On Isomorphism Types of Measure Algebras.

By

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We call a *measure* every σ-additive finite non-negative function μ defined on a Boolean σ-algebra [1]) $\boldsymbol{A}$.

Let $\boldsymbol{C}$ be a σ-subalgebra of $\boldsymbol{A}$. The measure μ restricted to elements $A \in \boldsymbol{C}$ will be denoted by $\mu|\boldsymbol{C}$.

A measure μ on $\boldsymbol{A}$ is said to be *strictly positive* if $\mu(A)>0$ for all $A \in \boldsymbol{A}$, $A \neq 0$.

Let μ be a measure on a Boolean σ-algebra $\boldsymbol{A}$. The σ-ideal of all $A \in \boldsymbol{A}$ with $\mu(A)=0$ will be denoted by $\boldsymbol{I}(\mu)$. The measure μ on $\boldsymbol{A}$ uniquely determines a strictly positive measure μ^0 on $\boldsymbol{A}^0=\boldsymbol{A}/\boldsymbol{I}(\mu)$, defined by the equation

$$\mu^0(A^0)=\mu(A) \quad \text{for} \quad A \in A^0 \in \boldsymbol{A}^0.$$

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two Boolean σ-algebras with measures μ and ν respectively. An isomorphism [2]) h of $\boldsymbol{A}$ into $\boldsymbol{B}$ is said to be *measure-preserving* if $\nu(h(A))=\mu(A)$ for each $A \in \boldsymbol{A}$. If there is a measure-preserving isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$, the measures μ and ν are said to be *isomorphic*. The measures μ and ν are said to be *almost isomorphic* if the strictly positive measures μ^0 and ν^0 determined by μ and ν respectively are isomorphic.

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[1]) Boolean algebras will be denoted by letters $\boldsymbol{A}, \boldsymbol{B}, \ldots$, their elements by $A, B, \ldots$; $A+B$, $\sum_\xi A_\xi$, AB, $A-B$ will denote the Boolean operations analogous to addition, multiplication and complementation of sets. If $A+B=B$, we write $A \subset B$. The symbol 0 denotes the least element of a Boolean algebra $\boldsymbol{A}$, i. e. $0 \subset A$ for every $A \in \boldsymbol{A}$.

[2]) That is, a σ-additive and complementative transformation such that $h(A) \neq 0$ for $A \neq 0$. The last condition is equivalent to the statement that h is one-one.

The purpose of this note is to prove the following theorem:

(T) *Let μ and ν be strictly positive measures defined on Boolean σ-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively, and let $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ be σ-subalgebras of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. If μ is isomorphic to $\nu|\boldsymbol{B}_0$ and ν is isomorphic to $\mu|\boldsymbol{A}_0$, then the measures μ and ν are isomorphic.*

Theorem (T) may be formulated in the following equivalent form:

(T′) *Let μ and ν be two measures on Boolean σ-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively, and let $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ be σ-subalgebras of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. If μ is almost isomorphic to $\nu|\boldsymbol{B}_0$ and ν is almost isomorphic to $\mu|\boldsymbol{A}_0$, then the measure μ and ν are almost isomorphic.*

It will be shown (§ 4) by giving a suitable counter-example that Theorem (T) is not true (even in the case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are σ-fields of sets) after rejection of the condition that μ and ν are strictly positive. Consequently the words „almost isomorphic" in the second formulation (T′) may not be replaced by „isomorphic".

Two ideas play an essential part in the proof of Theorem (A): the well-known topologizing of Boolean algebras with the help of measures, and two fundamental theorems of D. Maharam on strictly positive measures.

§ 1. Measure algebras as metric spaces.

A Boolean σ-algebra with a strictly positive measure will be called, for brevity, a *measure algebra* [3]).

In the sequel, the letters $\boldsymbol{A}$ and $\boldsymbol{B}$ *will always denote measure algebras with strictly positive measures μ and ν respectively.*

Every measure algebra $\boldsymbol{A}$ will be considered as a metric space [4]) with the distance

$$\varrho(A_1,A_2)=\mu(A_1-A_2)+\mu(A_2-A_1) \qquad (A_1, A_2 \in \boldsymbol{A}).$$

(i) *Every isomorphism [5]) h of $\boldsymbol{A}$ onto $\boldsymbol{B}$ is a homeomorphism between the metric spaces $\boldsymbol{A}$ and $\boldsymbol{B}$.*

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[3]) Every measure algebra is complete. See e. g. F. Wecken, *Abstrakte Integrale und fastperiodische Funktionen*, Mathematische Zeitschrift **45** (1939), pp. 377-404; in particular p. 380, Satz 2.

[4]) See e. g. O. Nikodym, *Sur une généralisation des intégrales de M. J. Radon*, Fund. Math. **15** (1930), pp. 131-179.

[5]) By an *isomorphism* we always understand a Boolean isomorphism mentioned in footnote [2]). Unless the contrary is explicitely stated, it is not assumed that the isomorphisms under consideration preserve measure.

This follows easily from the fact that two strictly positive measures on the same Boolean σ-algebra $\boldsymbol{A}$ determine the same topology in $\boldsymbol{A}$ [6]).

The greatest element of a Boolean algebra $\boldsymbol{C}$ will be denoted by $|\boldsymbol{C}|$. An element $A_0 \epsilon \boldsymbol{C}$ is called an *atom* if there is no element $C \epsilon \boldsymbol{C}$ such that $0 \neq C \subset A_0$, $C \neq A_0$. A Boolean algebra $\boldsymbol{C}$ is said to be *atomic* if, for every element $A \epsilon \boldsymbol{A}$ $(A \neq 0)$, there is an atom $A_0 \subset A$.

If a measure algebra A is atomic, then there exists an at most enumerable sequence $\{A_n\}$ of atoms such that $|\boldsymbol{A}| = A_1 + A_2 + \ldots$

(ii) *Every atomic measure algebra forms a compact space.*

This is obvious if $\boldsymbol{A}$ has only a finite number of atoms. If the sequence $A_1, A_2, \ldots$, formed of all the different atoms of $\boldsymbol{A}$, is infinite, then $\boldsymbol{A}$ is homeomorphic to the Cantor discontinuous set K. In fact, if a real number $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \ldots \epsilon K$, where $a_n = 0$ or 2, and of we let $f(x)$ be the sum of all atoms A_n such that $a_n = 2$, then clearly f is a one-one continuous mapping of K onto $\boldsymbol{A}$, thus a homeomorphism.

§ 2. Measure algebras of the type α. For every element E of a Boolean algebra $\boldsymbol{C}$, the symbol $E\boldsymbol{C}$ will denote the relativized Boolean algebra formed of all $A \subset E$, $A \epsilon \boldsymbol{C}$. If $\boldsymbol{C}$ is a measure algebra, then $E\boldsymbol{C}$ will also be considered as a measure algebra with the same measure restricted to elements $A \epsilon E\boldsymbol{C}$.

For any Boolean algebra $\boldsymbol{C}$, let $\tau(\boldsymbol{C})$ denote the least cardinal which is the power of a set $\boldsymbol{S} \subset \boldsymbol{C}$ such that the least σ-subalgebra containing $\boldsymbol{S}$ is $\boldsymbol{C}$ itself.

A Boolean algebra $\boldsymbol{C}$ is said to be *of the type* $\alpha \geqslant 0$ if $\tau(E\boldsymbol{C}) = \aleph_\alpha$ for every $E \epsilon \boldsymbol{C}$, $E \neq 0$. It is said to be *of the type* -1 if it is atomic.

Clearly, if $\boldsymbol{C}$ is of the type α, and $0 \neq E \epsilon \boldsymbol{C}$, then $E\boldsymbol{C}$ is also of the type α.

For a fixed ordinal $\alpha \geqslant 0$, let $K_\alpha = \mathbf{P}_{\xi < \omega_\alpha} I_\xi$ be the Cartesian product of $\aleph_\alpha$ sets I_ξ, each of which contains only two elements a and b. Let K_α^ξ be the set of all points $p \epsilon K_\alpha$ whose ξ-th coordinate is a, and let $\boldsymbol{K}_\alpha$ be the least σ-field generated by the sets K_α^ξ $(\xi < \omega_\alpha)$. In each I_ξ we define the measure λ putting $\lambda(a) = \lambda(b) = \frac{1}{2}$. These measures on I_ξ induce the product measure [7]) m_α on $\boldsymbol{K}_\alpha$. The σ-field $\boldsymbol{K}_\alpha$ and the measure m determine a measure algebra $\boldsymbol{K}_\alpha^0$ (the $\boldsymbol{K}_\alpha$-sets modulo sets of measure m_α zero) with the strictly positive measure m_α^0 (see p. 92).

D. Maharam [8]) has proved the two following fundamental theorems:

(M_1) *If a measure algebra $\boldsymbol{A}$ is of the type $\alpha \geqslant 0$, then there is a constant $c > 0$ such that μ is isomorphic to cm_α^0.*

(M_2) *For every measure algebra $\boldsymbol{A}$, there is a transfinite enumerable increasing [9]) sequence of ordinals $\alpha_\xi \geqslant -1$ $(\xi < \gamma)$ and there is a sequence of mutually disjoint elements $A_\xi \epsilon \boldsymbol{A}$ $(\xi < \gamma)$ such that $A_\xi \boldsymbol{A}$ is of the type α_ξ and $|\boldsymbol{A}| = \sum_{\xi < \gamma} A_\xi$.*

We call the sequence $\{A_\xi\}$ *Maharam's decomposition* of the measure algebra $\boldsymbol{A}$.

(iii) *If a measure algebra $\boldsymbol{A}$ is of the type $\alpha \geqslant 0$, then the metric space $\boldsymbol{A}$ is not compact and the least cardinal of a set dense in A is $\aleph_\alpha$.*

By (M_1), it is sufficient to prove our lemma in the case $\boldsymbol{A} = \boldsymbol{K}_\alpha^0$.

The least (finitely additive) field generated by K_α^ξ $(\xi < \omega_\alpha)$ is dense in $\boldsymbol{K}_\alpha^0$ and has the power $\aleph_\alpha$. On the other hand, the metric space $\boldsymbol{K}_\alpha^0$ contains $\aleph_\alpha$ elements, the distance between two of them is equal to $\frac{1}{2}$; the elements determined by K_α^ξ have this property.

(iv) *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are measure algebras of the types α and β respectively, and $\alpha > \beta \geqslant -1$, then there is no isomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$.*

Suppose the contrary, i. e. $\boldsymbol{A}$ is isomorphic to a σ-subalgebra $\boldsymbol{B}_1 \subset \boldsymbol{B}$. By (i), the metric space $\boldsymbol{A}$ is homeomorphic to $\boldsymbol{B}_1$. If $\beta = -1$, i. e. if $\boldsymbol{B}$ is atomic, then $\boldsymbol{B}_1$ is also atomic. By (ii), the metric space $\boldsymbol{B}_1$ is compact. Hence $\boldsymbol{A}$ is compact, which contradicts (iii). If $\beta \geqslant 0$, then, by (iii), the space $\boldsymbol{B}_1$ contains a dense subset with cardinal $\leqslant \aleph_\beta$. Therefore $\boldsymbol{A}$ also contains such a subset, which contradicts (iii).

[6]) See F. Wecken, l. c., p. 381, Satz 4.

[7]) See e. g. P. R. Halmos, *Measure Theory*, New York 1950, p. 157, Th. B, and p. 158, (2).

[8]) D. Maharam, *On homogeneous measure algebras*, Proceedings of the National Academy of Sciences **28** (1942), pp. 108-111.

[9]) That is, $\alpha_\xi < \alpha_\eta$ for $\xi < \eta < \gamma$.

§ 3. Proof of Theorem (T).

(v) *Let* $|\boldsymbol{A}|=\sum_{\xi<\gamma} A_\xi$ *where the* A_ξ *are disjoint, and let* h *be a measure-preserving isomorphism of* $\boldsymbol{A}$ *into itself such that*

$$h(A_\xi) \subset \sum_{\eta\geqslant\xi} A_\eta \quad \textit{for each} \quad \xi<\gamma.$$

Then $h(A_\xi)=A_\xi$ *for each* $\xi<\gamma$.

Suppose the contrary. Let ξ be the least ordinal such that $A_\xi \neq h(A_\xi)$. We have

$$|\boldsymbol{A}|=h(|\boldsymbol{A}|)=\sum_\eta h(A_\eta)=\sum_{\eta<\xi} h(A_\eta)+h(A_\xi)+\sum_{\eta>\xi} h(A_\eta)$$

$$\subset \sum_{\eta<\xi} A_\eta+h(A_\xi)+\sum_{\eta>\xi} A_\eta=(|\boldsymbol{A}|-A_\xi)+h(A_\xi).$$

Hence $A_\xi \subset h(A_\xi)$. Since $\mu(A_\xi)=\mu(h(A_\xi))$, we obtain $A_\xi=h(A_\xi)$ which contradicts our hypothesis.

(vi) *Let* $\{A_\xi\}_{\xi<\gamma}$ *be Maharam's decomposition of* $\boldsymbol{A}$. *Then* $h(A_\xi)=A_\xi$ *for every measure-preserving isomorphism* h *of* $\boldsymbol{A}$ *into itself.*

By (v) it is sufficient to prove that $h(A_\xi) \subset \sum_{\eta\geqslant\xi} A_\eta$ for each $\xi<\gamma$.

Suppose the contrary, i. e. that there are ordinals ξ and η, $\eta<\xi$, such that $h(A_\xi)A_\eta \neq 0$. Let B be the sum[3]) of all $A \in \boldsymbol{A}$ such that $h(A)\cdot A_\eta=0$, and let $C=A_\xi-B$. We have $C\neq 0$ and $C\boldsymbol{A}=C(A_\xi\boldsymbol{A})$ is of the type $a_\xi=\tau(A_\xi\boldsymbol{A})$. The equation

$$g(A)=A_\eta\cdot h(A) \quad \text{for} \quad A\subset C$$

defines an isomorphism of $C\boldsymbol{A}$ into $h(C)\boldsymbol{A}=h(C)(A_\eta\boldsymbol{A})$. This is impossible by (iv) since the last measure algebra is of the type $\tau(A_\eta\boldsymbol{A})=a_\eta<a_\xi$.

(vii) *Let* $\boldsymbol{A}$ *be a measure algebra of the type* $a\geqslant -1$, *and let* $\boldsymbol{A}_0, \boldsymbol{A}_1$ *be two* σ*-subalgebras of* $\boldsymbol{A}$, *such that* $\boldsymbol{A}_0\subset\boldsymbol{A}_1\subset\boldsymbol{A}$. *If* μ *is isomorphic to* $\mu|\boldsymbol{A}_0$, *then* μ *is also isomorphic to* $\mu|\boldsymbol{A}_1$.

Let h be a measure-preserving isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}_0$.

Consider first the case $a=-1$. Let $a_1>a_2>a_3>\dots$ be the decreasing sequence of all values which the measure μ assumes on atoms of $\boldsymbol{A}$. Denote by A_n the sum of all atoms $A\in\boldsymbol{A}$ with $\mu(A)=a_n$. Since $\mu(A)=\mu(h(A))$, we infer that, for every atom $A\subset A_n$, the element $h(A)$ is the sum of some atoms contained in $A_n+A_{n+1}+A_{n+2}+\dots$ Consequently $h(A_n)\subset A_n+A_{n+1}+A_{n+2}+\dots$

By (v), we have $h(A_n)=A_n$ for $n=1,2,\dots$ Since each A_n contains a finite number of atoms each of which has measure a_n, we infer $\boldsymbol{A}_0=\boldsymbol{A}$. Consequently $\boldsymbol{A}_1=\boldsymbol{A}$.

Suppose now $a\geqslant 0$. Let $\{A_\xi\}_{\xi<\gamma}$ be Maharam's decomposition of $\boldsymbol{A}_1$, i. e. $A_\xi\in\boldsymbol{A}_1$, $A_\xi\boldsymbol{A}_1$ is of the type a_ξ, $a_\xi<a_\eta$ and $A_\xi A_\eta=0$ for $\xi<\eta$.

Since $A_\xi\boldsymbol{A}_1\subset A_\xi\boldsymbol{A}$, it follows from (iv) that $a_\xi\leqslant a$. Suppose there is an ordinal ξ such that $a_\xi<a$. Let B be the sum[3]) of all $A\in\boldsymbol{A}_0$ such that $AA_\xi=0$, and let $C=|\boldsymbol{A}|-B\in\boldsymbol{A}_0$. Obviously $C\neq 0$. If $0\neq A\in C\boldsymbol{A}_0$, then $AA_\xi\neq 0$. Therefore the formula

$$g(A)=AA_\xi \quad \text{for} \quad A\in C\boldsymbol{A}_0$$

defines an isomorphism g of the algebra $C\boldsymbol{A}_0$ (of the type a) into $g(C)\boldsymbol{A}_1=g(C)(A_\xi\boldsymbol{A}_1)$ (of the type $a_\xi<a$) which is impossible by (iv). We infer thus that Maharam's decomposition of $\boldsymbol{A}_1$ contains exactly one element of the type a, i. e. $\boldsymbol{A}_1$ is of the type a. Therefore, by (M_1), μ is isomorphic to $\mu|\boldsymbol{A}_1$.

(viii) (Generalization of (vii)). *Let* $\boldsymbol{A}_0$ *and* $\boldsymbol{A}_1$ *be two* σ*-subalgebras of a measure algebra* $\boldsymbol{A}$, $\boldsymbol{A}_0\subset\boldsymbol{A}_1\subset\boldsymbol{A}$. *If* μ *is isomorphic to* $\mu|\boldsymbol{A}_0$, *then* μ *is also isomorphic to* $\mu|\boldsymbol{A}_1$.

Let $\{A_\xi\}_{\xi<\gamma}$ be Maharam's decomposition of $\boldsymbol{A}$ and let h be a measure preserving isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}_0$. By (vi), $A_\xi=h(A_\xi)\in\boldsymbol{A}_0\subset\boldsymbol{A}_1$. Since the three measure algebras $A_\xi\boldsymbol{A}_0$, $A_\xi\boldsymbol{A}_1$, $A_\xi\boldsymbol{A}$, satify the assumption of (vii), there is a measure-preserving isomorphism g_ξ of $A_\xi\boldsymbol{A}$ onto $A_\xi\boldsymbol{A}_1$. The formula

$$g(A)=\sum_{\xi<\gamma} g_\xi(AA_\xi) \quad \text{for} \quad A\in\boldsymbol{A}$$

defines a measure preserving isomorphism g of $\boldsymbol{A}$ onto $\boldsymbol{A}_1$, q. e. d.

Theorem (T) follows immediately from (viii).

§ 4. A counter-example.

Let $\boldsymbol{X}$ be the σ-field of all Borel subsets of $I\times I$, where I is the unit interval. Let $\boldsymbol{Y}_0$ be the σ-field of all sets $X\times I$, where X is a Borel subset of I. Further let $\boldsymbol{X}_0$ and $\boldsymbol{Y}$ denote the σ-field of all sets $Y-X_1+X_2$, where $Y\in\boldsymbol{Y}_0$, and X_1 and X_2 are at most enumerable subsets of $I\times I$. Clearly $\boldsymbol{X}_0\subset\boldsymbol{X}$ and $\boldsymbol{Y}_0\subset\boldsymbol{Y}$.

Let $\bar{\mu}$ be Lebesgue's plane measure on $\boldsymbol{X}$ and let $\bar{\nu}=\bar{\mu}|\boldsymbol{Y}$.

The measure $\overline{\mu}$ is isomorphic to $\overline{\nu}|\boldsymbol{Y}_0$ since there is a measure-preserving Baire transformation of $I\times I$ onto I. The measure $\overline{\nu}$ is trivially isomorphic to $\overline{\mu}|\boldsymbol{X}_0$. The measures $\overline{\mu}$ and $\overline{\nu}$, however, are not isomorphic since there is no Boolean isomorphism of $\boldsymbol{X}$ onto $\boldsymbol{Y}$.

In fact, suppose h is an isomorphism of $\boldsymbol{X}$ onto $\boldsymbol{Y}$. Since all one-point subsets of $I\times I$ belong to $\boldsymbol{X}$ and to $\boldsymbol{Y}$, there is a one-one mapping[10]) φ of $I\times I$ into $I\times I$ such that $h(X)=\varphi^{-1}(X)\in\boldsymbol{Y}_0$ for $X\in\boldsymbol{X}$. Thus φ is a Baire mapping. Consequently φ^{-1} is also a Baire mapping[11]). Let $X_0\in\boldsymbol{X}-\boldsymbol{Y}$. We have $\varphi(X_0)\in\boldsymbol{X}$ and $X_0=\varphi^{-1}(\varphi(X_0))=h(\varphi(X_0))\in\boldsymbol{Y}$ which is impossible.

The above example shows that the assumption in Theorem (T) that measures are strictly positive is essential.

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[10]) See e. g. E. Szpilrajn-Marczewski, *On the isomorphism and the equivalence of classes and sequences of sets*, Fund. Math. **32** (1939), pp. 133-148; in particular p. 137.

[11]) See e. g. C. Kuratowski, *Topologie I* (second edition), Warszawa-Wrocław 1948, p. 398, th. 3.

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Algebraic Treatment of the Functional Calculi of Heyting and Lewis [1]).

By

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Every formula a of a functional calculus can be interpreted as a functional on an abstract set I with values in a suitable abstract algebra $\mathfrak{A}$. This functional will be denoted by „Φ_a" and will be called the *$(I,\mathfrak{A})$-functional* determined by a[3]).

For instance, every formula of the ordinary functional calculus can be interpreted as an $(I,\mathfrak{A})$-functional, where $\mathfrak{A}$ is a complete Boolean algebra[4]); every formula of the functional calculus of Heyting can be interpreted as an $(I,\mathfrak{B})$-functional, where $\mathfrak{B}$ is a complete Brouwerian algebra[5]), and every formula of the functional calculus of Lewis[6]) can be interpreted as an $(I,\mathfrak{C})$-functional, where $\mathfrak{C}$ is a complete closure algebra[7]).

The above interpretation is a generalization of the well-known matrix method in sentential calculi. The connection between the

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[1]) This paper was presented to the Warsaw University in candidacy for the degree of Doctor of Philosophy and accepted in May 1950. The results were announced at the Polish-Czechoslovak Mathematical Congress in Prague in September 1949. The results of this paper together with that of „*A Proof of the Completeness Theorem of Gödel*" published by the author and R. Sikorski (Fundamenta Mathematicae **37** (1950), pp. 193-200) were announced at the meeting of the Association for Symbolic Logic in December 1949 The Journal of Symbolic Logic **15** (1950), p. 79).

[2]) The author wishes to thank Professor A. Mostowski for suggestions and criticisms in connection with the writing of this Thesis.

[3]) The notion of the $(I, \mathfrak{A})$-functional and the idea of treating the functional calculi algebraically is due to Mostowski [2]. For a definition of the $(I, \mathfrak{A})$-functional Φ_a see § 4, p. 113.

[4]) See Rasiowa and Sikorski [1].

[5]) See Mostowski [2].

[6]) The system considered here is based on the system S.4 of the sentential calculus of Lewis and Langford [1], p. 501.

[7]) See § 5, p. 119.