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Dimension Theory in Closure Algebras.

By

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This paper is the continuation of my paper *Closure Algebras* [1]) cited hereafter as CA.

The generalization of the concept of dimension to the case of closure algebras presents no difficulty. The definition assumed in this paper is inductive by means of separation of closed elements. For C-algebras [2]), this definition is equivalent (Theorem 3.5) to Lebesgue's definition which clearly can be formulated without difficulty for arbitrary closure algebras.

The generalization of fundamental theorems from Dimension Theory to the case of arbitrary C-algebras is easy. Some theorems can be proved analogously to the case of metric spaces (see § 1); their proofs are omitted. Other theorems follow from analogous statements for metric spaces (see e. g. 3.2).

The specification of all theorems which hold for C-algebras is not the purpose of this paper. As in my earlier paper CA, I shall only show the method of generalization. Roughly speaking, all theorems from Dimension Theory which hold for separable metric spaces are also true for arbitrary C-algebras.

It was stated in CA that every quotient algebra $\boldsymbol{A}/\boldsymbol{I}$, where $\boldsymbol{A}$ is a C-algebra and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{A}$, is also a C-algebra. This fact suggests the following general problem: Suppose the topological properties of $\boldsymbol{A}$ and $\boldsymbol{I}$ are known; what topological properties has the quotient algebra $\boldsymbol{A}/\boldsymbol{I}$?

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[1]) See *References* at the end of this paper.
The knowledge of Parts I and II of CA is assumed. Theorems from CA will be cited by their numbers together with the letters "CA".

[2]) See the definition on p. 154.

I shall discuss this problem in § 2 and § 4 from the point of view of Dimension Theory. In § 2 I shall calculate the dimension of $\boldsymbol{A}/\boldsymbol{I}$ for an arbitrary σ-ideal $\boldsymbol{I}$ (Theorems 2.9 and 2.10). In § 4 I shall examine some special σ-ideals. The main result is that $\dim \boldsymbol{A}/\boldsymbol{I} \leqslant \dim \boldsymbol{A}$. The subject of § 2 is related to Hurewicz's notion of a „Normalbereich".

The representation theorem proved in § 3 shows that the compact n-dimensional space universal for all n-dimensional spaces is also universal for arbitrary n-dimensional C-algebras.

Terminology and notation of this paper are the same as in CA. C-algebras will be denoted by the letters $\boldsymbol{A}, \boldsymbol{B}, \ldots$, their elements by $A, B, \ldots$ A C-algebra $\boldsymbol{A}$ is by definition a σ-complete Boolean algebra[3]) with the closure operation $\bar{A}$ defined for all $A \epsilon \boldsymbol{A}$ such that

$$\text{I. } \overline{A+B}=\bar{A}+\bar{B} \qquad \text{II. } \bar{0}=0$$
$$\text{III. } A \subset \bar{A} \epsilon \boldsymbol{A} \qquad \text{IV. } \bar{\bar{A}}=\bar{A}$$

V. there is an enumerable sequence $\{R_n\}$ (called the *C-basis* of $\boldsymbol{A}$) of open elements in $\boldsymbol{A}$ such that each open element $G \epsilon \boldsymbol{A}$ is the sum of all R_n with $\bar{R}_n \subset G$.

Clearly an element $A \epsilon \boldsymbol{A}$ is said to be
closed if $\bar{A}=A$;
open if its complement A' is closed;
an *F_σ-element* if it is the sum of an enumerable sequence of closed elements;
a *G_δ-element* if its complement is an F_σ-element.
We assume the following notations:

$$\mathrm{Fr}(A)=\bar{A}\cdot\overline{A'} \quad \text{and} \quad \mathrm{Int}(A)=(\overline{A'})' \quad \text{for any} \quad A \epsilon \boldsymbol{A}.$$

$\mathfrak{F}(\boldsymbol{A})$ is the class of all closed element in $\boldsymbol{A}$.
$\mathfrak{F}_\sigma(\boldsymbol{A})$ is the class of all F_σ-elements in $\boldsymbol{A}$.
$\mathfrak{B}(\boldsymbol{A})$ is the class of all *Borel elements* in $\boldsymbol{A}$, i. e. $\mathfrak{B}(\boldsymbol{A})$ is the least σ-subalgebra of $\boldsymbol{A}$ containing all closed elements. Clearly $\mathfrak{B}(\boldsymbol{A})$ is itself a C-algebra.
$E\boldsymbol{A}$ (where $E \epsilon \boldsymbol{A}$) is the relativized C-algebra formed of all elements $A \subset E$ with the closure operation $\bar{A}_E = E\bar{A}$.
$\boldsymbol{A}/\boldsymbol{I}$ (where $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{A}$) is a quotient C-algebra defined in CA 9. The element of $\boldsymbol{A}/\boldsymbol{I}$ determined by an element $A \epsilon \boldsymbol{A}$ will be denoted by $[A]$. By definition, $[A]=[B]$ if and only if $AB'+BA' \epsilon \boldsymbol{I}$.
$E\boldsymbol{I}$ (where $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{A}$ and $E \epsilon \boldsymbol{A}$) is the class of all $A \epsilon \boldsymbol{I}$ such that $A \subset E$. $E\boldsymbol{I}$ is thus a σ-ideal of $E\boldsymbol{A}$. We must distinguish between $E\boldsymbol{A}/E\boldsymbol{I}$ and $[E]\cdot\boldsymbol{A}/\boldsymbol{I}$. The construction of the first C-algebra is as follows: relativize $\boldsymbol{A}$ to the element E and divide $E\boldsymbol{A}$ by $E\boldsymbol{I}$; the construction of the second is: divide $\boldsymbol{A}$ by $\boldsymbol{I}$ and relativize to the element $[E] \epsilon \boldsymbol{A}/\boldsymbol{I}$.

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[3]) $A+B$, $\sum_{n=1}^{\infty} A_n$, AB, A' denote the Boolean operations analogous to addition, multiplication and complementation of sets.

$\mathfrak{S}(\mathfrak{X})$ is the C-algebra of all subsets of a separable metric space $\mathfrak{X}$.
$\mathfrak{B}(\mathfrak{X})$ is the C-algebra of all Borel subsets of a separable metric space $\mathfrak{X}$.
Two C-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are *homeomorphic* if there is a Boolean isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$ such that $h(\bar{A})=\overline{h(A)}$ for each $A \epsilon \boldsymbol{A}$.
Two C-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *weakly homeomorphic* provided $\mathfrak{B}(\boldsymbol{A})$ and $\mathfrak{B}(\boldsymbol{B})$ are homeomorphic.
0 and $|\boldsymbol{A}|$ denote respectively the greatest and the least element of $\boldsymbol{A}$ (i. e. $0 \subset A \subset |\boldsymbol{A}|$ for each $A \epsilon \boldsymbol{A}$).

§ 1. Definition and general properties. The dimension of a C-algebra $\boldsymbol{A}$ will be denoted by $\dim \boldsymbol{A}$. The inductive definition is the following:

$\dim \boldsymbol{A}=-1$ if $\boldsymbol{A}$ has only one element 0;

$\dim \boldsymbol{A} \leqslant n$ if for every pair of disjoint closed elements $F_1, F_2 \epsilon \boldsymbol{A}$ there is an open $G \epsilon \boldsymbol{A}$ such that

$$F_1 \subset G, \quad \bar{G}F_2=0, \quad \text{and} \quad \dim \mathrm{Fr}(G)\cdot\boldsymbol{A} \leqslant n-1.$$

Clearly $\dim \boldsymbol{A}$ is the least integer $n \geqslant -1$ such that $\dim \boldsymbol{A} \leqslant n$. If there exists no integer n with $\dim \boldsymbol{A} \leqslant n$, we write $\dim \boldsymbol{A}=\infty$.

If $\mathfrak{X}$ is a separable metric space, then $\dim \mathfrak{S}(\mathfrak{X})$ coincides with $\dim \mathfrak{X}$ in the usual sense.

1.1 $\dim \boldsymbol{A}=\dim \mathfrak{B}(\boldsymbol{A})$. *Consequently, if $\boldsymbol{A}$ is weakly homeomorphic to $\boldsymbol{B}$, then* $\dim \boldsymbol{A}=\dim \boldsymbol{B}$.

This follows immediately from the definition.

The following simple lemma will often be useful.

Lemma. Let $\{R_m\}$ be a C-basis of $\boldsymbol{A}$, and let $\{i_m, j_m\}$ be the sequence of all pairs of integers i, j such that $\bar{R}_i \subset R_j$. If a sequence $\{S_m\}$ of open elements has the property $\bar{R}_{i_m} \subset S_m \subset \bar{R}_{j_m}$ (in particular, if $\bar{R}_{i_m} \subset S_m \subset R_{j_m}$) for $m=1,2,\ldots$, then $\{S_m\}$ is a C-basis for $\boldsymbol{A}$.

1.2. *If* $\dim \boldsymbol{A} \leqslant n$, *then $\boldsymbol{A}$ has a C-basis $\{S_m\}$ such that* $\dim \mathrm{Fr}(S_m)\cdot\boldsymbol{A} \leqslant n-1$.

Let $\{R_m\}$ be a C-basis for $\boldsymbol{A}$, and let $\{i_m, j_m\}$ be a sequence of all pairs i, j such that $\bar{R}_i \subset R_j$. Since $\dim \boldsymbol{A} \leqslant n$, there is an open $S_m \epsilon \boldsymbol{A}$ such that $\bar{R}_{i_m} \subset S_m \subset \bar{S}_m \subset R_{j_m}$ and $\dim \mathrm{Fr}(S_m)\cdot\boldsymbol{A} \leqslant n-1$. The sequence $\{S_m\}$ is the required C-basis.

The converse theorem is also true and will be proved later (1.6). Now we can state only:

1.3. $\dim \boldsymbol{A} \leqslant 0$ *if and only if $\boldsymbol{A}$ has an enumerable basis $\{S_m\}$ with* $\mathrm{Fr}(S_m)=0$ *for* $m=1,2,\ldots$

The necessity follows from 1.2. The proof of the sufficiency is the same as in Topology of Metric Spaces [4]).

1.4. *If* $\dim \boldsymbol{A} \leqslant n$, *then there is a decomposition* $|\boldsymbol{A}| = A + \sum_{m=1}^{\infty} F_m$ *such that* $\dim A\boldsymbol{A} \leqslant 0$, F_m *is closed and* $\dim F_m\boldsymbol{A} \leqslant n-1$.

Let $\{S_m\}$ be a C-basis mentioned in 1.2. It is sufficient to assume $F_m = \mathrm{Fr}(S_m)$ and $A = (\sum_{m=1}^{\infty} F_m)'$. In fact, the elements $S_m A$ are simultaneously open and closed in $A\boldsymbol{A}$ and form a C-basis of $A\boldsymbol{A}$. Hence $\dim A\boldsymbol{A} \leqslant 0$ by 1.3.

1.5. (a_n) *If* $|\boldsymbol{A}| = A + B$, $\dim A\boldsymbol{A} \leqslant 0$ *and* $\dim B\boldsymbol{A} \leqslant n-1$, *then* $\dim \boldsymbol{A} \leqslant n$.

(b_n) *If* $|\boldsymbol{A}| = \sum_{m=1}^{\infty} F_m$, *where* F_m *are closed in* $\boldsymbol{A}$ *and* $\dim F_m\boldsymbol{A} \leqslant n$, *then* $\dim \boldsymbol{A} \leqslant n$.

(c_n) $\dim \boldsymbol{A} \leqslant n$ *if and only if there is a decomposition* $|\boldsymbol{A}| = \sum_{i=0}^{n} A_i$ *with* $\dim A_i\boldsymbol{A} \leqslant 0$ $(i = 0, 1, \ldots, n)$.

(d_n) *If* $\dim \boldsymbol{A} \leqslant n$, *then* $\dim E\boldsymbol{A} \leqslant n$ *for every* $E \in \boldsymbol{A}$. *Consequently, if* $A \subset B$, *then* $\dim A\boldsymbol{A} \leqslant \dim B\boldsymbol{A}$.

(e_n) *If* $|\boldsymbol{A}| = \sum_{m=1}^{\infty} F_m$, *where* $F_m \in \mathfrak{F}_\sigma(\boldsymbol{A})$ *and* $\dim F_m\boldsymbol{A} \leqslant n$, *then* $\dim \boldsymbol{A} \leqslant n$.

The proof is by induction on n.

(a_0) is true since then $B = 0$ and $A = |\boldsymbol{A}|$. The proof of (b_0) is the same as for separable metric spaces [5]). (c_0) is trivial. (d_0) follows easily from 1.3. (e_0) follows easily from (b_0) and (d_0).

Suppose now the statements (a_{n-1}), (b_{n-1}), (c_{n-1}), (d_{n-1}), (e_{n-1}) are true.

(a_n) follows from (d_{n-1}) and CA 11.2 (prove that for disjoint $F_1, F_2 \in \mathfrak{F}(\boldsymbol{A})$ there is an open $S \in \boldsymbol{A}$ such that $F_1 \subset G$, $\bar{G}F_2 = 0$ and $\mathrm{Fr}(G) \subset B$).

(b_n) follows from 1.4, (e_{n-1}), (b_0) and (a_n). The exact proof is the same as for metric spaces [6]).

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[4]) See e. g. Kuratowski [3], pp. 121-122.
[5]) See e. g. Kuratowski [4], p. 171; Hurewicz-Wallman [2], p. 18.
[6]) See e. g. Kuratowski [4], p. 176.

The necessity of (c_n) follows from 1.4, (b_{n-1}) and (c_{n-1}). The sufficiency follows from (c_{n-1}) and (a_n).

(d_n) follows from (c_n) and (d_0). (e_n) follows from (b_n) and (d_n).

1.6. $\dim E\boldsymbol{A} \leqslant n$ *if and only if for every pair of disjoint elements* $F_1, F_2 \in \mathfrak{F}(E\boldsymbol{A})$ *there is a* G *open in* $\boldsymbol{A}$ *such that* $F_1 \subset G$, $\bar{G}F_2 = 0$ *and* $\dim \mathrm{Fr}(G)E \cdot \boldsymbol{A} \leqslant n-1$.

1.7. *If* $\dim E\boldsymbol{A} \leqslant n$ *and* $F_1F_2 = 0$, $F_1, F_2 \in \mathfrak{F}(\boldsymbol{A})$, *then there is an element* $G \in \boldsymbol{A}$ *open in* $\boldsymbol{A}$ *such that* $F_1 \subset G$, $\bar{G}F_2 = 0$ *and* $\dim \mathrm{Fr}(G)E \cdot \boldsymbol{A} \leqslant n-1$.

The proof of 1.6 and of 1.7 is the same as in Metric Topology [7]). It is based on 1.4, 1.5 (d), and the separation theorem CA 11.2.

1.8. $\dim E\boldsymbol{A} \leqslant n$ *if and only if* $\boldsymbol{A}$ *has a* C-*basis* $\{S_m\}$ *such that* $\dim \mathrm{Fr}(S_m)E \cdot \boldsymbol{A} \leqslant n-1$.

The proof of the necessity is analogous to that of 1.2. It is based on 1.7. The sufficiency follows from 1.3, 1.5 (a, b) and from the decomposition $|E\boldsymbol{A}| = E = B + (E - B)$ where $B = E \cdot \sum_{m=1}^{\infty} \mathrm{Fr}(S_m)$.

1.9. *If* $\dim A\boldsymbol{A} \leqslant k$ *and* $\dim B\boldsymbol{A} \leqslant l$, *then* $\dim (A+B)\boldsymbol{A} \leqslant k+l+1$.

This follows directly from 1.5 (c).

1.10. *For every* $A \in \boldsymbol{A}$ *there is a* G_δ-*element* B *such that* $A \subset B$ *and* $\dim A\boldsymbol{A} = \dim B\boldsymbol{A}$.

The proof is the same as in Metric Topology.

§ 2. The dimension of $\boldsymbol{A}/\boldsymbol{I}$.

The letter $\boldsymbol{I}$ will denote in this section a σ-ideal of a C-algebra $\boldsymbol{A}$.

The least of the integers $\dim A'\boldsymbol{A}$ (∞ included), where $A \in \boldsymbol{I}$, will be denoted by $\dim(\boldsymbol{A}, \boldsymbol{I})$.

In the case of a relativized C-algebra $E\boldsymbol{A}$ ($E \in \boldsymbol{A}$) we shall often write, for brevity, $\dim(E\boldsymbol{A}, \boldsymbol{I})$ instead of $\dim(E\boldsymbol{A}, E\boldsymbol{I})$.

The following five lemmas are obvious.

2.1. $\dim(E\boldsymbol{A}, \boldsymbol{I})$ *is the least of the integers* $\dim A'E\boldsymbol{A}$ *where* $A \in \boldsymbol{I}$. *There always exists an* $A_0 \in \boldsymbol{I}$ *such that* $\dim(E\boldsymbol{A}, \boldsymbol{I}) = \dim A_0'E\boldsymbol{A}$.

2.2. $\dim(E\boldsymbol{A}, \boldsymbol{I}) \leqslant \dim E\boldsymbol{A}$.

2.3. *If* $A \subset B$, *then* $\dim(A\boldsymbol{A}, \boldsymbol{I}) \leqslant \dim(B\boldsymbol{A}, \boldsymbol{I})$.

2.4. $\dim(E\boldsymbol{A}, \boldsymbol{I}) = -1$ *if and only if* $E \in \boldsymbol{I}$. In particular, $\dim(\boldsymbol{A}, \boldsymbol{I}) = -1$ *if and only if* $\boldsymbol{I} = \boldsymbol{A}$.

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[7]) The proof is similar to that of th. (2) in Kuratowski [3], p. 118.

2.5. *If* $AB'+A'B\epsilon \boldsymbol{I}$ (i.e. if $[A]=[B]\epsilon \boldsymbol{A}/\boldsymbol{I}$), *then* $\dim(A\boldsymbol{A}, \boldsymbol{I}) = \dim(B\boldsymbol{A}, \boldsymbol{I})$.

2.6. $\dim(E\boldsymbol{A}, \boldsymbol{I})\leqslant n$ *if and only if* $\boldsymbol{A}$ *has a C-basis* $\{R_m\}$ *such that* $\dim(\mathrm{Fr}(R_m)E\cdot\boldsymbol{A}, \boldsymbol{I})\leqslant n-1$ $(m=1,2,\ldots)$.

In particular $(E=|\boldsymbol{A}|)$,

$\dim(\boldsymbol{A}, \boldsymbol{I})\leqslant n$ *if and only if* $\boldsymbol{A}$ *has a C-basis* $\{R_m\}$ *such that* $\dim(\mathrm{Fr}(R_m)\cdot\boldsymbol{A}, \boldsymbol{I})\leqslant n-1$.

Necessity. Let A_0 be such an element in $\boldsymbol{I}$ that $\dim A_0'E\boldsymbol{A}=\dim(E\boldsymbol{A}, \boldsymbol{I})\leqslant n$. On account of 1.8, the C-algebra $\boldsymbol{A}$ has a C-basis $\{R_m\}$ such that $\dim\mathrm{Fr}(R_m)A_0'E\cdot\boldsymbol{A}\leqslant n-1$. Consequently $\dim(\mathrm{Fr}(R_m)E\cdot\boldsymbol{A}, \boldsymbol{I})\leqslant n-1$.

Sufficiency. Let $\dim(\mathrm{Fr}(R_m)E\cdot\boldsymbol{A}, \boldsymbol{I})\leqslant n-1$ $(m=1,2,\ldots)$, where $\{R_m\}$ is a C-basis of $\boldsymbol{A}$. Let $A_m\epsilon \boldsymbol{I}$ satisfy the condition (see 2.1)

$$\dim A_m'\mathrm{Fr}(R_m)E\cdot\boldsymbol{A}=\dim(\mathrm{Fr}(R_m)E\cdot\boldsymbol{A}, \boldsymbol{I})\leqslant n-1.$$

Let $A=\sum_{m=1}^{\infty}A_m$. Clearly $A\epsilon \boldsymbol{I}$ and $\dim A'\mathrm{Fr}(R_m)E\cdot\boldsymbol{A}\leqslant n-1$. Hence $\dim A'E\cdot\boldsymbol{A}\leqslant n$ by 1.8, and $\dim(E\boldsymbol{A}, \boldsymbol{I})\leqslant n$.

2.7. *If* $\dim(\boldsymbol{A}, \boldsymbol{I})\leqslant n$, *then* $\dim\boldsymbol{A}/\boldsymbol{I}\leqslant n$.

The proof is by induction. The case $n=-1$ is trivial (see 2.4). Suppose theorem 2.7 is true for $n-1$.

By 2.6, the C-algebra $\boldsymbol{A}$ has a C-basis $\{R_m\}$ with $\dim(\mathrm{Fr}(R_m)\cdot\boldsymbol{A}, \boldsymbol{I})\leqslant n-1$ $(m=1,2,\ldots)$. By the inductive hypothesis, $\dim\mathrm{Fr}(R_m)\boldsymbol{A}/\mathrm{Fr}(R_m)\boldsymbol{I}\leqslant n-1$. The C-algebra $\mathrm{Fr}(R_m)\boldsymbol{A}/\mathrm{Fr}(R_m)\boldsymbol{I}$ being homeomorphic to $[\mathrm{Fr}(R_m)]\cdot\boldsymbol{A}/\boldsymbol{I}$ (see CA 9.5 (ii)), we have $\dim[\mathrm{Fr}(R_m)]\cdot\boldsymbol{A}/\boldsymbol{I}\leqslant n-1$. By CA 9.3 (iv), $\mathrm{Fr}([R_m])\subset[\mathrm{Fr}(R_m)]$. Hence $\dim\mathrm{Fr}([R_m])\cdot\boldsymbol{A}/\boldsymbol{I}\leqslant n-1$. Since $\{[R_m]\}$ is a C-basis for $\boldsymbol{A}/\boldsymbol{I}$ by CA 10.2, it follows from 1.8 that $\dim\boldsymbol{A}/\boldsymbol{I}\leqslant n$.

2.8. *If* $\dim\boldsymbol{A}/\boldsymbol{I}\leqslant n$, *then* $\dim(\boldsymbol{A}, \boldsymbol{I})\leqslant n$.

The proof is by induction. The case $n=-1$ follows from 2.4. Suppose theorem 2.8 is true for $n-1$. We shall prove it for n.

(A) Consider first the case where $\boldsymbol{I}$ is a boundary ideal [8].

Let $\{R_m\}$ be a C-basis of $\boldsymbol{A}$ and let $\{i_m, j_m\}$ be a sequence of all pair of integers i, j such that $\bar{R}_i\subset R_j$. Since $[\bar{R}_{i_m}]$ and $[R_{j_m}']$ are disjoint and closed in $\boldsymbol{A}/\boldsymbol{I}$, there is an open $H_m\epsilon\boldsymbol{A}/\boldsymbol{I}$ such that

(a) $$[\bar{R}_{i_m}]\subset H_m,\quad \bar{H}_m\cdot[R_{j_m}']=0,$$

(b) $$\dim\mathrm{Fr}(H_m)\boldsymbol{A}\leqslant n-1.$$

[8]) That is, no open element $G\neq 0$ belongs to $\boldsymbol{I}$. See CA 8, p. 179.

By CA 9.2, we may suppose $H_m=[G_m]$ where G_m is open in $\boldsymbol{A}$. By CA 8.1 (iii) and CA 9.3 (i), we have $\bar{H}_m=[\bar{G}_m]$. Hence

(c) $$\mathrm{Fr}(H_m)=[\mathrm{Fr}(G_m)].$$

It follows from (a) that

$$R_{i_m}\subset G_m+C_m \quad\text{and}\quad \bar{G}_m\subset R_{j_m}+D_m,\quad\text{where}\quad C_m, D_m\epsilon\boldsymbol{I}.$$

Let $S_m=\mathrm{Int}(\bar{G}_m)$. By CA 8.1 (iii), CA 7.2, and CA 7.1 (iv) [9],

$$R_{i_m}\subset\bar{R}_{i_m}=R_{i_m}^*\subset G_m^*+C_m^*=G_m^*=\bar{G}_m.$$

Hence $R_{i_m}\subset S_m$. On the other hand,

$$S_m\subset\bar{G}_m=G_m^*\subset R_{j_m}^*+D_m^*=R_{j_m}^*=\bar{R}_{j_m}.$$

Consequently $R_{i_m}\subset S_m\subset\bar{R}_{j_m}$, which proves (see Lemma in § 1) that $\{S_m\}$ is a C-basis of $\boldsymbol{A}$.

By (b) and (c), $\dim[\mathrm{Fr}(G_m)]\cdot\boldsymbol{A}/\boldsymbol{I}\leqslant n-1$. By CA 9.5 (ii), the C-algebra $[\mathrm{Fr}(G_m)]\cdot\boldsymbol{A}/\boldsymbol{I}$ is homeomorphic to $\mathrm{Fr}(G_m)\boldsymbol{A}/\mathrm{Fr}(G_m)\boldsymbol{I}$. Hence $\dim\mathrm{Fr}(G_m)\boldsymbol{A}/\mathrm{Fr}(G_m)\boldsymbol{I}\leqslant n-1$ and, by the inductive hypothesis,

$$\dim(\mathrm{Fr}(G_m)\boldsymbol{A}, \boldsymbol{I})=\dim(\mathrm{Fr}(G_m)\boldsymbol{A}, \mathrm{Fr}(G_m)\boldsymbol{I})\leqslant n-1.$$

Since $\mathrm{Fr}(S_m)\subset\mathrm{Fr}(G_m)$, we obtain by 2.3

$$\dim(\mathrm{Fr}(S_m)\boldsymbol{A}, \boldsymbol{I})\leqslant n-1,$$

and consequently $\dim(\boldsymbol{A}, \boldsymbol{I})\leqslant n$ on account of 2.6.

(B) Now consider the case in which $\boldsymbol{I}$ is an arbitrary σ-ideal. Let $E=|\boldsymbol{A}|^*$ (i. e. E is the complement of the sum of all open elements $G\epsilon\boldsymbol{I}$). Since $E\boldsymbol{A}/E\boldsymbol{I}$ is homeomorphic to $[E]\boldsymbol{A}/\boldsymbol{I}=\boldsymbol{A}/\boldsymbol{I}$ (see CA 9.5 (ii)), we have $\dim E\boldsymbol{A}/E\boldsymbol{I}\leqslant n$. Since $E\boldsymbol{I}$ is a boundary ideal of $E\boldsymbol{A}$ by CA 8.2, we may apply the proved part (A). Consequently $\dim(E\boldsymbol{A}, \boldsymbol{I})=\dim(E\boldsymbol{A}, E\boldsymbol{I})\leqslant n$. Since $E'\epsilon\boldsymbol{I}$, we obtain from 2.5 that $\dim(\boldsymbol{A}, \boldsymbol{I})\leqslant n$.

It follows directly from 2.7, 2.8 and 2.2 that

2.9. $\dim\boldsymbol{A}/\boldsymbol{I}=\dim(\boldsymbol{A}, \boldsymbol{I})\leqslant\dim\boldsymbol{A}$.

More generally,

2.10. $\dim[E]\cdot\boldsymbol{A}/\boldsymbol{I}=\dim(E\boldsymbol{A}, \boldsymbol{I})\leqslant\dim E\boldsymbol{A}$.

Theorem 2.10 follows from 2.9 and CA 9.5 (iii).

[9]) A^* is the complement of the sum of all open $G\epsilon A$ such that $GA\epsilon I$. See CA 7, p. 178.

One word was omitted in the formulation of CA 8.1 (iii). The correct formulation of CA 8.1 (iii) is: $G^*=\bar{G}$ *for every open element* G.

Notice that if I_0 is the σ-ideal generated by all closed elements in I (i. e. $A \in I_0$ if and only if $A \subset B \in I \cdot \mathfrak{F}_\sigma(A)$), then $\dim(A, I) = \dim(A, I_0)$ and consequently $\dim A/I = \dim A/I_0$. This remark follows immediatley from the definition and from 1.10.

Now we establish the connection between $\dim(A, I)$ and Hurewicz's „Normalbereich"[10]).

Let $A = \mathfrak{S}(\mathfrak{X})$ where $\mathfrak{X}$ is a separable metric space. Every σ-ideal I of $\mathfrak{S}(\mathfrak{X})$ is Hurewicz's „Normalbereich" and $\dim(\mathfrak{S}(\mathfrak{X}), I) \leqslant 0$ if and only if $\mathfrak{X}$ is „total discontinuous with respect to I" in Hurewicz's terminology. Conversely, if N is Hurewicz's „Normalbereich" of subsets of $\mathfrak{X}$, let I be the σ-ideal generated by closed sets in N. Then $\mathfrak{X}$ is „total discontinuous with respect to N" if and only if $\dim(\mathfrak{S}(\mathfrak{X}), I) \leqslant 0$. The easy proof is left to the reader.

§ 3. Representation theorems. Lebesgue's definition.

The representation theorem CA 15.1 can be formulated for n-dimensional C-algebras in the following form:

3.1. *For every n-dimensional C-algebra A there exist an n-dimensional separable metric space $\mathfrak{X}$ and a σ-ideal I of subsets of $\mathfrak{X}$ such that A is weakly homeomorphic to the C-algebra $\mathfrak{S}(\mathfrak{X})/I$ (i. e. $\mathfrak{B}(A)$ is homeomorphic to $\mathfrak{B}(\mathfrak{X})/I_0$, where $I_0 = I \cdot \mathfrak{B}(\mathfrak{X})$ is a σ-ideal of $\mathfrak{B}(\mathfrak{X})$)*[11]).

By CA 15.2, A is weakly homeomorphic to $\mathfrak{S}(\mathcal{H})/J$, where $\mathcal{H}$ is the Hilbert cube and J is a suitable σ-ideal. Hence, by 1.1, $\dim \mathfrak{S}(\mathcal{H})/J = n$ and consequently $\dim(\mathfrak{S}(\mathcal{H}), J) = n$ by 2.9. By 2.1, there is a set $\mathfrak{X} \subset \mathcal{H}$ such that $\mathcal{H} - \mathfrak{X} \in J$ and $\dim \mathfrak{X} = \dim \mathfrak{S}(\mathfrak{X}) = \dim \mathfrak{X} \cdot \mathfrak{S}(\mathcal{H}) = n$. Let $I = \mathfrak{X}J = J \cdot \mathfrak{S}(\mathfrak{X})$. The C-algebra $\mathfrak{S}(\mathfrak{X})/I$ is homeomorphic to $\mathfrak{S}(\mathcal{H})/J$, thus it is weakly homeomorphic to A.

It is known that the $(2n+1)$-dimensional Euclidean space contains an n-dimensional compact set $\mathfrak{U}_n$ which is universal for the class of all n-dimensional separable metric spaces, i. e. each such space is homeomorphic to a subset of $\mathfrak{U}_n$. The space $\mathfrak{U}_n$ is also universal for all n-dimensional C-algebras:

3.2. *Every n-dimensional C-algebra A is weakly homeomorphic to the C-algebra $\mathfrak{S}(\mathfrak{U}_n)/J$, where J is a suitable σ-ideal of $\mathfrak{S}(\mathfrak{U}_n)$ (i. e. $\mathfrak{B}(A)$ is homeomorphic to $\mathfrak{B}(\mathfrak{U}_n)/J_0$, where $J_0 = J \cdot \mathfrak{B}(\mathfrak{U}_n)$ is a σ-ideal of $\mathfrak{B}(\mathfrak{U}_n)$).*

[10]) Hurewicz [1], p. 754. See also Kuratowski [4], p. 187-188.

[11]) See CA 9.5 (iii).

Let $\mathfrak{X}$ and J have the same meanings as in 3.1. We may suppose $\mathfrak{X} \subset \mathfrak{U}_n$. Let J be the σ-ideal of all sets $X \subset \mathfrak{U}_n$ such that $X\mathfrak{X} \in I$. Since $I = \mathfrak{X}J$ and $\mathfrak{U}_n - \mathfrak{X} \in J$, the C-algebra $\mathfrak{S}(\mathfrak{U}_n)/J$ is homeomorphic to $\mathfrak{S}(\mathfrak{X})/I$, which proves 3.2.

Let $\mathfrak{X}$ be a separable metric space and let I be a σ-ideal of $\mathfrak{S}(\mathfrak{X})$. A finite sequence $\alpha = (G_1, \dots G_m)$ is said to be an *I-covering*[1]) of $\mathfrak{X}$ provided that:

(a) all sets G_i are open in $\mathfrak{X}$;

(b) $\mathfrak{X} - \sum_{i=1}^{m} G_i \in I$.

The space $\mathfrak{X}$ is said to *have the property* D_n^* (with respect to the ideal I) if, for every I-covering α, there is an I-covering $\beta = (H_1, H_2, \dots, H_k)$ such that

(i) β is a refinement of α (i. e. each H_i is entirely contained in some G_j);

(ii) $H_{i_0} \cdot \ldots \cdot H_{i_{n+1}} = 0$ for every sequence $i_0 < i_1 < \ldots < i_{n+1}$.

3.3. $\dim(\mathfrak{S}(\mathfrak{X}), I) \leqslant n$ *if and only if $\mathfrak{X}$ has the property* D_n^* *with respect to I.*

In the case where I contains only the empty set, theorem 3.3 is the well known theorem on the equivalence of Brouwer's definition of dimension with that of Lebesgue. The proof of 3.3 is a slight modification of this equivalence by the method of imbedding in the $(2n+1)$-dimensional Euclidean cube[12]). One proves by use of Baire's theorem on complete spaces that the condition D_n^* implies the existence of a homeomorphism φ of a subset $X \subset \mathfrak{X}$ ($\mathfrak{X} - X \in I$) into the set P_n of all points in the $(2n+1)$-dimensional Euclidean cube which have at most n rational coordinates. The space of continuous mappings must be, however, somewhat differently defined.

The idea of the proof is as follows:

Let $\mathfrak{Y}$ be a bounded metric space.

We shall consider the class Φ of all continuous mappings φ satisfying the following conditions:

(c) φ is defined on a set $X(\varphi) \subset \mathfrak{X}$ such that $\mathfrak{X} - X(\varphi) \in I$;

(d) values of φ are in $\mathfrak{Y}$.

Two mappings $\varphi, \psi \in \Phi$ are said to be *equivalent* provided they coincide on a set $Z \subset X(\varphi) \cdot X(\psi)$ such that $\mathfrak{X} - Z \in I$.

[12]) This method is due to Hurewicz and Kuratowski. The proof outlined below is similar to that in Hurewicz-Wallman [2], pp. 60-66.

The class of all mappings $\varphi' \in \Phi$ which are equivalent with a mapping $\varphi \in \Phi$ will be denoted by φ^*. The class Φ^* of all φ^*, where $\varphi \in \Phi$, is a metric space with the following definition of distance[13]):

$$d(\varphi^*, \psi^*) = \inf_X \sup_{x \in X} d(\varphi(x), \psi(x)),$$

where $d(p,q)$ is the distance between points p,q in $\mathcal{Y}$, and where X is an arbitrary set such that $X \subset X(\varphi)\cdot X(\psi)$ and $\mathfrak{X}-X \in \boldsymbol{I}$.

Lemma A). If $\mathcal{Y}$ is complete, then the space Φ^* is also complete.

Let $\alpha=(G_1,\dots,G_m)$ be an $\boldsymbol{I}$-covering of $\mathfrak{X}$ and let $\varphi \in \Phi$. The symbol φ_α will denote the mapping φ restricted to $x \in X(\varphi)\cdot\sum_{i=1}^m G_i$. A mapping $\varphi \in \Phi$ is called an *α-mapping* if every point $y \in \mathcal{Y}$ has a neighbourhood $V \subset \mathcal{Y}$ such that $\varphi_\alpha^{-1}(V)$ is entirely contained in some set G_i.

Lemma B). If $\mathcal{Y}$ is compact, the set of all $\varphi^* \in \Phi^*$, where φ is an α-mapping, is open in Φ^*.

Lemma C). Let $\mathcal{Y}$ be compact and $M=\overline{M}\subset\mathcal{Y}$. The set of all $\varphi^* \in \Phi^*$ such that[14]) $\varphi(\mathfrak{X})\cdot M=0$ is open in Φ^*.

Lemma D). *Let* $\mathcal{Y}$ be the $(2n+1)$-dimensional Euclidean cube and let M be the intersection of $\mathcal{Y}$ with an n-dimensional linear subspace. If $\mathfrak{X}$ has the property D_n^*, then, for every $\boldsymbol{I}$-covering α, the set of all $\varphi^* \in \Phi^*$, where φ is an α-mapping and $\varphi(X)\cdot M=0$, is a dense open subset in Φ^*.

Lemma E). Let $\mathcal{Y}$ be the $(2n+1)$-dimensional Euclidean cube and let $\mathfrak{X}$ have the property D_n^*. Then there exists a homeomorphism $\varphi \in \Phi$ such that $\varphi(\mathfrak{X})\subset P_n$.

Lemma E) implies immediately that if $\mathfrak{X}$ has the property D_n^*, then $\dim(\mathfrak{S}(\mathfrak{X}),\boldsymbol{I})\leqslant n$.

Suppose conversely that $\dim(\mathfrak{S}(\mathfrak{X}),\boldsymbol{I})\leqslant n$, and let $\alpha=(G_1,\dots,G_m)$ be an $\boldsymbol{I}$-covering of $\mathfrak{X}$. By 2.1 there is a set $X\subset\mathfrak{X}$ such that $\dim X\leqslant n$ and $\mathfrak{X}-X\in\boldsymbol{I}$. We may assume that $X\subset\sum_{i=1}^m G_i$. Consequently there are open sets $H_1,\dots,H_k$ such that $X\subset\sum_{i=1}^k H_i$, $H_i\subset G_i$ and the condition (ii) is satisfied[15]). The space $\mathfrak{X}$ possesses the property D_n^*.

A sequence $\alpha=(A_1,\dots,A_m)$ is said to be a *covering* of a C-algebra $\boldsymbol{A}$ if all A_i are open and $|\boldsymbol{A}|=\sum_{i=1}^m A_i$. A C-algebra $\boldsymbol{A}$ is said to *have the property* D_n if, for every covering $\alpha=(A_1,\dots,A_m)$, there is a covering $\beta=(B_1,\dots,B_k)$ which is a refinement of α (i. e. each B_i is contained in some A_j), and such that $B_{i_0}\cdot\ldots\cdot B_{i_{n+1}}=0$ for every sequence $i_0<i_1<\dots<i_{n+1}$.

[13]) Φ^* may be interpreted as the space of all continuous homomorphisms of $\mathfrak{S}(\mathcal{Y})$ into $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$. See CA 21.1.

[14]) Clearly $\varphi(\mathfrak{X})$ is the image of the set $X(\varphi)$.

[15]) See Kuratowski [4], p. 184.

3.4. *Let $\mathfrak{X}$ be a separable metric space, and let $\boldsymbol{I}$ be a σ-ideal of $\mathfrak{S}(\mathfrak{X})$. The C-algebra $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ has the property D_n if and only if $\mathfrak{X}$ has the property D_n^* with respect to $\boldsymbol{I}$.*

Suppose $\mathfrak{X}$ has the property D_n^*, and let $(A_1,\dots,A_m)$ be a covering of $\boldsymbol{A}$. We have $A_i=[G_i]$ where G_i is open in $\mathfrak{X}$. The sequence $(G_1,\dots,G_m)$ is an $\boldsymbol{I}$-covering of $\mathfrak{X}$. Let $(H_1,\dots,H_k)$ be an $\boldsymbol{I}$-covering of $\mathfrak{X}$ satisfying the conditions (i) and (ii). Consequently $B_i=[H_i]$ $(i=1,\dots,k)$ is a refinement of $(A_1,\dots,A_m)$ and $B_{i_0}\cdot\ldots\cdot B_{i_{n+1}}=0$. Therefore $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ has the property D_n.

Suppose now that $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ has the property D_n and let $\alpha=(G_1,\dots,G_m)$ be an $\boldsymbol{I}$-covering of $\mathfrak{X}$. Then $([G_1],\dots,[G_m])$ is a covering of $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ which has a refinement $(B_1,\dots,B_k)$ $(B_j\subset[G_{i(j)}])$ such that $B_{i_0}\cdot\ldots\cdot B_{i_{n+1}}=0$ whenever $i_0<i_1<\dots<i_{n+1}$. We have $B_j=[U_j]$ where U_j is open in $\mathfrak{X}$. Let $Q=\sum_{j=1}^k U_j-\sum(U_{i_0}\cdot\ldots\cdot U_{i_{n+1}})$, the last sign Σ being extended over all increasing sequences $i_0<\dots<i_{n+1}$. Clearly $\mathfrak{X}-Q\in\boldsymbol{I}$ and $Q(U_{i_0}\cdot\ldots\cdot U_{i_{n+1}})=0$. There are open sets[16]) V_j $(j=1,\dots,k)$ such that $V_jQ=U_j$ and $V_{i_0}\cdot\ldots\cdot V_{i_{n+1}}=0$ for each sequence $i_0<i_1<\dots<i_{n+1}$. Let $H_j=V_jG_{i(j)}$. The $\boldsymbol{I}$-covering $\beta=(H_1,\dots,H_k)$ satisfies the conditions (i) and (ii). This proves that $\mathfrak{X}$ has the property D_n^*.

3.5. *For every C-algebra $\boldsymbol{A}$, $\dim\boldsymbol{A}\leqslant n$ if and only if $\boldsymbol{A}$ has the property D_n.*

Theorem 3.5 follows immediately from 2.9, 3.1, 3.3 and 3.4.

§ 4. Ideals $\boldsymbol{A}^k$.

Let $A\in\boldsymbol{A}$. The symbol $\mathrm{Dim}(A,\boldsymbol{A})$ will denote the least integer $m\geqslant-1$ with the property: there is an element $F\in\mathfrak{F}_\sigma(\boldsymbol{A})$ such that $A\subset F$ and $\dim F\boldsymbol{A}=m$.

In this section the letter $\boldsymbol{A}$ will denote a fixed C-algebra of finite dimension. For brevity, we shall write „$\mathrm{Dim}\,A$" instead of „$\mathrm{Dim}(A,\boldsymbol{A})$".

The following lemmas are obvious.

4.1. *If* $A\in\mathfrak{F}_\sigma(\boldsymbol{A})$, *then* $\mathrm{Dim}\,A=\dim A\boldsymbol{A}$. *In particular,* $\mathrm{Dim}\,A=-1$ *if and only if* $A=0$.

4.2. $\mathrm{Dim}\,|\boldsymbol{A}|=\dim\boldsymbol{A}$.

4.3. *If* $A\subset B$, *then* $\mathrm{Dim}\,A\leqslant\mathrm{Dim}\,B$.

[16]) See Kuratowski [4], p. 122.

4.4. $\dim AA \leqslant \operatorname{Dim} A \leqslant \dim A$.

4.5. *Let* $A \subset E$. *Then* $\operatorname{Dim}(A, EA) \leqslant \operatorname{Dim} A$. *If* $E \in \mathfrak{F}_\sigma(A)$, *then* $\operatorname{Dim}(A, EA) = \operatorname{Dim} A$.

4.6. *If* $\operatorname{Dim} A_i < k$ *for* $i = 1, 2, \ldots$, *then* $\operatorname{Dim} \sum_{i=1}^{\infty} A_i < k$.

Let k be a non-negative integer. The σ-ideal (see 4.6) of all $A \in A$ such that $\operatorname{Dim} A < k$ will be denoted by A^k. Analogously $(EA)^k$ will denote the σ-ideal of all $A \in EA$ (i. e. $A \subset E$) such that $\operatorname{Dim}(A, EA) < k$. If $E \in \mathfrak{F}_\sigma(A)$, then $(EA)^k = EA^k$ by 4.5.

A sequence $N_0, N_1, \ldots, N_n \in A$ is said to be a *normal decomposition* of A provided that:

(a) $|A| = N_0 + N_1 + \ldots + N_n$;

(b) $N_0 + N_1 + \ldots + N_i \in \mathfrak{F}_\sigma(A)$ f r $i = 0, 1, \ldots, n$;

(c) $\dim N_i A = 0$ for $i = 0, 1, \ldots, n$;

(d) $\operatorname{Dim} N_i = i$ for $i = 0, 1, \ldots, n$.

The existence of a normal decomposition (a) implies $\dim A = n$. The converse statement is also true and will be proved later (4.8).

4.7. *Let* $N_0, \ldots, N_n$ *be a normal decomposition of* A, *let* $0 \leqslant i_0 < i_1 < i_2 < \ldots < i_r \leqslant n$, *and let* $E = N_{i_0} + N_{i_1} + \ldots + N_{i_r}$. *Then*

(i) $\operatorname{Dim} E = i_r$;

(ii) $\dim EA = r$;

(iii) *if* $i_{s-1} < k \leqslant i_s$, *then* $\dim(EA, A^k) \leqslant r - s$ [17]).

The property (i) follows from (d), 4.6 and 4.3.

It follows from 1.5 (c) that $\dim EA \leqslant r$ and $\dim E'A \leqslant n - r - 1$. Since $\dim A = n$, the equation (ii) holds (see 1.9).

We have $A = N_{i_0} + \ldots + N_{i_{s-1}} \in A^k$ and $\dim A'EA \leqslant r - s$ since $A'E$ is the sum of $r - s + 1$ null-dimensional elements. This proves (iii).

4.8. $\dim A = n$ *if and only if there is a normal decomposition* $N_0, \ldots, N_n$ *of* A.

Only the existence of a normal decomposition should be proved. The proof of this fact is by induction on $n = \dim A$.

The case $n = 0$ is trivial. Let $\dim A = n > 0$. By 1.4 and 1.5 (b), $|A| = N_n + B$, where $\dim N_n A = 0$, $B \in \mathfrak{F}_\sigma(A)$ and $\dim BA \leqslant n - 1$. By the induction hypothesis, BA has a normal decomposition $N_0, N_1, \ldots, N_{n-1}$. The sequence $N_0, N_1, \ldots, N_n$ is a normal decomposition of A.

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[17]) It follows from 4.16 that $\dim(EA, A^k) = r - s$.

4.9. *If* $\operatorname{Dim} E < k$ $(0 < k \leqslant n = \dim A)$, *then there is a normal decomposition* $N_0, \ldots, N_n$ *of* A *such that* $E \subset N_0 + \ldots + N_{k-1}$.

Let $M_0, \ldots, M_n$ be a normal decomposition of A, and let $F \in \mathfrak{F}_\sigma(A)$, $E \subset F$ and $\dim FA < k$. Let $A = F + M_0 + \ldots + M_{k-1}$. By 4.6 and 4.7, we have $\dim AA = k - 1$. Hence there is a normal decomposition $N_0, \ldots, N_{k-1}$ of AA. Let $N_j = M_j$ for $j = k, \ldots, n$. The sequence $N_0, \ldots, N_n$ is a normal decomposition of A since $A \in \mathfrak{F}_\sigma(A)$ (see (b)). The easy proof is omitted.

4.10 [18]). *If* $\operatorname{Dim} E \geqslant k$; *then* $\dim(EA, A^k) \leqslant \operatorname{Dim} E - k$.

Let $p = \operatorname{Dim} E$. By 4.9 there is a normal decomposition $N_0, \ldots, N_n$ of A (where $n = \dim A$) such that $E \subset N_0 + \ldots + N_p$. Hence, by 4.7 (iii),

$$\dim(EA, A^k) \leqslant \dim\big((N_0 + \ldots + N_p)A, A_k\big) \leqslant \dim(N_k + \ldots + N_p)A \leqslant p - k.$$

4.11. $\dim(EA, A^k) \geqslant \dim EA - k$.

Let $A \in A^k$ be such an element that $\dim(EA, A^k) = \dim A'EA$ (see 2.1). We have $\dim AEA \leqslant k - 1$ since $EA \in A^k$. Consequently, by 1.9,

$$\dim EA \leqslant \dim A'EA + \dim AEA + 1 \leqslant \dim(EA, A^k) + (k-1) + 1,$$

which proves 4.11.

4.12 [18]). *If* $\operatorname{Dim} E \geqslant k$, *then*

$$\max(0, \dim EA - k) \leqslant \dim(EA, A^k) \leqslant \min(\dim EA, \operatorname{Dim} E - k).$$

This follows from 4.10, 4.11 and 2.2.

Theorem 4.12 may be otherwise formulated as follows (see 2.9):

4.12′. *If* $\operatorname{Dim} E \geqslant k$, *then*

$$\max(0, \dim EA - k) \leqslant \dim [E] \cdot A / A^k \leqslant \min(\dim EA, \operatorname{Dim} E - k).$$

In particular, since $\operatorname{Dim} |A| = \dim A$,

4.13. $\dim A / A^k = \dim(A, A^k) = \dim A - k$.

More generally, by 4.1, 4.12 and 2.10,

4.14. *If* $E \in \mathfrak{F}_\sigma(A)$ *and* $\operatorname{Dim} E \geqslant k$, *then*

$$\dim [E] \cdot A / A^k = \dim EA - k.$$

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[18]) If $\operatorname{Dim} E < k$, then obviously $\dim(EA, A^k) = -1$. This case is not interesting.

Consequently, by CA 11.4,

4.15 [19]). *If* $\mathrm{Dim}\,E \geqslant k$, *then* $\mathrm{Dim}([E], \boldsymbol{A}/\boldsymbol{A}^k) = \mathrm{Dim}\,E - k$.

Theorems 4.15, 2.10, and CA 11.4 imply

4.16. *If* $N_0,\dots,N_n$ *is a normal decomposition of* $\boldsymbol{A}$ *(where* $n=\dim \boldsymbol{A}$*), then* $[N_k],\dots,[N_n]$ *is a normal decomposition of* $\boldsymbol{A}/\boldsymbol{A}^k$.

The evaluation given in 4.12 and 4.12′ is exact. In fact,

4.17. *If the integers* l, l', L *satisfy the inequalities*

$$l \leqslant L \leqslant n = \dim \boldsymbol{A}, \quad L \geqslant k, \quad \max(0, l-k) \leqslant l' \leqslant \min(l, L-k)$$

(where $k \leqslant n$*), then there is an element* $E \in \boldsymbol{A}$ *such that*

$$\dim E\boldsymbol{A} = l, \quad \mathrm{Dim}\,E = L \quad \text{and} \quad \dim(E\boldsymbol{A}, \boldsymbol{A}^k) = \dim [E]\cdot \boldsymbol{A}/\boldsymbol{A}^k = l'.$$

We have $l' \geqslant 0$ and $0 \leqslant l-l' \leqslant k \leqslant L-l' \leqslant L \leqslant n$.

Let $N_0,\dots,N_n$ be a normal decomposition of $\boldsymbol{A}$, and let $E_1 = N_{L-l'} + \dots + N_L$. Consequently $[E_1] = [N_{L-l'}] + \dots + [N_L] \in \boldsymbol{A}/\boldsymbol{A}^k$. By 4.16 and 4.7 (ii), $\dim [E_1]\cdot \boldsymbol{A}/\boldsymbol{A}^k = L-(L-l') = l'$.

If $l=l'$, let $E_2 = 0$; if $l > l'$, let $E_2 = N_0 + \dots + N_{l-l'-1}$.

The element $E = E_1 + E_2$ is the required one. In fact, it follows from 4.7 (i) and (ii) that $\mathrm{Dim}\,E = L$ and $\dim E\boldsymbol{A} = l$. Since $[E]=[E_1]$, we have $\dim [E]\cdot \boldsymbol{A}/\boldsymbol{A}^k = \dim [E_1]\cdot \boldsymbol{A}/\boldsymbol{A}^k = l'$, q. e. d.

4.18. $(\boldsymbol{A}/\boldsymbol{A}^k)/(\boldsymbol{A}/\boldsymbol{A}^k)^l$ *is homeomorphic to* $\boldsymbol{A}/\boldsymbol{A}^{k+l}$.

By CA 9.7, the C-algebra $(\boldsymbol{A}/\boldsymbol{A}^k)/(\boldsymbol{A}/\boldsymbol{A}^k)^l$ is homeomorphic to $\boldsymbol{A}/\boldsymbol{I}$ where $\boldsymbol{I}$ is the σ-ideal of all $A \in \boldsymbol{A}$ such that $\mathrm{Dim}([A], \boldsymbol{A}/\boldsymbol{A}^k) < l$. By 4.15, $\boldsymbol{I} = \boldsymbol{A}^{k+l}$, q. e. d.

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[19]) If $\mathrm{Dim}\,E < k$, then obviously $\mathrm{Dim}([E], \boldsymbol{A}/\boldsymbol{A}^k) = -1$.

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On Generalized Spheres.

By

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1. Let A_0 be a non-empty subset of a space [1]) A and r a positive number. By a *generalized sphere* with centre A_0 and radius r we understand the set

$$K_r(A_0, A) = \underset{x\in A}{E}[\varrho(x, A_0) \leqslant r]. \tag{1}$$

Frequently the topological structure of the generalized sphere is more simple than the topological structure of the set A_0.

For instance if A_0 is a compact subset of the Euclidean 1-dimensional space E_1, then every generalized sphere is a sum of a finite number of segments.

It follows by (1): If A is a convex space [2]) and $0 < r' < r$ then

$$K_r(A_0, A) = K_{r'}[K_{r-r'}(A_0, A), A]. \tag{2}$$

2. *Lemma.* *If* A_0 *is a compact subset of the Euclidean n-dimensional space* E_n *and* r *is a positive number, then for every* $a_0 \in K_r(A_0, E_n)$ *there exists a connected set* N *with diameter* $\delta(N) \leqslant 8r$, *constituting a neighbourhood of* a_0 *in* $K_r(A_0, E_n)$.

Proof. Let us put

$$M = \underset{x}{E}[x \in K_r(a, E_n), a \in A_0, \varrho(a, a_0) \leqslant r],$$

$$N = \underset{x}{E}[x \in K_r(a, E_n), a \in A_0, M\cdot K_r(a, E_n) \neq 0].$$

Evidently N is a connected subset of $K_r(A_0, E_n)$ and $\delta(N) \leqslant 8r$. It remains to be proved that N constitutes a neighbourhood of a_0 in $K_r(A_0, E_n)$.

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[1]) By *space* we always understand here a metric space.

[2]) A is *convex* if for every two points $a, b \in A$ and every positive number $0 < \alpha < \varrho(a, b)$ there exists a point $x \in A$ such that $\alpha = \varrho(a, x) = \varrho(a, b) - \varrho(b, x)$.