Consequently, by CA 11.4,

4.15 [19]). *If* $\operatorname{Dim} E \geqslant k$, *then* $\operatorname{Dim}([E], \boldsymbol{A}/\boldsymbol{A}^k) = \operatorname{Dim} E - k$.

Theorems 4.15, 2.10, and CA 11.4 imply

4.16. *If* $N_0, \dots, N_n$ *is a normal decomposition of* $\boldsymbol{A}$ *(where* $n = \dim \boldsymbol{A}$*), then* $[N_k], \dots, [N_n]$ *is a normal decomposition of* $\boldsymbol{A}/\boldsymbol{A}^k$.

The evaluation given in 4.12 and 4.12′ is exact. In fact,

4.17. *If the integers* l, l', L *satisfy the inequalities*

$$l \leqslant L \leqslant n = \dim \boldsymbol{A}, \quad L \geqslant k, \quad \max(0, l-k) \leqslant l' \leqslant \min(l, L-k)$$

(where $k \leqslant n$*), then there is an element* $E \in \boldsymbol{A}$ *such that*

$$\dim E\boldsymbol{A} = l, \quad \operatorname{Dim} E = L \quad \text{and} \quad \dim(E\boldsymbol{A}, \boldsymbol{A}^k) = \dim [E] \cdot \boldsymbol{A}/\boldsymbol{A}^k = l'.$$

We have $l' \geqslant 0$ and $0 \leqslant l - l' \leqslant k \leqslant L - l' \leqslant L \leqslant n$.

Let $N_0, \dots, N_n$ be a normal decomposition of $\boldsymbol{A}$, and let $E_1 = N_{L-l'} + \dots + N_L$. Consequently $[E_1] = [N_{L-l'}] + \dots + [N_L] \in \boldsymbol{A}/\boldsymbol{A}^k$. By 4.16 and 4.7 (ii), $\dim [E_1] \cdot \boldsymbol{A}/\boldsymbol{A}^k = L - (L - l') = l'$.

If $l = l'$, let $E_2 = 0$; if $l > l'$, let $E_2 = N_0 + \dots + N_{l-l'-1}$.

The element $E = E_1 + E_2$ is the required one. In fact, it follows from 4.7 (i) and (ii) that $\operatorname{Dim} E = L$ and $\dim E\boldsymbol{A} = l$. Since $[E] = [E_1]$, we have $\dim [E] \cdot \boldsymbol{A}/\boldsymbol{A}^k = \dim [E_1] \cdot \boldsymbol{A}/\boldsymbol{A}^k = l'$, q. e. d.

4.18. $(\boldsymbol{A}/\boldsymbol{A}^k)/(\boldsymbol{A}/\boldsymbol{A}^k)^l$ *is homeomorphic to* $\boldsymbol{A}/\boldsymbol{A}^{k+l}$.

By CA 9.7, the C-algebra $(\boldsymbol{A}/\boldsymbol{A}^k)/(\boldsymbol{A}/\boldsymbol{A}^k)^l$ is homeomorphic to $\boldsymbol{A}/\boldsymbol{I}$ where $\boldsymbol{I}$ is the σ-ideal of all $A \in \boldsymbol{A}$ such that $\operatorname{Dim}([A], \boldsymbol{A}/\boldsymbol{A}^k) < l$. By 4.15, $\boldsymbol{I} = \boldsymbol{A}^{k+l}$, q. e. d.

References.

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— and Wallman, H. [2] *Dimension Theory*, Princeton 1948.

Kuratowski, C. [3] *Topologie I* (first edition), Warszawa-Lwów 1933.

— [4] *Topologie I* (second edition), Warszawa-Wrocław 1948.

Sikorski, R. (CA) *Closure algebras*, Fund. Math. **36** (1949), pp. 165-206.

[19]) If $\operatorname{Dim} E < k$, then obviously $\operatorname{Dim}([E], \boldsymbol{A}/\boldsymbol{A}^k) = -1$.

Państwowy Instytut Matematyczny.

On Generalized Spheres.

By

Mieczysław Gindifer (Warszawa).

1. Let A_0 be a non-empty subset of a space [1]) A and r a positive number. By a *generalized sphere* with centre A_0 and radius r we understand the set

$$K_r(A_0, A) = \mathop{E}_{x \in A}[\varrho(x, A_0) \leqslant r]. \tag{1}$$

Frequently the topological structure of the generalized sphere is more simple than the topological structure of the set A_0.

For instance if A_0 is a compact subset of the Euclidean 1-dimensional space E_1, then every generalized sphere is a sum of a finite number of segments.

It follows by (1): If A is a convex space [2]) and $0 < r' < r$ then

$$K_r(A_0, A) = K_{r'}[K_{r-r'}(A_0, A), A]. \tag{2}$$

2. *Lemma.* *If A_0 is a compact subset of the Euclidean n-dimensional space E_n and r is a positive number, then for every $a_0 \in K_r(A_0, E_n)$ there exists a connected set N with diameter $\delta(N) \leqslant 8r$, constituting a neighbourhood of a_0 in $K_r(A_0, E_n)$.*

Proof. Let us put

$$M = \mathop{E}_{x}[x \in K_r(a, E_n), a \in A_0, \varrho(a, a_0) \leqslant r],$$

$$N = \mathop{E}_{x}[x \in K_r(a, E_n), a \in A_0, M \cdot K_r(a, E_n) \neq 0].$$

Evidently N is a connected subset of $K_r(A_0, E_n)$ and $\delta(N) \leqslant 8r$. It remains to be proved that N constitutes a neighbourhood of a_0 in $K_r(A_0, E_n)$.

[1]) By *space* we always understand here a metric space.

[2]) A is *convex* if for every two points $a, b \in A$ and every positive number $0 < \alpha < \varrho(a, b)$ there exists a point $x \in A$ such that $\alpha = \varrho(a, x) = \varrho(a, b) - \varrho(b, x)$.

Otherwise there would exist a sequence $\{a_\nu\} \epsilon A_0$ and a sequence $\{b_\nu\} \to a_0$ so that

(3) $$b_\nu \epsilon K_r(a_\nu, E_n) - N \quad \text{for every} \quad n = 1, 2, \ldots$$

Let $\{a_{\nu_k}\}$ be a subsequence of $\{a_\nu\}$ convergent to a point $\bar{a}_0 \epsilon A_0$. Then $a_0 \epsilon K_r(\bar{a}_0, E_n)$ and consequently $K_r(\bar{a}_0, E_n) \subset M$.

But for almost all indices k it is

$$K_r(a_{\nu_k}, E_n) \cdot K_r(\bar{a}_0, E_n) \neq 0$$

hence $K_r(a_{\nu_k}, E_n) \cdot M \neq 0$ and finally $K_r(a_{\nu_k}, E_n) \subset N$.

This contradicts 3.

3. *Theorem.* *If A is a compact subset of the Euclidean n-dimensional space E_n, then for every $r > 0$ the generalized sphere $K_r(A_0, E_n)$ is locally connected.*

Proof. It is to be shown that for every point $a_0 \epsilon K_r(A_0, E_n)$ and every $\varepsilon > 0$ there exists a connected neighbourhood N_ε of a_0 in $K_r(A_0, E_n)$ with diameter $\leqslant \varepsilon$. Let r' be a positive number, such that

$$r' < \operatorname{Min}(r, \tfrac{1}{8}\varepsilon).$$

Then by (2)

$$K_r(A_0, E_n) = K_{r'}[K_{r-r'}(A_0, E_n), E_n].$$

By the lemma of the section 2 there exists a connected neighbourhood N of a_0 in $K_r(A_0, E_n)$ with diameter $\delta(N) \leqslant 8r' \leqslant \varepsilon$.

4. A positive number r will be called a *singular radius* for the set $A_0 \subset E_n$ if there exists a point $p \epsilon K_r(A_0, E_n)$ such that $K_r(A_0, E_n)$ is not locally contractible in p [3]).

The purpose of this paper is to show that there exist in the Euclidean plane E_2 compact sets having continuum singular radii.

Let $\{i_\nu\}$, $\nu = 0, 1, 2, \ldots$, be a sequence such that for every ν it is $i_\nu = 0$ or $i_\nu = 1$. Consider all systems of the form

$$[\{i_\nu\}, l],$$

[3]) A is *locally contractible in the point* $a \epsilon A$ if for every neighbourhood N of a in A there exists a neighbourhood $N_0 \subset N$ of a in A and a continuous mapping $f(x,t)$ defined in the Cartesian product of N_0 and of the interval $0 \leqslant t \leqslant 1$, with values lying in N, such that $f(x,0) = x$ and $f(x,1) = a$ for every $x \epsilon N$.

where $l = 0$ if in the sequence $\{i_\nu\}$ the equality $i_\nu = 1$ holds for an infinite set of indices ν, and where l is an arbitrary integer $\geqslant 0$ if in the sequence $\{i_\nu\}$ the equality $i_\nu = 1$ holds only for a finite set of indices ν.

Let $k = k(\{i_\nu\})$ denote 0 if $i_\nu = 0$ for every $\nu = 0, 1, \ldots$ or if $i_\nu = 1$ for an infinite set of indices ν. In all other cases $k = k(\{i_\nu\})$ denotes the maximal index ν such that $i_\nu = 1$. Putting

$$s_\nu = \sum_{\mu=0}^{\nu} i_\mu$$

consider the set X_0 composed of all numbers $x[\{i_\nu\}, l]$ given by the formula

(4) $$x[\{i_\nu\}, l] = 48 \cdot \sum_{\nu=0}^{\infty} 3^{-2^\nu} i_\nu (-1)^{1+s_\nu} + l \cdot 3^{-2^{k+l+1}} (-1)^{1+s_k}.$$

Lemma. *If the systems $[\{i_\nu\}, l]$ and $[\{i'_\nu\}, l']$ are different, then $x[\{i_\nu\}, l] \neq x[\{i'_\nu\}, l']$.*

Proof. Putting

$$s'_\nu = \sum_{\mu=0}^{\nu} i'_\mu,$$

$$P = 48\Big[\sum_{\nu=0}^{\infty} 3^{-2^\nu} i_\nu (-1)^{1+s_\nu} - \sum_{\nu=0}^{\infty} 3^{-2^\nu} i'_\nu (-1)^{1+s'_\nu}\Big],$$

$$Q = l \cdot 3^{-2^{k+l+1}} (-1)^{1+s_k}, \qquad Q' = l' \cdot 3^{-2^{k'+l'+1}} (-1)^{1+s'_{k'}},$$

we have

$$x[\{i_\nu\}, l] - x[\{i'_\nu\}, l'] = P + Q - Q'.$$

In order to prove that $x[\{i_\nu\}, l] \neq x[\{i'_\nu\}, l']$ we distinguish two cases:

1. *The case in which* $\{i_\nu\} \neq \{i'_\nu\}$.

Let j denote the minimal index, such that $i_j \neq i'_j$; for instance $i_j = 1$ and $i'_j = 0$. Hence $i_\nu = i'_\nu$ for every $\nu < j$. Then

$$|P| = \Big|48 \cdot 3^{-2^j} (-1)^{1+s_j} + 48 \sum_{\nu=j+1}^{\infty} 3^{-2^\nu} [i_\nu (-1)^{1+s_\nu} - i'_\nu (-1)^{1+s'_\nu}]\Big| >$$

$$48(3^{-2^j} - 2 \cdot 3^{-2^{j+1}} - 2 \cdot 3^{-2^{j+2}} - \ldots) > 48 \cdot 3^{-2^j}\Big(1 - \frac{2}{3} - \frac{2}{3^4} - \ldots\Big) >$$

$$48 \cdot 3^{-2^j} \cdot \Big[1 - \frac{2}{3} - \frac{2}{9}\Big(1 + \frac{1}{3^2} + \frac{1}{3^4} + \ldots\Big)\Big] = 48 \cdot 3^{-2^j} \cdot \frac{1}{12} = 4 \cdot 3^{-2^j}.$$

Moreover, let us observe that the sign of P is equal to $(-1)^{1+s_j}$ (where $i_j=1$, $i'_j=0$). The absolute value of Q is $\leqslant 3^{-2^{j+2}}$ (since $k\geqslant j$ or $l=0$). Also the absolute value of Q' is $\leqslant 3^{-2^{j+2}}$, if there exists an index $\nu>j$ such that $i'_\nu=1$, or if $l'=0$. If $i'_\nu=0$ for $\nu>j$ and $l'\neq 0$ then the sign of Q' is opposite to the sign of P, since the sum $1+\sum_{\mu=0}^{k'} i'_\mu$ contains exactly one unity less than the sum $1+\sum_{\mu=0}^{j} i_\mu$ ($i_\mu=i'_\mu$ for $\mu<j$, $i'_j=0$, $i_j=1$). It follows that if P and $Q-Q'$ have opposite signs then

$$|Q-Q'|\leqslant 2\cdot 3^{-2^{j+2}}.$$

Hence

$$(5)\quad |x[\{i_\nu\},l]-x[\{i'_\nu\},l']|=|P+Q-Q'|>4\cdot 3^{-2^j}-2\cdot 3^{-2^{j+2}}>34\cdot 3^{-2^{j-2}},$$

where j denotes the minimal index ν, such that $i_\nu\neq i'_\nu$.

Moreover, let us observe that

(5′) the sign of $x[\{i_\nu\},l]-x[\{i'_\nu\},l']$

is the same as the sign of P and equal to $(-1)^{1+s_j}$ (where $i_j=1$, $i'_j=0$).

2. *The case in which* $\{i_\nu\}=\{i'_\nu\}$.

Since the systems $[\{i_\nu\},l]$ and $[\{i'_\nu\},l']$ are different, $l\neq l'$. In this case $k=k'$ and $P=0$. Consequently

$$|x[\{i_\nu\},l]-x[\{i'_\nu\},l']|=|l\cdot 3^{-2^{k+l+1}}-l'\cdot 3^{-2^{k+l'+1}}|\neq 0.$$

5. Lemma. *If* $x[\{i_\nu\},l]\neq x[\{i_\nu\},\bar{l}]$ *and* $\{i'_\nu\}\neq\{i_\nu\}$, *then* $x[\{i'_\nu\},l']$ *does not lie between* $x[\{i_\nu\},l]$ *and* $x[\{i_\nu\},\bar{l}]$.

Proof. Since $x[\{i_\nu\},l]\neq x[\{i_\nu\},\bar{l}]$ the sequence $\{i_\nu\}$ is finite. Let j denote the minimal index ν, such that $i_\nu\neq i'_\nu$. If there exists an index $\nu\geqslant j$, such that $i_\nu=1$ then $k\geqslant j$ and we have

$$|x[\{i_\nu\},l]-x[\{i_\nu\},\bar{l}]|=|l\cdot 3^{-2^{k+l+1}}-\bar{l}\cdot 3^{-2^{k+\bar{l}+1}}|\leqslant 3^{-2^{j-2}}.$$

It follows by (5) that $x[\{i'_\nu\},l']$ does not lie between $x[\{i_\nu\},l]$ and $x[\{i_\nu\},\bar{l}]$. If $i_\nu=0$ for every $\nu\geqslant j$ then $i_j=0$ and $i'_j=1$ and, by (5′), the sign of the difference $x[\{i'_\nu\},l']-x[\{i_\nu\},0]$ is $(-1)^{1+s'_j}$ and consequently (since $i_\mu=i'_\mu$ for $\mu<j$ and next unity in the sequence $\{i'_\nu\}$ after i'_k is i'_j) opposite to the sign of the difference

$$x[\{i_\nu\},l]-x[\{i_\nu\},0]=l\cdot 3^{-2^{k+l+1}}\cdot(-1)^{1+s_k}=l\cdot 3^{-2^{k+l+1}}\cdot(-1)^{1+s'_k}.$$

It follows that $x[\{i_\nu\},l]$ and $x[\{i_\nu\},\bar{l}]$ lie on one side of $x[\{i_\nu\},0]$ and $x[\{i'_\nu\},l']$ on the opposite side. Hence the lemma is also true in this case.

6. Lemma. *Between* $x[\{i_\nu\},l]$ *and* $x[\{i_\nu\},l+1]$ *there does not lie any number of* X_0.

Proof. With regard to the lemma of the section 5 it is enough to observe that the sign of the difference

$$x[\{i_\nu\},l]-x[\{i_\nu\},0]=l\cdot 3^{-2^{k+l+1}}\cdot(-1)^{1+s_k}$$

is independent of l and the absolute value of this difference diminishes when l increases.

7. For every $x=x[\{i_\nu\},l]\in X_0$ let us put

$$(6)\qquad y=f(x)=1+\sum_{\nu=0}^{\infty}3^{-2^{\nu+1}}i_\nu.$$

Let Y_0 denote the set composed by all the numbers

$$y=f(x)\quad\text{where}\quad x\in X_0.$$

According to (6) we have

$$(7)\qquad f(x)\leqslant 1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^8}+\ldots<\frac{1}{1-\frac{1}{9}}=\frac{9}{8}.$$

Let us denote by $Z_j(x_0)$ for every $x_0=x[\{i^0_\nu\},l_0]\in X_0$ and every natural j the set composed of all points $x=x[\{i_\nu\},l]\in X_0$ such that $i_\nu=i^0_\nu$ for every $\nu\leqslant j$.

Lemma. *Let* $x_0=x[\{i^0_\nu\},l_0]$ *and* $\bar{x}_0=x[\{\bar{i}^0_\nu\},\bar{l}_0]$ *be two points of* X_0 *such that* $\{i^0_\nu\}\neq\{\bar{i}^0_\nu\}$ *and let* j *denote the minimal index* ν, *such that* $i^0_\nu\neq\bar{i}^0_\nu$. *Then for every* $x\in Z_j(x_0)$ *and* $\bar{x}\in Z(\bar{x}_0)$

$$(8)\qquad f(x_0)<\sqrt{[f(\bar{x}_0)]^2+\left(\frac{\bar{x}-x}{2}\right)^2}.$$

Proof. According to (6) we have

$$|f(x_0)-f(\bar{x}_0)|<3^{-2^{j+1}}+3^{-2^{j+2}}+3^{-2^{j+3}}+\ldots<$$
$$3^{-2^{j+1}}(1+3^{-2}+3^{-4}+3^{-8}+\ldots)<\tfrac{9}{8}\cdot 3^{-2^{j+1}}.$$

Hence

$$(9)\qquad f(x_0)<f(\bar{x}_0)+\tfrac{9}{8}\cdot 3^{-2^{j+1}}.$$

In order to prove (8) it is sufficient to show by (5) and (9), that

$$(f(\overline{x}_0)+\tfrac{9}{8}\cdot 3^{-2^{j+1}})^2<[f(\overline{x}_0)]^2+(\tfrac{17}{9}\cdot 3^{-2^j})^2,$$

i. e. that

$$[f(\overline{x}_0)]^2+\tfrac{9}{4}f(\overline{x}_0)\cdot 3^{-2^{j+1}}+\tfrac{81}{64}\cdot 3^{-2^{j+2}}<[f(x_0)]^2+\tfrac{289}{81}\cdot 3^{-2^{j+1}}.$$

But the above inequality is the result of (7) and the inequality $\frac{11}{12}+\frac{81}{64}\cdot 3^{-2^{j+1}}<\frac{289}{81}$ which holds for every $j=0,1\dots$

8. *Lemma.* *The set Y_0 is of potency $2^{\aleph_0}$.*

Proof. It is sufficient to show that if $\{i_\nu\}\neq\{i'_\nu\}$ and $x=x[\{i_\nu\},l]$, $x'=x[\{i'_\nu\},l']$ then $f(x)\neq f(x')$.

Let j denote the minimal index ν such that $i_\nu\neq i'_\nu$. For instance let be $i_j=0$ and $i'_j=1$. Then

$$f(x')-f(x)=3^{-2^{j+1}}+\sum_{\nu=j+1}^{\infty}3^{-2^{\nu+1}}(i'_\nu-i_\nu)>\tag{10}$$
$$3^{-2^{j+1}}-3^{-2^{j+2}}(1+3^{-2}+3^{-4}+\dots)>0.$$

9. *Theorem.* *There exists a compactum $A_0\subset E_2$, such that the set of singular radii of A_0 is of potency $2^{\aleph_0}$.*

Proof. We shall prove that the set

$$A_0=\mathop{E}_{(xy)}[x\in X_0,|y|=f(x)]$$

has the property required. It is enough to show that every number $y\in Y_0$ is a singular radius of A_0.

Case 1. Let $y_0=f(x_0)$ where $x_0=x[\{i_\nu^0\},l_0]$ and $i_\nu^0=1$ holds only for a finite number of indices ν.

If we put $x_{0l}=x[\{i_\nu^0\},l]$ we can see at once that all points of the form

$$p_{0l}=\big(x_{0l},f(x_{0l})\big)\quad\text{or}\quad p'_{0l}=\big(x_{0l},-f(x_{0l})\big)$$

belong to A_0 and are at the same distance $y_0=f(x_{0l})$ from the x-axis. The circles with centers p_{0l} and p'_{0l} and radius y_0 are tangent to the x-axis at the point $(x_{0l},0)$.

Let us show that none of the points

$$q_{0l}=[\tfrac{1}{2}(x_{0l}+x_{0l+1}),0]\qquad\text{for}\quad l=0,1,\dots$$

belong to the generalized sphere $K_{y_0}(A_0,E_2)$. Let $x=x[\{i_\nu\},l]$ and $y=\pm f(x)$ be the coordinates of an arbitrary point $p=(xy)$ of A_0.

If $\{i_\nu\}=\{i_\nu^0\}$ then $f(x)=y_0$ and by the lemma of the section 6 $x\neq\frac{1}{2}(x_{0l}+x_{0l+1})$, consequently $\varrho(q_{0l},p)>y_0$.

If $\{i_\nu\}\neq\{i_\nu^0\}$ then let j denote the minimal index ν, such that $i_\nu\neq i_\nu^0$. Now let us observe that $x_{0l}\in Z_j(x_0)$. According to the lemma of the section 7 we have

$$y_0=f(x_0)<\sqrt{[f(x)]^2+\left(\frac{x-x_{0l}}{2}\right)^2}.$$

Therefore the distance between p and $(x_{0l},0)$ is greater than y_0. The lemma of the section 6 leads us to the conclusion that the points q_{0l}, $l=0,1,2,\dots$ do not belong to $K_{y_0}(A_0,E_2)$. Moreover it is evident that for $l\neq l'$ the points q_{0l} and $q_{0l'}$ belong to the different components of $E_2-K_{y_0}(A_0,E_2)$, because the circles

$$K_{y_0}(p_{0l}),\quad K_{y_0}(p'_{0l}),\quad K_{y_0}(p_{0l+1})\quad\text{and}\quad K_{y_0}(p'_{0l+1})$$

cut the plane E_2 between these points. Hence $E_2-K_{y_0}(A_0,E_2)$ contains an infinite number of components. Consequently [4]) the set $K_{y_0}(A_0,E_2)$ is not locally contractible.

Case 2. Let $y_0=f(x_0)$ where $x_0=x[\{i_\nu^0\},l]$ and $i_\nu^0=1$ holds for an infinite number of indices ν. By the definition of the systems $[\{i_\nu^0\},l_0]$ we infer that $l_0=0$.

Let $\{\nu_j\}$ denote the increasing sequence composed of all the natural ν such that $i_\nu^0=1$.

Furthermore, let us denote by $\{i_\nu^{(j)}\}$ for every $j=1,2,\dots$ the sequence defined by the formulae

$$i_\nu^{(j)}=i_\nu^0\quad\text{for}\quad\nu\leqslant\nu_j,$$
$$i_\nu^{(j)}=0\quad\text{for}\quad\nu>\nu_j.$$

Let us put

$$x_j=x[\{i_\nu^{(j)}\},0]\quad\text{for every}\quad j=1,2,\dots$$

Since for x_0 it is $l=l_0=0$ and by (4) x_j is the j-th partial-sum of an alternating series convergent at x_0.

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[4]) K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), p. 230 and 240.

Thereby, since $3^{-2^{\nu_j}} > 2\cdot 3^{-2^{\nu_{j+1}}}$ we may conclude that

$$x_{2j+1} - \frac{x_{2j+1}+x_{2j+2}}{2} = \frac{x_{2j+1}-x_{2j+2}}{2} = \tfrac{1}{2}\cdot 48\cdot 3^{-2^{\nu_{2j+2}}} > 0,$$

$$\frac{x_{2j+1}+x_{2j+2}}{2} - x_{2j+3} = \tfrac{1}{2}(x_{2j+2} + 48\cdot 3^{-2^{\nu_{2j+2}}} + x_{2j+2}) - x_{2j+2} - 48\cdot 3^{-2^{\nu_{2j+3}}} =$$

$$= \tfrac{1}{2}\cdot 48\cdot 3^{-2^{\nu_{2j+2}}} - 48\cdot 3^{-2^{\nu_{2j+3}}} > 0,$$

$$x_{2j+2} - \frac{x_{2j}+x_{2j+1}}{2} = x_{2j+1} - 48\cdot 3^{-2^{\nu_{2j+2}}} - \tfrac{1}{2}\cdot(x_{2j+1} - 48\cdot 3^{-2^{\nu_{2j+1}}} + x_{2j+1}) =$$

$$= \tfrac{1}{2}\cdot 48\cdot 3^{-2^{\nu_{2j+1}}} - 48\cdot 3^{-2^{\nu_{2j+2}}} > 0,$$

$$\frac{x_{2j}+x_{2j+1}}{2} - x_{2j} = \tfrac{1}{2}\cdot(x_{2j} + x_{2j} + 48\cdot 3^{-2^{\nu_{2j+1}}}) - x_{2j} = \tfrac{1}{2}\cdot 48\cdot 3^{-2^{\nu_{2j+1}}} > 0.$$

Hence

$$x_1 > \frac{x_1+x_2}{2} > x_3 > \frac{x_3+x_4}{2} > x_5 > \frac{x_5+x_6}{2} > \ldots > x_{2j+1} > \frac{x_{2j+1}+x_{2j+2}}{2} > x_{2j+3}$$

$$\ldots > x_0 > \ldots > x_{2j+2} > \frac{x_{2j}+x_{2j+1}}{2} > \ldots > x_4 > \frac{x_2+x_3}{2} > x_2. \tag{11}$$

We shall show that the points

$$\left(\frac{x_j+x_{j+1}}{2}, 0\right)$$

do not belong to $K_{y_0}(A_0, E_2)$. Let $x[\{i_\nu\}, l]$ and $y = \pm f(x)$ be the coordinates of an arbitrary point $p = p(x,y)$ of A_0. It is sufficient to prove that

$$f(x_0) < \sqrt{[f(x)]^2 + \left(x - \frac{x_j+x_{j+1}}{2}\right)^2} \quad \text{for every } j = 1, 2, \ldots \tag{12}$$

The proof of the inequality 12 will be divided into three cases:

(a) $x \in Z_{\nu_j}(x_0) - Z_{\nu_{j+1}}(x_0)$,

(b) $x \in Z_{\nu_j}(x_0)\cdot Z_{\nu_{j+1}}(x_0)$,

(c) $x \,\bar{\in}\, Z_{\nu_j}(x_0)$.

The case (a) may be subdivided into two cases:

(a_1) $x \in Z_{\nu_j}(x_0) - Z_{\nu_{j+1}}(x_0)$ and there exists an index $\bar{\nu}$ such that $\nu_1 < \bar{\nu} < \nu_{j+1}$ and $i_\nu = 0$ for $\nu_j < \nu < \bar{\nu}$, $i_\nu = 1$;

(a_2) $x \in Z_{\nu_j}(x_0) - Z_{\nu_{j+1}}(x_0)$ and $i_\nu = 0$ for $\nu_j < \nu < \nu_{j+1}$ and $i_{\nu_{j+1}} = 0$.

In case (a_1) according to (10), $f(x) > f(x_0)$ and consequently also (12) is true.

In the case of (a_2) let us denote by x' the number $x[\{i'_\nu\}, l]$ where $i'_\nu = i_\nu$ for $\nu \neq \nu_{j+1}$ and $i'_{\nu_{j+1}} = 1 = i^0_{\nu_{j+1}}$ (while $i_{\nu_{j+1}} = 0$), l is the same as for x. Applying the lemma of the section 7 (where we preserve the sense of x_0, and replace j by ν_{j+1}, $\bar{x}_0$ and $\bar{x}$ by x and x by $x' \in Z_{\nu_{j+1}}(x_0)$), we obtain

$$f(x_0) < \sqrt{[f(x)]^2 + \left(\frac{x-x'}{2}\right)^2}. \tag{13}$$

Let us observe now that in the formula (4) defining the numbers $x = x[\{i_\nu\}, l]$ and $x' = x[\{i'_\nu\}, l]$ the coefficients $(-1)^{1+s_\nu}$ and $(-1)^{1+s'_\nu}$ have opposite signs for $\nu > \nu_j$. Therefore if there exists an index $\nu' > \nu_{j+1}$ (that is $\nu > \nu_j$), such that $i_{\nu'} = i_\nu = 1$, then (according to (4))

$$\left|\frac{x-x'}{2}\right| = \left|x - \frac{x+x'}{2}\right| = \left|x - \frac{x_j+x_{j+1}}{2}\right|$$

and consequently (13) implies (12).

If $i_\nu = 0$ for $\nu > \nu_j$ and $j = 2j_1 + 1$ then in the formula (4) defining x, the coefficient

$$(-1)^{1+s_k} = (-1)^{1+s_{\nu_j}} = (-1)^{1+s^0_{\nu_j}} \quad \text{where} \quad s^0_\nu = \sum_{\mu=0}^{\nu} i^0_\mu$$

is positive and in the formula (4) defining x' the coefficient

$$(-1)^{1+s'_{k'}} = (-1)^{1+s'_{\nu_{j+1}}} = (-1)^{1+s^0_{\nu_{j+1}}} \quad \text{is negative.}$$

Consequently

$$x > x_j > x_{j+1} > x',$$

and

$$0 < \frac{x-x'}{2} = x - \frac{x+x'}{2} = x - \tfrac{1}{2}(x_j + l\cdot 3^{-2^{\nu_j}+l+1} + x_{j+1} - l\cdot 3^{-2^{\nu_{j+1}}+l+1} =$$

$$x - \frac{x_j+x_{j+1}}{2} - \frac{l}{2}\cdot 3^{-2^{\nu_j}+l+1} + \frac{l}{2}\cdot 3^{-2^{\nu_{j+1}}+l+1} < x - \frac{x_j+x_{j+1}}{2}.$$

Hence

$$\left|\frac{x-x'}{2}\right| < \left|x - \frac{x_j+x_{j+1}}{2}\right|. \tag{14}$$

If $i_\nu = 0$ for $\nu > \nu_j$ and $j = 2j_1$ then in the formula (4) defining x the coefficient $(-1)^{1+s_k} = (-1)^{1+s_{\nu_j}} = (-1)^{1+s^0_{\nu_j}}$ is negative and in the formula (4) defining x' the coefficient

$$(-1)^{1+s'_{k'}} = (-1)^{1+s'_{\nu_{j+1}}} = (-1)^{1+s^0_{\nu_{j+1}}}$$

is positive.

Consequently

$$x < x_j < x_{j+1} < x'$$

and

$$0 > \frac{x-x'}{2} = x - \frac{x+x'}{2} = x - \tfrac{1}{2}(x_j - l\cdot 3^{-2^{\nu_j}+l+1} + x_{j+1} + l\cdot 3^{-2^{\nu_{j+1}}+l+1}) =$$

$$x - \frac{x_j + x_{j+1}}{2} + l\cdot 3^{-2^{\nu_j}+l+1} - l\cdot 3^{-2^{\nu_{j+1}}+l+1} > x - \frac{x_j + x_{j+1}}{2}.$$

Therefore

$$(14') \qquad \left|\frac{x-x'}{2}\right| < \left|x - \frac{x_j + x_{j+1}}{2}\right|.$$

From (13) and (14) or (14′) follows (12).

In case (b), let us denote by x'' the number $x = x[\{i''_\nu\}, l'']$ where $i'' = i_\nu$ for $\nu \neq \nu_{j+1}$ and $i''_{\nu_{j+1}} = 0$ (while $i_{\nu_{j+1}} = 1$), $l'' = l$, when $i_{\nu_{j+1}}$ is not the last unity in the sequence $\{i_\nu\}$ and $l'' = 0$ if it is the last one.

Applying the lemma of the section 7 (where we preserve the sense of x_0 and $x \in Z_{\nu_{j+1}}(x_0)$ and replace j by ν_{j+1}, and $\bar{x}_0$ and $\bar{x}$ by x'') we obtain

$$(15) \qquad f(x_0) < \sqrt{[f(x'')]^2 + \left(\frac{x-x''}{2}\right)^2}.$$

From (10) may be derived

$$(16) \qquad f(x'') < f(x).$$

As in the case (a_2), we may say that in the formula (4) defining the numbers $x = [\{i_\nu\}, l]$ and $x'' = x[\{i''_\nu\}, l'']$ the coefficients $(-1)^{1+s_\nu}$ and $(-1)^{1+s''_\nu}$, where $s''_\nu = \sum_{\mu=0}^{\nu} i''_\mu$, have opposite signs for $\nu > \nu_j$.

Consequently if an index $\nu > \nu_{j+1}$ then an $i_\nu = 1$ exists

$$(17) \qquad \left|\frac{x-x''}{2}\right| = \left|x - \frac{x+x''}{2}\right| = \left|x - \frac{x_j + x_{j+1}}{2}\right|.$$

If $i_\nu = 0$ for $\nu > \nu_{j+1}$ and $j = 2j_1$ then in the formula (4) defining x the coefficient $(-1)^{1+s_k} = (-1)^{1+s_{\nu_{j+1}}} = (-1)^{1+s^0_{\nu_{j+1}}}$ is positive.

Hence

$$x'' = x_j < x_{j+1} < x$$

and

$$0 < \frac{x-x''}{2} = x - \frac{x+x''}{2} = x - \frac{x_{j+1}+x_j}{2} - \frac{l}{2\cdot 3^{2^{\nu_{j+1}}+l+1}} < x - \frac{x_{j+1}+x_j}{2}.$$

So

$$(17') \qquad \left|\frac{x-x''}{2}\right| < \left|x - \frac{x_j + x_{j+1}}{2}\right|.$$

If $i_\nu = 0$ for $\nu > \nu_{j+1}$ and $j = 2j_1 + 1$ then in the formula (4) defining x the coefficient

$$(-1)^{1+s_k} = (-1)^{1+s_{\nu_{j+1}}} = (-1)^{1+s^0_{\nu_{j+1}}}$$

is negative.

Hence

$$x'' = x_j > x_{j+1} > x$$

and

$$0 > \frac{x-x''}{2} = x - \frac{x+x''}{2} = x - \tfrac{1}{2}(x_{j+1} - l\cdot 3^{-2^{\nu_{j+1}}+l+1} + x_j) =$$

$$x - \frac{x_j + x_{j+1}}{2} + \frac{l}{2}\cdot 3^{-2^{\nu_{j+1}}+l+1} > x - \frac{x_j + x_{j+1}}{2}.$$

Hence

$$(17'') \qquad \left|\frac{x-x''}{2}\right| < \left|x - \frac{x_j + x_{j+1}}{2}\right|.$$

The formulae (15), (16) and (17) or (17′), or (17″) imply (12).

In case (c) there exists an index $\nu < \nu_j$ that $x \in Z_\nu(x_0)$ and $x \,\bar{\epsilon}\, Z_{\nu+1}(x_0)$. Applying the lemma of the section 7 where we preserve the sense of x_0 and replace j by ν, and $\bar{x}_0$ and $\bar{x}$ by x, and x firstly by $x_j \in Z_{\nu+1}(x_0)$, secondly by $x_{j+1} \in Z_{\nu+1}(x_0)$ we obtain two inequalities:

$$(18) \qquad f(x_0) < \sqrt{[f(x)]^2 + \left(\frac{x-x_j}{2}\right)^2} \quad \text{and} \quad f(x_0) < \sqrt{[f(x)]^2 + \left(\frac{x-x_{j+1}}{2}\right)^2}.$$

The numbers $x - x_j$ and $x - x_{j+1}$ appearing in (5′), have the same sign because the differences have the same sign of the term P.

Hence either

$$x-x_j<x-\frac{x_j+x_{j+1}}{2}<x-x_{j+1} \quad \text{or} \quad x-x_j>x-\frac{x_j+x_{j+1}}{2}>x-x_{j+1}$$

then either

$$(x-x_j)^2<\left(x-\frac{x_j+x_{j+1}}{2}\right)^2 \text{ or } (x-x_{j+1})^2<\left(x-\frac{x_j+x_{j+1}}{2}\right)^2.$$

Consequently it is

$$(19)\quad \text{either } \left(\frac{x-x_j}{2}\right)^2<\left(x-\frac{x_j+x_{j+1}}{2}\right)^2 \text{ or } \left(\frac{x-x_{j+1}}{2}\right)^2<\left(x-\frac{x_j+x_{j+1}}{2}\right)^2.$$

The inequalities (18) and (19) imply (12). Therefore the inequality (12) is true in all cases.

Thus we have shown that no point $\left(\frac{x_j+x_{j+1}}{2}, 0\right)$ belongs to $K_{y_0}(A_0, E_2)$.

Formula (11) implies that between two points

$$\left(\frac{x_{j_1}+x_{j_1+1}}{2}, 0\right) \quad \text{and} \quad \left(\frac{x_{j_2}+x_{j_2+1}}{2}, 0\right)$$

for $j_1 \neq j_2$ lies at least one point $x_{j'}$ belonging to $K_{y_0}(A_0, E_2)$ since according to (6) $f(x_{j'})<f(x_0)$. Consequently the points $\left(\frac{x_{j_1}+x_{j_1+1}}{2}, 0\right)$ and $\left(\frac{x_{j_2}+x_{j_2+1}}{2}, 0\right)$ for $j_1 \neq j_2$ lie in the different components of $E_2-K_{y_0}(A_0, E_2)$.

Consequently $E_2-K_{y_0}(A_0, E_2)$ contains an infinite number of components. Hence $K_{y_0}(A_0, E_2)$ is not locally contractible.

10. Problem. *Let A_0 be a compact subset of the n-dimensional Euclidean space E_n. Let R denote the set of all positive numbers r, such that $K_r(A_0, E_n)$ is not homeomorphic to a polytope. The problem is, whether the set R is necessarily of first category (in the sense of Baire) and of measure zero (in the sense of Lebesgue)?*

Simply connected spaces.

By

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1. There are two ways of defining simple connectedness for topological spaces.

The first way is based on closed paths and their deformation:

An arcwise and locally arcwise connected topological space is termed *simply connected* whenever each of its closed paths is homotopic to a point ([9], p. 310; [10], p. 221) [1]). Such spaces will be referred to hereafter as *pathwise simply connected.*

Another way of defining simple connectedness makes use of the idea of a covering space:

A connected and locally connected topological space is termed *simply connected* whenever it admits only a trivial covering space ([5], p. 44). These will be referred to merely as *simply connected* spaces.

The first definition requires arcwise connectedness, while the second has a meaning even for Hausdorff-Lennes connected and locally connected spaces.

Similarly, the fundamental group of a space may be defined either as the group of paths, or as the group of covering homeomorphisms of the simply connected covering space (*Deckbewegungsgruppe*).

2. It is the purpose of this paper to state some theorems on simply connected spaces, which do not hold true for pathwise simple connectedness. As a consequence, it will be shown that, without further local assumptions, the two definitions are not equivalent [2]).

Our main goal is the proof of two kinds of approximation theorems: one related to the so-called ε-mappings, the other concerning convergent families of sets.

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[1]) Numbers in brackets refer to the bibliography at the end of the paper.

[2]) Pathwise simple connectedness implies simple connectedness, but less than that is needed.