A Characterization of $L$ Spaces$^1$.

by

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1. Introduction. Kakutani [3] has characterized a space of integrable functions as a Banach lattice satisfying the following three conditions:

(1) there exists a unit element $e > 0$ such that $x > 0$ implies $e \wedge x > 0$;

(2) $x > 0$, $y > 0$ imply $||x + y|| = ||x|| + ||y||$;

(3) $x \wedge y = 0$ implies $||x - y|| = ||x + y||$.

The set of points, $\Omega$, over which the $L$ space is defined can be assumed to have measure 1. Kakutani [3] has also given a similar type of characterization for Banach lattices of functions continuous over a bicomplete Hausdorff space. More recently, Clarkson [1] has characterized a Banach space of continuous functions in terms of the shape of the unit sphere. In this characterization an order relation is introduced by means of a certain type of cone used in the construction of the unit sphere, and under this ordering the space is shown to be an $M$ space and hence equivalent to a space of continuous functions. In this paper spaces of integrable functions will be characterized by the shape of their unit spheres, making use of methods similar to those of Clarkson. The Borel field of measurable subsets of the space $\Omega$ will be shown to correspond to the family of maximal convex subsets of the unit sphere in a manner similar to the role played by this family in the case of a space of continuous functions as investigated by Eilenberg [4].

The case in which the measure is completely atomic is of particular interest and will be treated in more detail.

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2. The Characterization Theorem. A cone in a linear space $X$ is a set $C$ containing a point $v$ such that if $w \in C$ and $t$ is any non-negative real number, then $v + tw \in C$. A cone will be called a $C$-cone if for any two points $x, y$ of $X$ there exists a point $z \in C$ such that $(z + C) \cap (y + C) = z + C$. This type of cone was used by Clarkson in his paper and has received more recent attention in a paper by Krein and Rutman [5], who have called it a minihedral cone. As they have pointed out, Clarkson's theorem shows that a necessary and sufficient condition that a Banach space $X$ be isomorphic to a space of real-valued functions continuous over a bi-compact set $S$ is that $X$ contain a proper $C$-cone with an interior point.

**Theorem 1.** Necessary and sufficient conditions that a Banach space $X$ be equivalent to a space $L(Q, m)$ of all real-valued functions defined on a set $Q$, integrable with respect to a completely additive measure $m$ with $m(Q) = 1$, are:

1. There exists on the surface of the unit sphere a maximal closed convex set $F$ such that the cone $C = (F + x, x \in C, x = 0)$ is a $C$-cone;

2. The unit sphere consists of all points of the form $x + y$, $x \in C, y \in C$;

3. $F$ contains an element $e$ such that if $x \in C$, $x \neq 0$, there exists $y + e + e$ such that $x + e \in C$.

**Proof.** Let $L(Q, m)$ be the space described above. If $x, y \in L(Q, m)$, $x \geqslant y$ means $x(t) \geqslant y(t)$ almost everywhere. The norm is defined in the usual manner:

$$
\|x\| = \int |x(t)| \, dm(t).
$$

Let $F$ be the set of all essentially non-negative functions of unit norm. Evidently $F$ is both closed and convex. To show that $F$ is a maximal convex subset of the surface of the unit sphere, suppose $F \subset F'$, where $F'$ is a convex set of elements of unit norm. Suppose $x' \in F'$ but $x' \notin F$. Then there exists a measurable subset $C$ of $Q$ such that $x'(t) < 0$ for $t \in C$. Let $e$ be the set of all points for which $x'(t) < 0$, and consider $\bar{C}$, the complement of $e$ in $Q$. Let $x' \in F'$, $\frac{x' + x}{2} \in F'$ and

$$
\|x + x'\| = \frac{1}{2} \int |x'(t) + x(t)| \, dm(t).
$$

Let $x \in F$, $\frac{x + x'}{2} \in F$ and

$$
1 = \frac{1}{2} \int |x'(t) + x(t)| \, dm(t) = \frac{1}{2} \int |x'(t) + x(t)| \, dm(t)
$$

This contradiction shows that $F' = F$, and $F$ is maximal.

Let $Q(\theta, x) = \{x = ax + b, x \in Q, x \neq 0\}$. This is a cone with vertex $\theta$ generated by the set $F$. Let $y_1, y_2, y_3 \in L(Q, m)$. Then $y_1 + C, y_2 + C$ are evidently cones with vertices $y_1$ and $y_2$, respectively, and consist of all elements of $L(Q, m)$ not less than $y_1$ and $y_2$, respectively. If $y_1 \cup y_2 = \max(y_1, y_2)$, then $(y_1 + C) \cap (y_2 + C)$ is the set of all elements not less than $y_1$ and $y_2$, and $y_1 + C \cap (y_2 + C) = (y_1 \cup y_2) + C$. Thus $C$ is a $C$-cone.

To prove (2), assume $|x| = 1$, $x = a_1 + a_2$, where $a_1 = x \cap \theta$, $a_2 = x \setminus \theta$. Then $|x| = |a_1| + |a_2|$. Let $y$ be any non-negative element with $|y| = 1 - |x|$. Let $s_1 = a_1 + \frac{y}{2}$, $s_2 = a_2 - \frac{y}{2}$. Then

$$
|s_1| + |s_2| = |a_1| + |a_2| - |y| = 1 - |x| = 1.
$$

Thus $s_1 + s_2 = a_1 + a_2 = x$. Since $\frac{s_1}{|s_1|} + \frac{s_2}{|s_2|} = \frac{x}{|x|}$, (2) is true.

If $e(t)$ is almost everywhere positive on $Q$, (3) is obvious if we let $y = e \wedge |x| > 0$.

In the proof of the sufficiency it will be shown that if (1), (2), and (3), hold, it is possible to introduce an order relation in the space under which the space becomes an abstract $L$ space with a unit and by Kakutani's theorem, is equivalent to a space $L(Q, m)$ as described.

An abstract $L$ space is a Banach space with an order relation satisfying the following postulates:

1. $x \geqslant y, y \geqslant x$, imply $y = x$;
2. $x \geqslant y, y \geqslant x$, imply $x = y$.

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Let $w_n = x_n - y_n$, $w = x - y$. If $w = 0$, $K_5$ is obvious. If $w + \delta$, we may assume without loss of generality that $|w|=1$. It must be shown that $w \in F$. Since $\lim |w_n|=1$, it may be assumed that $||w_n||$ is bounded away from zero. Let $w'_n = \frac{w_n}{||w_n||}$. Then $w'_n \in F$.

$$||w'_n - w|| \leq ||w'_n - w_n|| + ||w_n - w|| = \left|\frac{w_n}{||w_n||} - w_n\right| + ||w_n - w|| = \frac{1}{||w_n||} - 1 - ||w_n - w|| = 1 - ||w_n|| - ||w_n - w|| \to 0.$$  

Hence $\lim ||w'_n|| = ||w||$. Since $w'_n \in F$ and $F$ is closed, $w \in F$.

$K_6$. Let $(a + \delta) \cap (y + \delta) = z + \delta$. Then $z \geq a$, $z \geq y$, and if $w \geq a$, $w \geq y$, then $w \in (a + \delta) \cap (y + \delta) = z + \delta$ and $w \in F$. Thus $\chi \vee y = z$.

$K_7$ is dual to $K_6$ where the infimum is defined by $x \wedge y = -(z \vee (-y))$.

$K_8$. Assume $y \geq 0$. Then $x = x \cdot x_1$, $y = y \cdot y_1$, where $x_1 \in F$, $y_1 \in F$.

$$||x + y|| = ||x|| \cdot ||y|| \cdot ||x_1 + y_1|| = ||x|| \cdot ||y|| \left(\frac{||x||}{||x||} + \frac{||y||}{||y||}\right) = ||x|| + ||y||$$  

for, since $F$ is convex, the second quantity of the third identity is equal to 1.

$K_9$. $x \wedge y = 0$ means that $x \in \text{C}$, $y \in \text{C}$, but $x \not< z + \delta$, $y \not< z + \delta$ for any $z \in \text{C}$, $z \not= 0$. Assume first that $|x| = |y| = 1$. Let

$$L = \{ax + by : a + b = 1\}$$  

and let $L' = L - x$. $L'$ is a line in $\mathbf{X}$ passing through the origin and a translate of $z$. Let $u$ and $-u$ be the points of $L'$ for which $|u| = 1$. By hypothesis (2) there exists a line $\mathbf{M}$ through $z$, which intersects $F$ and $-F$. It is evident that $u \not\in F$. Since $\mathbf{M}$ contains three points of unit norm, the entire segment $S$ containing these three points lies on the surface of the unit sphere. Since the sphere is symmetric, $-S$ has the same properties. Let $v_1$ and $v_2$ be the points in which $\mathbf{M}$ and $-\mathbf{M}$ intersect $F$. Let $L'_1 = L' + v_1$, and $L'_2 = L' + v_2$. Since $F$ is convex, at least one of these lines will contain no points of norm less than one. Assume that $L'_2$ is this line. Let $v_3$ and $v_4$ be the points of intersection of $L'_2$ with $\mathbf{M}$ and $-\mathbf{M}$. Evidently $||v_1 - w|| = ||u + w|| = 2$. Since $|x| = |y| = 1$, $|x - y| \leq 2$.  

\[ \]
Let \( T \) be the segment \( \{a_1 t + \beta \}, a, \beta \geq 0, a + \beta = 1 \). Let
\[
T_1 = T + (x - v_1), \quad T_2 = T + (x - w).
\]

\( T_1 \) and \( T_2 \) are included in \( \overline{C} \) and of length 2 with \( v_1 \) and \( w \) translated to \( x \) respectively. Hence either \( T_1 \) or \( T_2 \) contains the segment joining \( x \) and \( y \). Assume that \( T_1 \) is this set. Then \( x \leq (x - v_1) + \overline{C} \) and \( y \leq (x - v_1) + \overline{C} \). Hence \( x \) and \( y \) are in \( (x - v_1) + \overline{C} \cap \overline{C} = x + \overline{C} \), where \( x \geq \theta \) and \( x \geq \theta \), only if \( x - v_1 \leq 0 \). If \( x = v_1 \) and \( x > 0 \), and \( 1 = |v_1 - x| + |y - x| - |x - y| + |x| > 1 \) by \( K \). Hence \( x = v_1 \) since \( x \leq \theta \) by assumption. Similarly it can be shown that \( w = w \). Hence \( |x - y| = 2 \). Since evidently \( |x + y| = 2, |x + y| = |x + y| \) and \( |x - y| = |x - y| \), the entire segments \( (x + \beta y) \) and \( (x - \beta y) \), \( a - \beta, 1 \geq 0, a + \beta = 1, y = \beta = 1 \), consist of points of unit norm.

Assume that \( x \) and \( y \) are not necessarily of unit norm and \( a \wedge y = 0 \). Then
\[
1 = \left| \frac{x}{x + y} \right| = \frac{1}{\left| x + y \right|} \left| x + y \right|
\]

and
\[
|x + y| = |x - y|.
\]

K.10 is hypothesis (3).

This shows that all the postulates for a Kakutani L space with a unit are satisfied and that the space is equivalent to a space of integrable real functions over a totally disconnected bicomplete Hausdorff space \( \Omega \), where the field of measurable subsets is the field of Boorei sets and \( m(\Omega) = 1 \).

Remark. It is to be noted that, although \( C \) is convex and closed and generates the entire space, it need not have any interior points.

3. Further properties of the unit sphere.

Theorem 2. Let \( x \) be an L space with a unit. If \( x \in \overline{C}, |x| = 1 \), there exists a maximal closed convex subset \( H \) of the surface of the unit sphere containing \( x \) such that the unit sphere consists of the closure of the convex set determined by \( H \) and \( -H \).

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Proof. Let \( x \in \mathbb{R}, |x| = 1 \). \( x \) is representable in \( L(\Omega, m) \) as an integrable real function \( \alpha(t) \) with
\[
\int_\Omega |\alpha(t)| \, d\mu(t) = 1.
\]

If \( \alpha(t) \geq 0 \) or \( \alpha(t) \leq 0 \) almost everywhere, the theorem follows from theorem 1. If \( \alpha(t) \geq 0 \) for \( t \in \Omega \), \( x(t) < 0 \) for \( t \in \Omega \), where \( m(\Omega) > 0 \), \( m(\overline{\Omega}) > 0 \), let \( H \) be the set of all functions essentially non negative on \( \Omega \) and essentially positive on \( \overline{\Omega} \) and of unit norm. By methods similar to those used in the proof of the necessity in theorem 1, \( H \) can be shown to be both convex and closed. To prove that \( H \) and \( -H \) generate the entire unit sphere let \( y(t) \) be any positive function of \( L(\Omega, m) \), \( |y| = 1 \). Let \( y(t) = y(t) \) for \( t \in \Omega \) and \( y(t) = 0 \) otherwise and let \( y(t) = y(t) \) for \( t \in \overline{\Omega} \) and \( y(t) = 0 \) elsewhere. Let
\[
z_1 = \frac{y_{1}}{y_1}, \quad z_2 = \frac{y_2}{y_2},
\]
then
\[
z_1 \in H, \quad z_2 \in -H,
\]
and
\[
|y| = |y_1| + |y_2| = 1, \quad y = y_1 \cdot z_1 + y_2 \cdot z_2,
\]

which is in the closed convex set determined by \( H \) and \( -H \).

Similarly all negative functions of unit norm are in the set and, by theorem 1, the closed convex set determined by \( H \) and \( -H \) is the unit sphere of \( X \).

Corollary. Every maximal convex set on the surface of the unit sphere in an L space with a unit generates a C-descent with vertex \( \theta \) which satisfies the conditions of theorem 1.

4. Construction of the function space. Convergent convex subsets of the surface of the unit sphere will be called faces of the unit sphere. To each face of the unit sphere in an L space with a unit corresponds the set of integrable functions of \( L(\Omega, m) \) which are essentially non negative on a measurable subset of \( \Omega \) and essentially non positive on \( \overline{\Omega} \). In this way to each face of the unit sphere can be associated a class of measurable subsets of \( \Omega \), each differing from the other on a set of measure zero on which the elements of the face \( F \) are essentially non negative. The positive face \( F^+ \) corresponds to \( \Omega \), and the negative face \( F^- = F_0 \) to the sets of measure zero. If \( m(\Omega) = 0 \), then \( F_0 \cap F = F_0 \cap F \).
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The space $L$ is an $L$ space determined over a completely atomic measure space where $m(2)=\infty$. A completely atomic measure space is a Borel field of measurable sets such that each set of positive measure contains a subset of minimal positive measure.

**Theorem 3.** Necessary and sufficient conditions that a Banach space $X$ be equivalent to the space $L$ are that the conditions of Theorem 1 be satisfied and in addition that there exist a countable set of extreme points $\{v_i\}$, $\|v_i\|=1$, $i=1,2,3,...$, such that the unit sphere in $X$ is the closed convex set determined by the $\{\pm v_i\}$.

**Proof.** Let $\mathbb{X}=L$ and let $v_i=(0,0,...,0,1,0,...)$ with 1 in the $i$-th place and 0 elsewhere. It can easily be shown that the set $\mathcal{F}$ of theorem 1 is the convex closure of the $\{\pm v_i\}$, and hence that the conditions are necessary.

To prove the sufficiency of the conditions, suppose that the unit sphere in the space is the smallest closed convex set determined by the $\{\pm v_i\}$. Since these are extreme points, they lie in $\mathcal{F}$ or $-\mathcal{F}$. Assume that all $v_i \in \mathcal{F}$. Also, $v_i \wedge v_j = 0$ for $i \neq j$, and for any $v \in \mathcal{F}$, $\sum v_i \in \mathcal{F}$, and $v \in \mathcal{F}$.

In fact, if $\pi$ and $\pi'$ are two disjoint sets of integers and $\{\xi_1, \xi_2, \ldots, \phi_1, \phi_2, \ldots\}$ are sets of non-negative numbers, since the $\{v_i\}$ are integrable functions, it is possible to deduce from this that

$$\left(\sum_{i=1}^{\infty} v_i\right) \wedge \left(\sum_{j=1}^{\infty} v_j\right) = 0.$$

Hence by K 9

$$\left|\sum_{i=1}^{\infty} v_i - \sum_{j=1}^{\infty} v_j\right| = \sum_{i=1}^{\infty} v_i - \sum_{j=1}^{\infty} v_j.$$

Consider two disjoint sets of integers $\pi$ and $\pi'$ and let $\{v_\pi\}$, $\{\phi_\pi\}$ be finite sets of non-negative numbers, where $i \in \pi$ and $j \in \pi'$ with

$$\sum_{i \in \pi} v_i + \sum_{j \in \pi'} \phi_j = 1,$$

and if $S_{i,\pi'}$ is the closure of the convex set determined by $\{v_i\}$, $i \in \pi$; $\{-v_i\}$, $j \in \pi'$, then $S_{i,\pi'}$ consists of all elements of the form

$$\sum_{i \in \pi} v_i - \sum_{j \in \pi'} \phi_j.$$
plus all limit points of such elements. However
\[ \left\| \sum_{i \in \pi} a_i v_i - \sum_{j \in \pi'} \beta_j v_j \right\| = \left\| \sum_{i \in \pi} a_i v_i + \sum_{j \in \pi'} \beta_j v_j \right\| = 1 \]
and the same holds for the limit points. Hence each \( S_{\pi, \pi'} \) lies on the surface of the unit sphere. If \( \pi \cup \pi' \) is all the positive integers, then \( S_{\pi, \pi'} \) is a face of the unit sphere. Also every point of unit norm is in such a set, for otherwise it would be in a set \( S_{\pi, \pi'} \) with \( \pi \cap \pi' \) non empty and could be approximated by a point of the form
\[
Z = \sum_{i \in \pi} \alpha_i v_i - \sum_{j \in \pi'} \beta_j v_j
= \sum_{i \in \pi} \alpha_i v_i + \sum_{i \in \pi'} (a_i - \beta_i) v_i - \sum_{i \in \pi} \beta_i v_i
\]
with \( (\alpha_i), (\beta_i) \), finite sets of non negative numbers such that
\[ \sum_i \alpha_i + \sum_j \beta_j = 1. \]

By properties K8 and K9 it can be seen that \( \| v \| < 1. \) Hence the \( S_{\pi, \pi'} \) with \( \pi \cap \pi' = \emptyset \) and \( \pi \cup \pi' = \emptyset \), consist of all the faces of the unit sphere.

The faces \( S_k \) determined by \( v_k \) and \( (-v_k) \), \( k \neq k' \) represent subsets \( S_k \) of minimal positive measure, since the only face for which \( k = k' \) is \( L_k. \) If \( S_k \) is any face, then \( S_k \leq S_k \) for all \( k \) for which \( k \neq k. \)

Thus every measurable subset contains an atomic subset, and since the space \( L \) is of finite measure, the space of integrable functions represented is the space \( L. \)

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References.


1) The undecidability of the closure algebra has been proved in another way by Stanislaw Jaskowski in 1939. See [3].
3) See Moskowitz [9].