde{X},f)$, such that $W\subset\varphi^{-1}(V)\subset U$. If $\tilde{W}$ is now a component of $f^{-1}(W)$ and $\tilde{U}$ the component of $f^{-1}(U)$ containing $\tilde{W}$, then $\tilde{W}\subset\tilde{U}\cap f^{-1}\varphi^{-1}(V)$, hence

(*) $$\gamma\hat{\varphi}(\tilde{W})\subset\gamma\hat{\varphi}[\tilde{U}\cap f^{-1}\varphi^{-1}(V)].$$

Taking into account the fact that the components of $h^{-1}(V)$ are sets $\hat{\varphi}[\tilde{U}_\alpha\cap f^{-1}\varphi^{-1}(V)]$, with $\tilde{U}_\alpha$ a component of $f^{-1}(U)$, and since $\gamma\in\mathcal{G}$ carries components of $h^{-1}(V)$ onto such components, it follows that:

(⁑) $$\gamma\hat{\varphi}[\tilde{U}\cap f^{-1}\varphi^{-1}(V)]=\hat{\varphi}[\tilde{U}'\cap f^{-1}\varphi^{-1}(V)],$$

where $\tilde{U}'$ is also a component of $f^{-1}(U)$. From (*) and (⁑) it follows that

$$\xi(\tilde{W})\subset f^{-1}f(\tilde{W})\cap\hat{\varphi}^{-1}\gamma\hat{\varphi}(\tilde{W})\subset f^{-1}(W)\cap\hat{\varphi}^{-1}\hat{\varphi}[\tilde{U}'\cap f^{-1}\varphi^{-1}(V)]=$$
$$=f^{-1}(W)\cap\tilde{U}'\cap f^{-1}\varphi^{-1}(V)=\tilde{U}'\cap f^{-1}(W)=\tilde{W}',$$

where $\tilde{W}'$ is again a component of $f^{-1}(W)$. Since $\tilde{X}$ is locally connected, $\tilde{W}$ and $\tilde{W}'$ are open and ξ is proved to be continuous. The same holds for $\xi'=f^{-1}f\cap\hat{\varphi}^{-1}\gamma^{-1}\hat{\varphi}$, and from $\xi'=\xi^{-1}$ it follows that ξ is a homeomorphism of $\tilde{X}$ onto itself, that is $\xi\in\mathcal{F}$.

As can easily be seen, from $\xi=f^{-1}f\cap\hat{\varphi}^{-1}\gamma\hat{\varphi}$ it follows that $\hat{\varphi}\xi\hat{\varphi}^{-1}=\gamma$, hence each $\gamma\in\mathcal{G}$ is the image under Φ of some $\xi\in\mathcal{F}$.

$\gamma_1\gamma_2=\hat{\varphi}\xi_1\hat{\varphi}^{-1}\hat{\varphi}\xi_2\hat{\varphi}^{-1}=\hat{\varphi}\xi_1\xi_2\hat{\varphi}^{-1}$ follows from $\gamma_i=\hat{\varphi}\xi_i\hat{\varphi}^{-1}$, $i=1,2$, and $\xi_1=\xi_2$ follows from $\hat{\varphi}\xi_1\hat{\varphi}^{-1}=\gamma_1=\gamma_2=\hat{\varphi}\xi_2\hat{\varphi}^{-1}$.

We have thus produced an onto-isomorphism $\Phi:\mathcal{F}\to\mathcal{G}$, which on account of $h\hat{\varphi}=\varphi f$ and $\gamma\hat{\varphi}=\hat{\varphi}\xi$, is paired to φ.

Suppose now that $(\tilde{X},f)$ is regular, and let $\hat{y}_1,\hat{y}_2\in\hat{Y}$, $h(\hat{y}_1)=h(\hat{y}_2)=y$. It follows that $\hat{\varphi}^{-1}(\hat{y}_i)=\tilde{U}_i\cap f^{-1}\varphi^{-1}(y)$, with $\varphi^{-1}(y)\subset\varphi^{-1}(V)\subset U$ for some U and V, $\tilde{U}_i$ denoting a component of $f^{-1}(U)$, $i=1,2$. If $x\in\varphi^{-1}(y)$ and $\tilde{x}_i\in\tilde{U}_i\cap f^{-1}(x)\subset\hat{\varphi}^{-1}(\hat{y}_i)$, $i=1,2$, there is a $\xi\in\mathcal{F}$ with $\xi(\tilde{x}_1)=\tilde{x}_2$ and $\gamma=\Phi(\xi)$ satisfies $\gamma(\hat{y}_1)=\hat{y}_2$. Conversely, if $(\hat{Y},h)$ is regular and $f(\tilde{x}_1)=f(\tilde{x}_2)=x$, then $h\hat{\varphi}(\tilde{x}_1)=h\hat{\varphi}(\tilde{x}_2)$ follows and there is a $\gamma\in\mathcal{G}$ with $\gamma\hat{\varphi}(\tilde{x}_1)=\gamma\hat{\varphi}(\tilde{x}_2)$. Then $\xi=\Phi^{-1}(\gamma)$ satisfies $\xi(\tilde{x}_1)=\tilde{x}_2$.

Now assume b). Then, with $\varphi^{-1}(y)\subset\varphi^{-1}(V)\subset U$ for some U, V, and $\tilde{U}$ denoting a component of $f^{-1}(U)$, $\hat{\varphi}^{-1}(\hat{y})=\tilde{U}\cap f^{-1}\varphi^{-1}(y)$ is a component ([5], p. 41, L. 2) of $f^{-1}\varphi^{-1}(y)$, hence connected. In addition, for any $\hat{G}$ open in $\hat{Y}$, $\hat{\varphi}^{-1}(\hat{G})$ is a union of connected open sets $\tilde{U}\cap f^{-1}\varphi^{-1}(V)$, with $\varphi^{-1}(V)\subset U$ and $\tilde{U}$ a component of $f^{-1}(U)$, and the connectedness of $\hat{\varphi}^{-1}(\hat{G})$ is an easy consequence of that of $\hat{G}$.

6.4. Let $(\tilde{X},f)$ be a covering space of the connected, locally connected *paracompact* [7]) space X. Let $\{W\}$ be an open covering of X, whose members are evenly covered by $(\tilde{X},f)$. $\{W\}$ has a *barycentric open refinement* [8]) $\{N\}$. Let $\{U\}$ denote the family of all the components of the sets N. Since X is locally connected, $\{U\}$ is an open covering of X, and from $U_1,U_2\in\{U\}$ with $U_1\cap U_2\neq 0$ it follows that $U_1\cup U_2$ is connected and contained in some W, hence ([5], p. 41, L. 2) is evenly covered by $(\tilde{X},f)$.

Our theorems on ε-mappings, now follow easily from (6.3):

6.5. **Theorem.** *If for each open covering ε, the connected, locally connected, paracompact space X has a simply connected Hausdorff ε-image (depending on ε), then X is itself simply connected.*

Proof. Let $(\tilde{X},f)$ be a covering space of X, and $\varepsilon=\{U\}$ be an open covering such that $U_1\cup U_2$ is evenly covered by $(\tilde{X},f)$, if $U_1\cap U_2\neq 0$ (6.4). If Y is a simply connected ε-image of X, by (6.3) Y possesses a covering space $(\hat{Y},h)$ with the same number of leaves as $(\tilde{X},f)$. Since Y is simply connected, this number equals unity, hence f is univalent and X simply connected.

6.6. **Theorem.** *Quasi-homeomorphic connected, locally connected, paracompact spaces X and Y are both simply connected or not.*

6.7. **Theorem.** *Let X be a paracompact, connected, locally connected space, admitting a simply connected covering space $(\tilde{X},f)$. There exists an open covering ε of X, depending solely on X, such that for each Hausdorff ε-image Y of X, possessing a simply connected covering space, $\pi_1(Y)$ contains a retract which is isomorphic to $\pi_1(X)$.*

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[7]) A Hausdorff space is paracompact if each open covering has a neighbourhood-finite ([9], p. 13) open refinement ([6]). Paracompact spaces are normal. Each compact or metric (even non-separable) space is paracompact ([11]).

[8]) That is, for each $x\in X$, the union of all the N containing x, is included in a W. Each open covering of a paracompact space has a barycentric refinement ([11]).

Proof. Let $\varepsilon=\{U\}$ be an open covering of X, such that $U_1\cup U_2$ is evenly covered by $(\tilde{X},f)$, for any two $U_1, U_2\in\varepsilon$ with $U_1\cap U_2\neq 0$, (6.4). Since the simply connected covering space, if it exists, is unique, up to an isomorphism ([5], p. 51, P. 2), ε depends solely upon X, i. e. $\varepsilon=\varepsilon(X)$. Now let $Y=\varphi(X)$ be an ε-image of X, admitting a simply connected covering space $(\tilde{Y},g)$. By (6.3. ii), Y admits a covering space $(\hat{Y},h)$, whose group $\mathcal{G}$ of automorphisms is the isomorphic image of $\pi_1(X)$, under an onto-isomorphism $\Phi:\pi_1(X)\to\mathcal{G}$, paired to φ. Since $(\tilde{X},f)$ is regular ([5], p. 52, P. 3), so is $(\hat{Y},h)$, according to (6.3. iii), and our statement is now an immediate consequence of (4.2).

6.8. **Theorem.** *Let X and Y be quasi-homeomorphic connected, locally connected paracompact spaces, both admitting simply connected covering spaces. Then each of the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$, contains a retract isomorphic to the other.*

I have been unable to decide whether or not the fundamental groups in question are isomorphic. Purely algebraic methods give partial results:

6.9. **Corollary.** *If one of the groups in question is a Q-group* ([2], p. 267), *in particular if it is finite, or free and finitely generated, the Poincaré groups $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.*

On the other hand, by use of topological methods, the problem has been answered in the affirmative for orientable manifolds ([7], p. 95).

Finally, a mere restatement of (6.5) is the

6.10. **Corollary.** *Let X be a compact, connected, locally connected subspace of a metric space M. If for each positive number ε, X can be ε-displaced (ε-verschoben,* ([1], p. 110)) *in M, onto a simply connected subspace $Y_\varepsilon\subset M$, then X is itself simply-connected.*

7. Monotone transformations are now subjected to investigation, owing to their property of inducing, as proved below, a homomorphism of the fundamental group of their range onto that of the image space.

Combining the results of this section with those of the preceding one, yields a class of transformations inducing isomorphisms of fundamental groups.

7.1. **Lemma.** *Let $(\tilde{X},f)$, $(\tilde{Y},g)$ be covering spaces of the connected, locally connected spaces X and $Y=\varphi(X)$. Let $\varphi:X\to Y$ be quasi-monotone (6.2) and $\tilde{\varphi}:\tilde{X}\to\tilde{Y}$ continuous, into, satisfying $g\tilde{\varphi}=\varphi f$ on the whole of $\tilde{X}$.*

Then $\tilde{\varphi}$ is onto. In addition, for each open $V\subset Y$, evenly covered by $(\tilde{Y},g)$, and any component $\tilde{V}$ of $g^{-1}(V)$, there exists a component $\tilde{U}$ of $f^{-1}\varphi^{-1}(V)$ with $\tilde{\varphi}(\tilde{U})=\tilde{V}$.

Proof. Let: $\tilde{y}\in\tilde{Y}$ be adherent to $\tilde{\varphi}(\tilde{X})$, $y=g(\tilde{y})$, $V\ni y$ open in Y, evenly covered by $(\tilde{Y},g)$; finally let $\tilde{V}$ be the component of $g^{-1}(V)$ containing $\tilde{y}$. Since $\tilde{Y}$ is locally connected, $\tilde{V}$ is a neighborhood of $\tilde{y}$, and as such there exists $\tilde{z}\in\tilde{V}\cap\tilde{\varphi}(\tilde{X})$, i. e. $\tilde{z}=\tilde{\varphi}(\tilde{x})$, $\tilde{x}\in\tilde{X}$. It follows that:

$$\varphi f(\tilde{x})=g\tilde{\varphi}(\tilde{x})=g(\tilde{z})\in g(\tilde{V})\subset V$$

hence, with $x=f(\tilde{x})$, $x\in\varphi^{-1}(V)$. Let U denote the component of $\varphi^{-1}(V)$ containing x and $\tilde{U}$ the component of $f^{-1}(U)$ containing $\tilde{x}$. Since U and V are connected and open, hence locally connected, from [5] (p. 42, L. 5) it follows that $f(\tilde{U})=U$ and $g(\tilde{V})=V$. In addition, the quasi-monotony of φ implies $\varphi(U)=\tilde{V}$. Thus:

$$g\tilde{\varphi}(\tilde{U})=\varphi f(\tilde{U})=\varphi(U)=V.$$

Since $\tilde{\varphi}(\tilde{U})$ is connected, $\tilde{V}$ is a component of $g^{-1}(V)$, $\tilde{z}=\tilde{\varphi}(\tilde{x})\in\tilde{\varphi}(\tilde{U})$ and $\tilde{z}\in\tilde{V}$, if follows that $\tilde{\varphi}(\tilde{U})\subset V$. Finally, since g is univalent on $\tilde{V}$, $\tilde{\varphi}(\tilde{U})=\tilde{V}$ and $\tilde{y}\in\tilde{V}\subset\tilde{\varphi}(\tilde{X})$.

The set $\tilde{\varphi}(\tilde{X})$ is thus open and closed in $\tilde{Y}$, whose connectedness implies $\tilde{\varphi}(\tilde{X})=\tilde{Y}$. Since now any $\tilde{y}\in\tilde{Y}$ is adherent to $\tilde{\varphi}(\tilde{X})$ and $\tilde{U}$ is easily seen to be a component of $f^{-1}\varphi^{-1}(V)$, the assertion is completely proved.

An easy by-product is the

7.2. **Corollary.** *Let X and $Y=\varphi(X)$ be compact, connected, locally connected Hausdorff spaces, admitting simply connected covering spaces $(\tilde{X},f)$ and $(\tilde{Y},g)$. If φ is quasi-monotone, in particular if it is open* ([14], p. 152), *and $\pi_1(X)$ is finite, so is $\pi_1(Y)$.*

Proof. Since $\pi_1(X)$ is finite and X compact, so is $\tilde{X}$. There exists (4.1. i) a map $\tilde{\varphi}:\tilde{X}\to\tilde{Y}$, with $\varphi f=g\tilde{\varphi}$ on $\tilde{X}$. Lemma (7.1) implies $\tilde{\varphi}(\tilde{X})=\tilde{Y}$, hence $\tilde{Y}$ is compact and $\pi_1(Y)$ finite.

7.3. **Theorem.** *Let $Y=\varphi(X)$, where X is simply connected, Y Hausdorff, connected, locally connected and φ monotone. Then Y is itself simply connected.*

Proof. Let $(\tilde{Y},g)$ be a covering space of Y. According to [5] (p. 50, P. 1) there is a continuous, into, $\tilde{\varphi}:X\to\tilde{Y}$ with $g\tilde{\varphi}=\varphi$

on X. Since φ is also quasi-monotone, (7.1) implies $\tilde{\varphi}(X)=\tilde{Y}$, hence $g^{-1}(y)=\tilde{\varphi}\varphi^{-1}(y)$ for any $y\in Y$. Since φ is monotone and $\tilde{\varphi}$ continuous, $g^{-1}(y)$ is connected. On the other hand, as g is the covering map, $g^{-1}(y)$ is discrete. These two properties imply that $g^{-1}(y)$ is a single point, hence g is univalent and (7.3) proved.

7.4. ***Theorem.*** *Let $Y=\varphi(X)$, where X and Y are both connected, locally connected, Hausdorff spaces, admitting simply connected covering spaces $(\tilde{X},f)$ and $(\tilde{Y},g)$. If φ is monotone, $\pi_1(Y)$ is the homomorphic image of $\pi_1(X)$, under an onto-homomorphism paired to φ.*

Proof. According to (4.1.i) there is a map $\tilde{\varphi}:\tilde{X}\to\tilde{Y}$ with $g\tilde{\varphi}=\varphi f$ on $\tilde{X}$, and, by (4.1. ii), the relation $\eta\tilde{\varphi}=\tilde{\varphi}\xi$, where $\xi\in\pi_1(X)$, $\eta\in\pi_1(Y)$, defines an into-homomorphism $\Phi:\pi_1(X)\to\pi_1(Y)$ paired to φ. Choose $\eta\in\pi_1(Y)$, $\tilde{x}_1\in\tilde{X}$; let $\tilde{y}_1=\tilde{\varphi}(\tilde{x}_1)$, $\tilde{y}_2=\eta(\tilde{y}_1)$, hence $g(\tilde{y}_1)=g(\tilde{y}_2)=y$, and put $x=f(\tilde{x}_1)$. Now let $V\ni y$ be open, evenly covered by $(\tilde{Y},g)$, and let $\tilde{V}$ denote the component of $g^{-1}(V)$ containing $\tilde{y}_2$. By (7.1), since φ is also quasi-monotone, there is a component $\tilde{U}$ of $f^{-1}\varphi^{-1}(V)$ with $\tilde{\varphi}(\tilde{U})=\tilde{V}$, and since $\varphi^{-1}(V)$ is connected and open, hence locally connected, $f(\tilde{U})=\varphi^{-1}(V)$. It follows that

$$\varphi(x)=\varphi f(\tilde{x}_1)=g\tilde{\varphi}(\tilde{x}_1)=g(\tilde{y}_1)=y\in V$$

hence $x\in\varphi^{-1}(V)$. There is a $\tilde{x}_2\in\tilde{U}$ with $f(\tilde{x}_2)=x$ and so:

$$g\tilde{\varphi}(\tilde{x}_2)=\varphi f(\tilde{x}_2)=\varphi(x)=y\in V \quad \text{with} \quad \tilde{\varphi}(\tilde{x}_2)\in\tilde{V}.$$

Since g is univalent on $\tilde{V}$, $\tilde{\varphi}(\tilde{x}_2)\in\tilde{V}\cap g^{-1}(y)=\tilde{y}_2$. From $f(\tilde{x}_1)=f(\tilde{x}_2)$ follows the existence of a $\xi\in\pi_1(X)$ with $\xi(\tilde{x}_1)=\tilde{x}_2$. If now $\eta'=\Phi(\xi)$, i. e. $\eta'\tilde{\varphi}=\tilde{\varphi}\xi$, then:

$$\eta'(\tilde{y}_1)=\eta'\tilde{\varphi}(\tilde{x}_1)=\tilde{\varphi}\xi(\tilde{x}_1)=\tilde{\varphi}(\tilde{x}_2)=\tilde{y}_2,$$

hence $\eta=\eta'$ is the image of a $\xi\in\pi_1(X)$ under Φ.

Lemma (6.3), together with (7.3), now yields the following isomorphism condition:

7.5. ***Theorem.*** *Let $Y=\varphi(X)$, where X and Y are connected, locally connected Hausdorff spaces. Assume X to admit a simply connected covering space $(\tilde{X},f)$ and let $\varphi:X\to Y$ be monotone.*

The following two assertions are then equivalent:

a) *Each point $y\in Y$ has a neighbourhood $V\subset Y$ with $\varphi^{-1}(V)$ evenly covered by $(\tilde{X},f)$;*

b) *The space Y possesses a simply connected covering space $(\tilde{Y},g)$, and each into-homomorphism $\Phi:\pi_1(X)\to\pi_1(Y)$, paired to φ, is an onto-isomorphism.*

Proof. a) implies b).

We may as well assume the collection of all the sets V to be a base $\{V\}$ for Y, and let $\{U\}=\{\varphi^{-1}(V)\}$. By (6.3. ii), Y then admits a covering space $(\hat{Y},h)$, regular by (6.3. iii), whose group $\mathcal{G}$ of automorphisms is the isomorphic image, paired to φ, of $\pi_1(X)$. In addition, by (6.3. iv), there is a $\hat{\varphi}:\tilde{X}\to\hat{Y}$, paired to φ, which is onto and monotone. From (7.3), it follows now that $\hat{Y}=\hat{\varphi}(\tilde{X})$ is simply connected, and with $\tilde{Y}=\hat{Y}$, $g=h$, $\pi_1(Y)=\mathcal{G}$, b) is proved.

b) implies a).

There is, (4.1. i), a $\tilde{\varphi}:\tilde{X}\to\tilde{Y}$, paired to φ, and by (7.1), since φ is also quasi-monotone, $\tilde{\varphi}(\tilde{X})=\tilde{Y}$. We shall prove that, for each open $V\subset Y$, evenly covered by $(\tilde{Y},g)$, f is univalent on any component $\tilde{U}$ of $f^{-1}\varphi^{-1}(V)$. Since $\varphi^{-1}(V)$ is connected and open, hence locally connected, by [5] (p. 42, L. 5), $f(\tilde{U})=\varphi^{-1}(V)$ and so

$$g\tilde{\varphi}(\tilde{U})=\varphi f(\tilde{U})=\varphi\varphi^{-1}(V)=V.$$

$\tilde{\varphi}(\tilde{U})$ is connected, and as such contained in a component $\tilde{V}$ of $g^{-1}(V)$. Now let $\tilde{x}_i\in\tilde{U}$, with $f(\tilde{x}_i)=x\in\varphi^{-1}(V)$, $i=1,2$. There exists in consequence, a $\xi\in\pi_1(X)$ with $\xi(\tilde{x}_1)=\tilde{x}_2$. Let $\tilde{y}_i=\tilde{\varphi}(\tilde{x}_i)$, hence:

$$\tilde{y}_i\in\tilde{\varphi}(\tilde{U})\subset\tilde{V} \quad \text{and} \quad g(\tilde{y}_i)=g\tilde{\varphi}(\tilde{x}_i)=\varphi f(\tilde{x}_i)=\varphi(x)\in V, \quad i=1,2.$$

Since V is evenly covered by $(\tilde{Y},g)$, g is univalent on $\tilde{V}$, and from $\tilde{y}_1,\tilde{y}_2\in\tilde{V}$ with $g(\tilde{y}_1)=g(\tilde{y}_2)$ follows $\tilde{y}_1=\tilde{y}_2$.

If now $\eta=\Phi(\xi)$, i. e. $\eta\tilde{\varphi}=\tilde{\varphi}\xi$, it follows that:

$$\eta(\tilde{y}_1)=\eta\tilde{\varphi}(\tilde{x}_1)=\tilde{\varphi}\xi(\tilde{x}_1)=\tilde{\varphi}(\tilde{x}_2)=\tilde{y}_2=\tilde{y}_1$$

and η is the neutral element of $\pi_1(Y)$. Since Φ is assumed an isomorphism, it follows that ξ is the neutral element of $\pi_1(X)$, hence $\tilde{x}_2=\xi(\tilde{x}_1)=\tilde{x}_1$ and the univalence of f on $\tilde{U}$ is proved.

In case X is compact, we may assume, instead of a) in (7.5), each $\varphi^{-1}(y)$, $y\in Y$, to be locally connected and evenly covered by $(\tilde{X},f)$. More precisely:

7.6. ***Theorem.*** *Let $Y=\varphi(X)$, where X and Y are compact, connected, locally connected Hausdorff spaces. Assume X to admit a simply connected covering space $(\tilde{X},f)$ and let $\varphi:X\to Y$ be monotone.*

If for each point $y\in Y$, $\varphi^{-1}(y)$ is locally connected and evenly covered by $(\tilde{X},f)$, hence in particular, if it is simply connected, then Y admits a simply connected covering space and $\pi_1(Y)$ is the isomorphic image of $\pi_1(X)$.

Proof. On account of (7.5) it suffices to show that each $y \in Y$ has a neighborhood V, with $\varphi^{-1}(V)$ evenly covered by $(\tilde{X},f)$. This will now be accomplished, chiefly with the aid of (5.3).

Since $\varphi^{-1}(y)$ is connected, locally connected, closed in X, hence compact, and evenly covered by $(\tilde{X},f)$, there exists, by (5.3), an open, connected set $G\supset\varphi^{-1}(y)$, evenly covered by $(\tilde{X},f)$ too. Let W denote the union of all the $\varphi^{-1}(z)$ contained in G. From $\varphi^{-1}(y)\subset G$ it follows that $W\neq 0$, and since φ is a closed, continuous transformation, W is easily seen to be open in X. Now let U denote the component of W containing the connected set $\varphi^{-1}(y)$. Since X is locally connected and W open, U is open in X. As it is connected, contained in W, hence in G, U is evenly covered by $(\tilde{X},f)$ ([5], p. 41, L. 2). Let $V=\varphi(U)$ and suppose $x\in\varphi^{-1}(V)$. It follows that $\varphi(x)=\varphi(s)$ for some $s\in U\subset W$, hence $s\in\varphi^{-1}(z)$ for some $\varphi^{-1}(z)\subset G$, and so $x\in\varphi^{-1}(z)$. Since $\varphi^{-1}(z)$ is connected, contained in G, hence in W, and since $\varphi^{-1}(z)$ meets the component U of W, it follows that $x\in\varphi^{-1}(z)\subset U$, i. e. $\varphi^{-1}(V)=U$. Thus $y=\varphi\varphi^{-1}(y)\in\varphi(U)=V$ with $U=\varphi^{-1}(V)$ evenly covered by $(\tilde{X},f)$. In addition, as φ is closed and continuous, since $\varphi^{-1}(V)$ is open in X, so is V in Y.

Finally:

7.7. ***Theorem.*** *Let $Y=\varphi(X)$, where φ is continuous and X, Y are both connected, locally connected, compact Hausdorff spaces. If Y and $\varphi^{-1}(y)$, for each $y\in Y$, are simply connected, then X itself is simply connected.*

Proof. Let $(\tilde{X},f)$ be a covering space of X. Each $\varphi^{-1}(y)$ is evenly covered by $(\tilde{X},f)$, and since it is closed in X, hence compact, it possesses on account of (5.3) a neighboorhod G, evenly covered by $(\tilde{X},f)$. Let W be the union of all the $\varphi^{-1}(z)$ contained in G, and let U denote the component of W that contains the connected set $\varphi^{-1}(y)$. As before one realizes that U is evenly covered by $(\tilde{X},f)$, $U=\varphi^{-1}\varphi(U)$ and $V=\varphi(U)$ is open in Y. By (6.3. b, i) Y now possesses a covering space $(\hat{Y},h)$ with the same number of leaves as $(\tilde{X},f)$. Since Y is simply connected, this numbers equals unity, and so X is itself simply connected.

8. The object of our investigation now is to find the fundamental group of the intersection of a family of sets. The results of this section depend very much upon those of § 5.

Let A be a compact, connected, locally connected subspace of the normal, locally connected Hausdorff space E. Then (5.2), together with (4.3), yield at once the

8.1. ***Proposition.*** *If A is not simply connected, there exists a set $U\supset A$, open in E and connected, such that any connected, locally connected subspace C of E, with $A\subset C\subset U$, is not simply connected.*

An immediate consequence of (8.1) is the

8.2. ***Theorem.*** *Let $\{F_\lambda\}$ be a family of non-empty, simply connected, closed subsets of the locally connected, compact Hausdorff space E. Suppose each finite intersection of sets F_λ to contain a member of the family. Suppose in addition the intersection F of all the F_λ be locally connected. Then F is simply connected.*

With respect to the fundamental group, we prove first the

8.3. ***Theorem.*** *Let A and E be as in* (8.1). *Suppose in addition that A possesses a simply connected covering space $(\tilde{A},f)$. There exists then a set $G\supset A$, open in E and connected, such that $\pi_1(C)$ contains a retract isomorphic to $\pi_1(A)$, for any connected, locally connected subspace C of E, satisfying $A\subset C\subset G$ and possessing a simply connected covering space.*

Proof. By (5.2), there exists a set $U\supset A$, open in E and connected, possessing a covering space $(\tilde{U},g)$, such that $g^{-1}(A)$ is connected and the two covering spaces $(\tilde{A},f)$ and $(g^{-1}(A),g)$ of the space A are isomorphic.

Notice first that $g^{-1}(C)$ is connected and contains $g^{-1}(A)$, for any connected, locally connected subspace C of E, with $A\subset C\subset U$.

Since $(g^{-1}(A),g)$ is a simply connected, hence regular, covering space of A, by (5.1), there is a set G, open in E and connected, satisfying $A\subset G\subset U$, hence with connected $g^{-1}(G)$, and such that its covering space $(g^{-1}(G),g)$ is still regular.

Now let C be any connected, locally connected subspace of E, with $A\subset C\subset G\subset U$, hence with connected $g^{-1}(C)$. Since $(g^{-1}(G),g)$ is regular, so is by (4.3) the covering space $(g^{-1}(C),g)$ of C.

Since $g^{-1}(A)$ is connected, simply connected, and contained in $g^{-1}(C)$, the group of automorphisms of the regular covering space $(g^{-1}(C),g)$ of C, is by (4.3) the isomorphic image, paired to the inclusion map $A\to A\subset C$, of the group $\pi_1(A)$. Under the assumption that C possesses a simply connected covering space, (8.3) is now an immediate consequence of (4.2).

For our final result, we shall be dealing with inverse systems of groups. They are taken with the discrete topology, and are *not* supposed commutative. The terminology used is the one in [9] (p. 54, 55), and the few propositions thereof, used in the sequel, hold also for the noncommutative case.

We need first the following algebraic

8.4. ***Proposition.*** *Assume:*

(i) $\{\mathcal{H}_\lambda; \Omega^\lambda_\mu\}$ *is an inverse system of discrete, not necessarily commutative groups;*

(ii) *For each* λ, *there is a retraction* $P_\lambda: \mathcal{H}_\lambda \to \mathcal{R}_\lambda \subset \mathcal{H}_\lambda$;

(iii) *For each pair* $\lambda \succ \mu$, *the contraction* $\Omega^\lambda_\mu: \mathcal{R}_\lambda \to \mathcal{H}_\mu$ *is an isomorphism of* $\mathcal{R}_\lambda$ *onto* $\mathcal{R}_\mu \subset \mathcal{H}_\mu$;

(iv) *For each pair* $\lambda \succ \mu$, $P_\mu \Omega^\lambda_\mu = \Omega^\lambda_\mu P_\lambda$ *holds on* $\mathcal{H}_\lambda$.

Then:

$\{\mathcal{R}_\lambda; \Omega^\lambda_\mu\}$, *with the* Ω^λ_μ *contracted to* $\mathcal{R}_\lambda$, *is an inverse system of groups. Its limit* $\mathcal{R}$ *is a retract of the limit* $\mathcal{H}$ *of* $\{\mathcal{H}_\lambda; \Omega^\lambda_\mu\}$. *For each* λ, $\mathcal{R}_\lambda$ *is isomorphic to* $\mathcal{R}$, *in the algebraic sense.*

Proof. Since $\Omega^\lambda_\mu(\mathcal{R}_\lambda) \subset \mathcal{R}_\mu$ and the contractions obviously satisfy $\Omega^\mu_\nu \Omega^\lambda_\mu = \Omega^\lambda_\nu$ for $\lambda \succ \mu \succ \nu$, $\{\mathcal{R}_\lambda; \Omega^\lambda_\mu\}$ is an inverse system of groups. For $h = \{h_\lambda\} \in \mathcal{H}$, let $P(h) = \{P_\lambda(h_\lambda)\}$. If $\lambda \succ \mu$, $P_\mu(h_\mu) = P_\mu \Omega^\lambda_\mu(h_\lambda) = \Omega^\lambda_\mu P_\lambda(h_\lambda)$ so that $P(\mathcal{H}) \subset \mathcal{R}$ and clearly P is a homomorphism. For $r = \{r_\lambda\} \in \mathcal{R}$, $P(r) = \{P_\lambda(r_\lambda)\} = \{r_\lambda\} = r$ which proves the retraction. That $\mathcal{R}_\lambda$ and $\mathcal{R}$ are isomorphic follows from (iii) and [9] (p. 55).

8.5. ***Theorem.*** *Let* $\{F_\lambda\}$ *be a family of closed, connected, locally connected non-empty subsets of the locally connected, compact Hausdorff space* E. *Suppose each finite intersection of sets* F_λ *to contain a member of the family, and partially order the aggregate* $\{\lambda\}$ *of subscripts by:* $\lambda \succ \mu$ *whenever* $F_\lambda \subset F_\mu$. *Suppose in addition all the* F_λ *as well as* $F = \cap F_\lambda$ *to posses simply connected covering spaces. There exists then a family of intohomomorphisms* $\Omega^\lambda_\mu: \pi_1(F_\lambda) \to \pi_1(F_\mu)$, *for* $\lambda \succ \mu$, *paired to the corresponding inclusion maps, such that:*

$\{\pi_1(F_\lambda); \Omega^\lambda_\mu\}$ *is an inverse system of groups, whose limit contains a retract isomorphic to* $\pi_1(F)$.

Proof. Notice first that the partial ordering imposed on $\{\lambda\}$ turns it into a directed set ([9], p. 4).

Now let $(\widetilde{F}, f)$ and $(\widetilde{F}_\lambda, f_\lambda)$ be simply connected covering spaces of F and each F_λ. Since $(\widetilde{F}, f)$ is regular, by successive application of (5.2) and (5.1), as in the proof of (8.3), one can find a set G, open in E and connected, possessing a regular covering space $(\hat{G}, g)$, such that $g^{-1}(F)$ is connected and the two covering spaces $(\widetilde{F}, f)$ and $(g^{-1}(F), g)$ of F are isomorphic. Let $\hat{F} = g^{-1}(F)$. There is an onto-homeomorphism

$$\psi: \widetilde{F} \to \hat{F} \quad \textit{with} \quad g\psi = f \quad \textit{on} \quad \widetilde{F}.$$

Denote by $\mathcal{G}$ the group of automorphisms of $(\hat{F}, g)$; its isomorphism to $\pi_1(F)$ will be taken into account later.

Replacing, if necessary, $\{F_\lambda\}$ by a cofinal subfamily, since E is compact, G open and each F_λ closed, we may admit each $F_\lambda \subset G$. Let $\hat{F}_\lambda = g^{-1}(F_\lambda) \subset \hat{G}$ and let g_λ be the contraction to $\hat{F}_\lambda$ of $g: \hat{G} \to G$. Each $\hat{F}_\lambda$ is connected, and by (4.3), $(\hat{F}_\lambda, g_\lambda)$ is a regular covering space of F_λ, with $\mathcal{G}_\lambda$ as its group of automorphisms. By [5] (p. 50, P. 1), there exists for each λ, a map

$$\psi_\lambda: \widetilde{F}_\lambda \to \hat{F}_\lambda \quad \textit{with} \quad g_\lambda \psi_\lambda = f_\lambda \quad \textit{on} \quad \widetilde{F}_\lambda,$$

and as can be seen, ψ_λ is onto and $(\widetilde{F}_\lambda, \psi_\lambda)$ is a covering space of $\hat{F}_\lambda$. For any λ and $\mu \prec \lambda$, let

$$\omega_\lambda: F \to F_\lambda, \quad \omega^\lambda_\mu: F_\lambda \to F_\mu, \quad \theta_\lambda: \hat{F} \to \hat{F}_\lambda, \quad \theta^\lambda_\mu: \hat{F}_\lambda \to \hat{F}_\mu,$$

be inclusion maps. It follows that for any λ and $\mu \prec \lambda$:

$$g_\lambda \theta_\lambda = \omega_\lambda g \quad \textit{on} \quad \hat{F}, \quad g_\mu \theta^\lambda_\mu = \omega^\lambda_\mu g_\lambda \quad \textit{on} \quad \hat{F}_\lambda.$$

Now choose $\hat{a} \in \hat{F}$, $\widetilde{a} \in \psi^{-1}(\hat{a}) \subset \widetilde{F}$, and for any λ, let $\hat{a}_\lambda = \theta_\lambda(\hat{a})$ and take a $\widetilde{a}_\lambda \in \psi_\lambda^{-1}(\hat{a}_\lambda) \subset \widetilde{F}_\lambda$. Since $(\widetilde{F}_\lambda, \psi_\lambda)$ is a covering space of $\hat{F}_\lambda$, while $\widetilde{F}$ and $\widetilde{F}_\lambda$ are simply connected, by (4.1. i) there exist, for any λ and $\mu \prec \lambda$, into, continuous maps:

$$\widetilde{\omega}_\lambda: \widetilde{F} \to \widetilde{F}_\lambda \quad \textit{with} \quad \widetilde{\omega}_\lambda(\widetilde{a}) = \widetilde{a}_\lambda \quad \textit{and} \quad \psi_\lambda \widetilde{\omega}_\lambda = \theta_\lambda \psi \quad \textit{on} \quad \widetilde{F};$$
$$\widetilde{\omega}^\lambda_\mu: \widetilde{F}_\lambda \to \widetilde{F}_\mu \quad \textit{with} \quad \widetilde{\omega}^\lambda_\mu(\widetilde{a}_\lambda) = \widetilde{a}_\mu \quad \textit{and} \quad \psi_\mu \widetilde{\omega}^\lambda_\mu = \theta^\lambda_\mu \psi_\lambda \quad \textit{on} \quad \widetilde{F}_\lambda.$$

Next, on account of (4.1), for any λ and $\mu \prec \lambda$, we have the into-homomorphisms

$\Psi: \pi_1(F) \to \mathcal{G}$	*defined by*	$\gamma\psi = \psi\xi$, $\xi \in \pi_1(F)$, $\gamma \in \mathcal{G}$
$\Psi_\lambda: \pi_1(F_\lambda) \to \mathcal{G}_\lambda$	,, ,,	$\gamma_\lambda \psi_\lambda = \psi_\lambda \xi_\lambda$, $\xi_\lambda \in \pi_1(F_\lambda)$, $\gamma_\lambda \in \mathcal{G}_\lambda$
$\Gamma_\lambda: \mathcal{G} \to \mathcal{G}_\lambda$	,, ,,	$\gamma_\lambda \theta_\lambda = \theta_\lambda \gamma$, $\gamma \in \mathcal{G}$, $\gamma_\lambda \in \mathcal{G}_\lambda$
$\Gamma^\lambda_\mu: \mathcal{G}_\lambda \to \mathcal{G}_\mu$	,, ,,	$\gamma_\mu \theta^\lambda_\mu = \theta^\lambda_\mu \gamma_\lambda$, $\gamma_\lambda \in \mathcal{G}_\lambda$, $\gamma_\mu \in \mathcal{G}_\mu$
$\Omega_\lambda: \pi_1(F) \to \pi_1(F_\lambda)$	,, ,,	$\xi_\lambda \widetilde{\omega}_\lambda = \widetilde{\omega}_\lambda \xi$, $\xi \in \pi_1(F)$, $\xi_\lambda \in \pi_1(F_\lambda)$
$\Omega^\lambda_\mu: \pi_1(F_\lambda) \to \pi_1(F_\mu)$	,, ,,	$\xi_\mu \widetilde{\omega}^\lambda_\mu = \widetilde{\omega}^\lambda_\mu \xi_\lambda$, $\xi_\lambda \in \pi_1(F_\lambda)$, $\xi_\mu \in \pi_1(F_\mu)$.

Since ψ produces an isomorphism of covering spaces:

Ψ *is an isomorphism onto.*

Taking advantage of the fact that $(\widetilde{F}_\lambda, \psi_\lambda)$ is a simply connected covering space of $\hat{F}_\lambda$, it is readily seen that:

Ψ_λ *is an homomorphism onto.*

Since $\hat{F}=g^{-1}(F)=g_\lambda^{-1}(F)$ and $\hat{F}_\lambda=g_\lambda^{-1}(F_\lambda)=g_\mu^{-1}(F_\lambda)$ are connected subsets, respectively, of $\hat{F}_\lambda$ and $\hat{F}_\mu$, (4.3) implies that:

Γ_λ *and* Γ_μ^λ *are isomorphisms onto.*

A mere calculation yields, for $\lambda \succ \mu \succ \nu$:

$$f_\mu \widetilde{\omega}_\mu^\lambda = \omega_\mu^\lambda f_\lambda, \quad \widetilde{\omega}_\nu^\mu \widetilde{\omega}_\mu^\lambda(\widetilde{a}_\lambda) = \widetilde{\omega}_\nu^\lambda(\widetilde{a}_\lambda), \quad \psi_\nu \widetilde{\omega}_\nu^\mu \widetilde{\omega}_\mu^\lambda = \psi_\nu \widetilde{\omega}_\nu^\lambda \quad \text{on} \quad \widetilde{F}_\lambda;$$

hence the $\widetilde{\omega}_\mu^\lambda$ are paired to the inclusion maps ω_μ^λ, and by [5] (p. 51, L. 1) it follows that $\widetilde{\omega}_\nu^\mu \widetilde{\omega}_\mu^\lambda = \widetilde{\omega}_\nu^\lambda$.

As a consequence:

The homomorphisms Ω_μ^λ *are paired to the inclusion maps* ω_μ^λ *and* $\Omega_\nu^\mu \Omega_\mu^\lambda = \Omega_\nu^\lambda$ *holds for any* $\lambda \succ \mu \succ \nu$.

Thus:

(i) $\{\pi_1(F_\lambda); \Omega_\mu^\lambda\}$ *is an inverse system of groups.*

As above, we realize the pairing of the $\widetilde{\omega}_\lambda$ to the inclusion maps ω_λ and $\widetilde{\omega}_\mu^\lambda \widetilde{\omega}_\lambda = \widetilde{\omega}_\mu$ for $\lambda \succ \mu$. As a consequence:

The homomorphisms Ω_λ *are paired to the* ω_λ *and* $\Omega_\mu^\lambda \Omega_\lambda = \Omega_\mu$ *for* $\lambda \succ \mu$.

Paraphrasing the reasoning in the proof of (4.2) we see that:

Ω_λ *is an isomorphism into,*

and with $\mathcal{R}_\lambda = \Omega_\lambda[\pi_1(F)] \subset \pi_1(F_\lambda)$:

(ii) $P_\lambda = \Omega_\lambda \Psi^{-1} \Gamma_\lambda^{-1} \Psi_\lambda$ *is a retraction of* $\pi_1(F_\lambda)$ *onto* $\mathcal{R}_\lambda$.

Since they are isomorphisms-into and the Ω_λ satisfy $\Omega_\mu^\lambda \Omega_\lambda = \Omega_\mu$, it follows that:

(iii) *For each pair* $\lambda \succ \mu$, *the contraction* $\Omega_\mu^\lambda: \mathcal{R}_\lambda \to \pi_1(F_\mu)$ *is an isomorphism of* $\mathcal{R}_\lambda$ *onto* $\mathcal{R}_\mu \subset \pi_1(F_\mu)$.

From $\psi_\mu \widetilde{\omega}_\mu^\lambda = \theta_\mu^\lambda \psi_\lambda$ and $\theta_\mu^\lambda \theta_\lambda = \theta_\mu$, it follows that:

$$\Psi_\mu \Omega_\mu^\lambda = \Gamma_\mu^\lambda \Psi_\lambda \quad \text{and} \quad \Gamma_\mu^\lambda \Gamma_\lambda = \Gamma_\mu, \quad \lambda \succ \mu.$$

Since Γ_λ, Γ_μ and Γ_μ^λ are isomorphisms onto, it follows that $\Gamma_\mu^\lambda = \Gamma_\mu \Gamma_\lambda^{-1}$, and, again using $\Omega_\mu = \Omega_\mu^\lambda \Omega_\lambda$, we finally obtain:

(iv) $P_\mu \Omega_\mu^\lambda = \Omega_\mu \Psi^{-1} \Gamma_\mu^{-1} \Psi_\mu \Omega_\mu^\lambda = \Omega_\mu \Psi^{-1} \Gamma_\mu^{-1} \Gamma_\mu^\lambda \Psi_\lambda =$
$= \Omega_\mu \Psi^{-1} \Gamma_\mu^{-1} \Gamma_\mu \Gamma_\lambda^{-1} \Psi_\lambda = \Omega_\mu \Psi^{-1} \Gamma_\lambda^{-1} \Psi_\lambda = \Omega_\mu^\lambda \Omega_\lambda \Psi^{-1} \Gamma_\lambda^{-1} \Psi_\lambda = \Omega_\mu^\lambda P_\lambda.$

Theorem (8.5) is now a consequence of (8.4).

9. Consider now Borsuk's well-known acyclic continuum, as described in [3]. This is an arcwise and locally arcwise connected, compact subset of Euclidian 3-space, which is not pathwise simply connected [3]. It may be viewed as an intersection of a decreasing sequence of 3-cells in the 3-space [3], and is in addition, quasi-homeomorphic to such a cell [4].

This startling example makes it clear that the theorems in sections 6 and 8 of this paper, do not hold true for pathwise simple connectedness, nor for the fundamental group defined in terms of paths. By these very theorems, Borsuk's acyclic continuum is simply connected, in terms of covering spaces, and thus the two definitions are not equivalent, in the absence of further local assumptions.

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