Nous avons par suite de 4.2, 5.1, 5.2 et 5.3:

$$\varphi[x,\varphi(x,x)]=\gamma_1(x), \quad \varphi[\varphi(x,x),x]=\gamma_2(x).$$

La proposition 5.5 résulte de la définition des fonctions γ_1, γ_2.

6. ***Démonstration de la proposition 3.1.*** Soit f_n une fonction de la suite considérée, n_1 et n_2 deux nombres naturels tels que $|n_1-n_2|=n$. On a d'après 5.4

$$\varphi[g_{n_1}(x), g_{n_2}(y)]=f_{|n_1-n_2|}\{g_{n_1}^{-1}[g_{n_1}(x)], g_{n_2}^{-1}[g_{n_2}(y)]\}=f_n(x,y).$$

La proposition 3.1 et notre théorème sont ainsi démontrés.

A Construction for Models of Consistent Systems[1]).

By

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This paper describes a method for constructing a model (S'_m) of an extension (S') of a given system (S) within the syntax of (S). § 1 states the conditions which (S) must satisfy to have a model of this type and also explains the relation between (S) and (S'). This is analogous to that existing between Zermelo-Fraenkel set theory and von Neumann-Bernays set theory [2]). In § 2 the formal syntax of (S) is built up. It involves no notion of truth or satisfaction and is essentially equivalent to arithmetic based on the 5 Peano axioms together with the hypothesis that (S) is consistent. It is therefore a denumerable system. Within this syntax it is possible to define a predicate „T" of statements having some of the properties of „is true". This predicate plays an essential role in the construction of the model. The notion of „model" is defined in § 3 and the construction of the model of (S') in the syntax of (S) described in § 4. The existence of this model establishes the consistency of (S') relative to the syntax of (S). Since in syntax there is only a denumerable number of expressions and the range of the variables of (S'_m) is restricted to expressions of a certain type of this syntax, (S'_m) is a denumerable model of (S'). Therefore (S'_m) contains a „subsystem" forming a denumerable model of (S). Such a construction is possible for every system (S) satisfying the assumptions of § 1 and can be carried out entirely within syntax without semantical concepts. It therefore gives another proof of the Skolem-Löwenheim theorem.

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[1]) Presented to the American Mathematical Society December 30, 1948 and April 30, 1949. The author wishes to express sincere thanks to Profs. L. H. Loomis, A. Mostowski, W. V. Quine and to Dr. J. Myhill for their excellent suggestions and help. This paper is based in part on a thesis submitted for the degree of Ph. D. at Radcliffe College in June 1948, which was written while the author held the M. E. Maltby fellowship (AAUW). Prof. Mostowski has recently improved the results of this paper and shown that every theorem of (S') which can be expressed in (S) is a theorem of (S).

[2]) Cf. Wang [13] where the relations between the two systems are discussed.

§ 1. The systems (S), (S') and (S_ε).

(S) may be any system of axioms satisfying the following conditions:

(i) it contains an infinite number of variables $\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{x}', \mathrm{y}', \mathrm{z}', \mathrm{x}'', \ldots$ whose range is a universe V. (Of course, the variables need not be allowed to assume V as value).

(ii) it contains only a finite number of primitive connectives $\Delta, \S_1, \ldots, \S_{n-2}, \star$ one of which, say $\S_1$, is $=$, in addition to the usual truth-functional ones and quantification over the variables of the system [3]).

(iii) it is based on a finite number of axioms and axiom schemata, including those for sentential calculus, quantification theory and the theory of identity. (More generally, it is sufficient to assume that the sentence ⌜Φ is an axiom of (S)⌝ be expressible in syntax).

(S') is an extension of (S). It contains not only one, but two kinds of variables. The first kind, coinciding with the variables of (S) and also denoted by small Latin letters (with or without accents and subscripts) ranges over V, the second, denoted by capital Latin letters (with or without accents and subscripts) ranges over a more inclusive universe W. Elements represented by variables of the first kind thus satisfy all the axioms of (S), but the elements of W do not necessarily satisfy them all. (S') further contains a new primitive connective „∘" such that for every A in W and for every x in V, $\mathrm{A} \circ \mathrm{x}$ is a formula [4]). Variables of the second kind satisfy an axiom schema which can be stated with help of the notion of *primitive propositional formula* (abbrev. ppf), defined as follows [5]):

Let $\mathrm{x}, \mathrm{y}, \mathrm{z}$ denote variables of the first kind, $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ variables of the second kind, then

(i) $\Delta\mathrm{x}, \Delta\mathrm{X}, \S_1\mathrm{XY}, \S_1\mathrm{xY}, \S_1\mathrm{Xy}, \S_1\mathrm{xy}, \ldots, \star\mathrm{xyZ}, \star\mathrm{xyz}$ are ppfs.

(ii) for each X of the second kind, $\mathrm{X} \circ \mathrm{y}$ is a ppf.

(iii) if Φ and Ψ are ppfs, so are $\sim\Phi$ and $\Phi \cdot \Psi$.

(iv) If Φ is a ppf, then $(\exists \mathrm{x})\Phi$ is a ppf, and so is any result obtained by replacing x by another variable of the first kind.

(v) only formulas so obtained are ppfs.

The axiom schema in question then states:

(1.1) If Φ is a ppf whose only free variable is v, then the closure of

$$(\exists \mathrm{Z})(\mathrm{v})(\mathrm{Z} \circ \mathrm{v} \cdot \equiv \Phi)$$

is a theorem [6]).

(S') further contains axioms of identity and one axiom relating the two kinds of variables:

$$\mathrm{X} = \mathrm{Y} \cdot \equiv (\mathrm{z})(\mathrm{X} \circ \mathrm{z} \equiv \mathrm{Y} \circ \mathrm{z}), \tag{1.2}$$

$$\mathrm{x} = \mathrm{y} \cdot \supset \cdot \mathrm{Z} \circ \mathrm{x} \equiv \mathrm{Z} \circ \mathrm{y}, \tag{1.3}$$

$$(\mathrm{x})(\exists \mathrm{X})(\mathrm{x} = \mathrm{X}). \tag{1.4}$$

The system (S_ε) is obtained from (S) by adjoining to it the Hilbert ε-operator [7]). If Φ is a statement, $(\varepsilon \mathrm{x})\Phi$ can be read as „an x such that Φ", in analogy with the reading of $(\iota \mathrm{x})\Phi$ as „the x such that Φ". The axiom schema govering ε is:

$$(\exists \mathrm{x})\Phi \supset \Phi' \tag{1.5}$$

where Φ' is like Φ except for containing occurrences of $(\varepsilon \mathrm{x})\Phi$ wherever Φ contained occurrences of x (x is assumed not to be bound in Φ).

It is known that if (S) is consistent, then (S_ε) is also consistent. The proof of this fact consists in giving an explicit construction for a contradiction in (S) once a contradiction in (S_ε) has been discovered. This proof can be carried out within a syntax such as the one described in § 2. From now on we shall therefore assume that (S_ε) is consistent and build the model of (S') in the syntax of (S_ε). If a contradiction can then be derived in (S'), a corresponding one can be derived in the syntax of (S_ε), so that this syntax would then be inconsistent. But since this is merely the syntax of (S) plus the statement that (S_ε) is consistent and we know that the consistency of (S) implies that of (S_ε), this would assert that the syntax of (S) is inconsistent. This would contradict the basic hypothesis throughout, namely that the syntax of (S) is consistent. Constructing the model as indicated therefore establishes the consistency of (S') relative to that of the syntax of (S). Since however (S') also contains (S), this method allows us to build a model of (S).

[3]) We shall here, for the sake of definiteness assume Δ to be monadic, $\S_1, \ldots, \S_n$ diadic, $\star$ triadic. It should perhaps be noted here that in this paper the convention is made that expressions in Roman type are names of corresponding (i. e. similarly shaped) expressions in italics. (See Note 1 on p. 109).

[4]) „$A \circ x$" is defined for every A in W. For values of A in V, „$A \circ x$" may well be equivalent to „$\S_i Ax$", (for some i), so that „∘" may be considered an extension of „$\S_i$" over a wider range of functionality. This is the case if „$\S_i$" is „ε", e.g. if (S) is the Zermelo-Fraenkel system, (S') the von Neumann-Bernays system.

[5]) Cf. Gödel [2], p. 8.

[6]) Cf. Gödel [2], p. 8, theorem M 1, or Bernays [1], p. 72.

[7]) Cf. Hilbert-Bernays [5], § 1.

The von Neumann-Bernays system is related to the Zermelo-Fraenkel system as (S') is to (S). This can be seen as follows. As primitive connective in Zermelo's system we may take ε and define $=$ in terms of ε by extensionality. Then the axioms are (cf. W. Ackermann, *Mengentheoretische Begründung der Logik*, Math. Ann., vol. 115 (1937), p. 1 or H. Wang [13]):

Z 1: $x=y\supset(x\,\varepsilon\,z\equiv y\,\varepsilon\,z)$ Z 2: $(\exists w)(x)(x\,\varepsilon\,w\equiv(x=y\vee x=z))$,

Z 3: $(\exists w)((\exists u)(u\,\varepsilon\,w)\cdot(x)(x\,\varepsilon\,w\supset(\exists y)(y\,\varepsilon\,w\cdot x\subset y\cdot x\neq y)))$,

Z 4: $(\exists w)(x)(x\,\varepsilon\,w\equiv(\exists y)(x\,\varepsilon\,y\cdot y\,\varepsilon\,z))$, Z 5: $(\exists w)(x)(x\,\varepsilon\,w\equiv x\subset z)$,

Z 6: *If* Φxy *is a formula in which* x *and* y *are free but* z, w *and* u *are not free, then* $(x)(y)(z)((\Phi xy\cdot\Phi xz)\supset y=z)\supset(\exists w)(x)(x\,\varepsilon\,w\equiv(\exists y)(y\,\varepsilon\,u\cdot\Phi yx))$ *is an axiom.*

If we then add new variables A, B, ... and axiom schemata (1.1)-(1.5), we can easily show that Mostowski's or Gödel's axioms for the von Neumann-Bernays set theory hold (cf. Gödel [2] or Mostowski [6a]). To do this, we write Cls X for $(\exists Y)(Y=X)$, $\mathfrak{M}X$ for $(\exists Y)(X\,\varepsilon\,Y)$. A1 then follows immediately from (1.4). As the new connective $\circ$ we also take ε, i. e. instead of $A\circ X$ we shall have $X\,\varepsilon\,A$. A 2 then follows immediately from the definition of $\mathfrak{M}X$, A 3 from the definition of $=$ by extensionality. A 4, C 1, C 2, C 3 do not involve any concepts other then those already present in the Zermelo-Fraenkel system nor class-variables, and therefore hold in the extension of this system. B 1-B 8 follow from schema (1.1). To show B 1: $(\exists A)(x)(y)(\langle xy\rangle\,\varepsilon\,A\cdot\equiv\cdot x\,\varepsilon\,y)$, note that the existence of A is asserted by taking Φ in (1.1) as $(\exists x)(\exists y)(v=\langle xy\rangle\cdot x\,\varepsilon\,y)$. To show the existence of C in B 2, take Φ as $v\,\varepsilon\,A\cdot v\,\varepsilon\,B$, and similarly for B 3-B 8. C 4 then follows from Z 6 by (1.5), since by (1.1), there exists a class A of pairs $\langle xy\rangle$ such that $\langle xy\rangle\,\varepsilon\,A\equiv\Phi xy$.

The following additional remarks were suggested to me by Dr. Hao Wang: It may be noted that by generalizing the Bernays-Gödel procedure (cf. [2], pp. 3-15) of proving general principles of class existence and the proof of relative consistency in this paper, we can obtain a general theorem on the finitization of proper axioms (axioms not belonging to the theory of truth-functions and quantifiers) of a large class of formal systems. Thus the system of [2] may be considered as an „equivalent extension" of the Zermelo-Fraenkel system ZF in the sense that all formulas and proofs of the latter can be expressed and carried out in the former and that we can prove that if the latter is consistent, then the former is. There is one difference between these two systems: while ZF contains infinitely many proper axioms, the system of [2] contains only a finite number. Hence we may say that in [2] we find an „equivalent extension" of ZF in which the axioms are finitized. Now it is not difficult to see that we can prove the following generalization of the above special case: If A is a system with one kind of variables, finitely many primitive predicates, finitely many proper axioms and axiom schemata, and with n-tuplets available, and if each axiom-schema is of the form „*if* Φ *is a formula, then* $F(\Phi)$ *is an axiom*" (not restricting Φ to any special kind except for possibly specifying that certain variables are free or not free in it, and $F(\Phi)$ being a fixed formula of the system for every fixed Φ); then we can construct a system which is an „equivalent extension" of the given system and in which there are only finitely many proper axioms. In particular, the theorem applies to the system (Z^*) (including the axioms for identity) on p. 465 of vol. I of [5], to all systems obtained from [7] when we substitute finitely many axioms for its principle *200 of set existence, and to the system NQ in [13]; all these systems have „equivalent extensions" with only a finite number of proper axioms.

§ 2. The formal syntax of (S) and (S_ε).

By „syntax" will here be meant the part of the metalanguage which is built up from the names of finite number of basic signs by means of a primitive syntactic 3-place predicate „M" (whose properties are described below), joint denial and quantification [8]). If (S) contains n primitive connectives Δ, $\S_1,\dots,\S_{n-2}$, $*$ and Δ is monadic (i. e. Δx is a formula), $\S_1,\dots,\S_{n-2}$ are diadic (i. e. $\S_i xy$ is a formula) and $*$ is triadic (i. e. $*xyz$ is a formula), the list of basic names will be:

$$
\begin{array}{lll}
S_1=\text{„}w\text{"}=\mathrm{w}, & S_2=\text{„}x\text{"}=\mathrm{x}, & S_3=\text{„}y\text{"}=\mathrm{y}\\
S_4=\text{„}z\text{"}=\mathrm{z}, & S_5=\text{„}'\text{"}=' & S_6=\text{„}(\text{"}=(\\
S_7=\text{„})\text{"}=) & S_8=\text{„}\downarrow\text{"}=\downarrow & S_9=\text{„}\Delta\text{"}=\Delta\\
S_{10}=\text{„}\S_1\text{"}=\S_1 & S_{11}=\text{„}\S_2\text{"}=\S_2 & \dots\dots\\
S_{8+n}=\text{„}*\text{"}=* & S_{9+n}=\text{„}\varepsilon\text{"}=\varepsilon. &
\end{array}
$$

These form a finite $9+n$ sign alphabet, the last sign being the Hilbert ε-operator.

The primitive syntactic predicate „M" can be thought of as having the following properties:

(i) If x is a single sign, (i. e. one of $S_1,\dots,S_{9+n}$), then $Mxyz$ if x is the alphabetic successor of y.

(ii) If x is a complex expression, $Mxyz$ if x is the result of writing y followed by z and is written $y^\frown z$.

(iii) If x is not an expression at all, $Mxyz$ if $x=y$.

In terms of this alphabet and this predicate, we can define what we mean by „*x is a formula*", „*y is a free variable of x*" and so on. Since this was carried out in great detail by Quine [9]), the definitions will not be repeated here. A number of additions and changes are however necessary, as we may have more than one primitive connective in the object language and shall also make use of the Hilbert ε-operator. They are listed below:

$\ulcorner\zeta$ *is an atomic logical formula*$\urcorner$:

$\Delta L\,Fmla_0$: $\ulcorner L\,Fmla_0\,\zeta\urcorner$ for $\ulcorner(\exists\alpha)(\exists\beta)(\exists\gamma)(Vbl\,\alpha\cdot Vbl\,\beta\cdot Vbl\,\gamma\cdot\zeta= =S_9{}^\frown\alpha\cdot\vee\cdot\zeta=S_{10}{}^\frown\alpha^\frown\beta\cdot\vee\cdot\zeta=S_{11}{}^\frown\alpha^\frown\beta\cdot\vee\cdot\dots\cdot\vee\cdot\zeta=S_{8+n}{}^\frown\alpha^\frown\beta^\frown\gamma)\urcorner$.

The dots in this definition as in many following ones do not refer to an infinite process, but on the contrary, once the axioms

8) Cf. Quine [7], p. 291, § 55. Quines notation of Greek letters and corners is followed here.

9) Cf. Quine [7], Ch. VII, pp. 283-305.

of (S) are given, we can pick out the primitive connectives and fill in the dots in this definition. The result will of course be an expression of finite length. A similar remark could be made concerning the dots in the list of basic names.

The following list of definitions repeats several of those given by Quine, but is given here because the notation is slightly different.

$$\varDelta\downarrow:\quad \ulcorner(\zeta\downarrow\eta)\urcorner \quad\text{for}\quad \ulcorner(S_6{}^\frown\zeta{}^\frown S_8{}^\frown\eta{}^\frown S_7)\urcorner,$$
$$\varDelta(\,):\quad \ulcorner(\zeta)\eta\urcorner \quad\text{for}\quad \ulcorner(S_6{}^\frown\zeta{}^\frown S_7{}^\frown\eta)\urcorner,$$
$$\varDelta\sim:\quad \ulcorner(\sim\zeta)\urcorner \quad\text{for}\quad \ulcorner(\zeta\downarrow\zeta)\urcorner,$$
$$\varDelta\exists:\quad \ulcorner(\exists\zeta)\eta\urcorner \quad\text{for}\quad \ulcorner(\sim(\zeta)\sim\eta)\urcorner,$$
$$\varDelta\vee:\quad \ulcorner(\zeta\vee\eta)\urcorner \quad\text{for}\quad \ulcorner\sim(\zeta\downarrow\eta)\urcorner,$$
$$\varDelta\supset:\quad \ulcorner(\zeta\supset\eta)\urcorner \quad\text{for}\quad \ulcorner(\eta\vee\cdot\sim\zeta)\urcorner,$$
$$\varDelta\cdot:\quad \ulcorner(\zeta\cdot\eta)\urcorner \quad\text{for}\quad \ulcorner(\sim\zeta\cdot\downarrow\cdot\sim\eta)\urcorner,$$
$$\varDelta\equiv:\quad \ulcorner(\zeta\equiv\eta)\urcorner \quad\text{for}\quad \ulcorner((\zeta\supset\eta)\cdot(\eta\supset\zeta))\urcorner.$$

Here „$\downarrow$“, „$\vee$“, „$\supset$“, „$\cdot$“, „$\sim$“ are name connectives, forming a name of a statement when applied to names of statements. (See note 2 on p. 109).

Definitions of the following expressions may be taken directly from Quine [7]:

$$\ulcorner\zeta \textit{ is a bound occurrence of } \eta \textit{ in } \theta\urcorner \quad\text{for}\quad \ulcorner(\zeta\, BO_\theta\, \eta)\urcorner.$$
$$\ulcorner\zeta \textit{ is a free occurrence of } \eta \textit{ in } \theta\urcorner \quad\text{for}\quad \ulcorner(\zeta\, FO_\theta\, \eta)\urcorner.$$
$$\ulcorner\zeta \textit{ is free in } \eta\urcorner \quad\text{for}\quad \ulcorner(\zeta\, F_\eta)\urcorner.$$
$$\ulcorner\zeta \textit{ is a formula}\urcorner \quad\text{for}\quad \ulcorner Fmla\,\zeta\urcorner.$$
$$\ulcorner\zeta \textit{ is a matrix}\urcorner \quad\text{for}\quad \ulcorner Mat\,\zeta\urcorner.$$
$$\ulcorner\zeta \textit{ is a statement}\urcorner \quad\text{for}\quad \ulcorner Stat\,\zeta\urcorner.$$
$$\ulcorner\zeta \textit{ is a tautology}\urcorner \quad\text{for}\quad \ulcorner Taut\,\zeta\urcorner.$$
$$\ulcorner\textit{the closure of } \zeta\urcorner \quad\text{for}\quad \ulcorner cl\,\zeta\urcorner.$$

By means of above definitions, it is possible to form a name of any statement of the object language. In particular, each axiom will have a name, and if they are finite in number, we can explicitly write down a definition of $\ulcorner\zeta$ *is an axiom*$\urcorner$. If the system contains an infinite number of axioms it is impossible to enumerate them all, but it may still be possible to use the vocabulary of this syntax to characterize the form of those statements which we shall take as axioms in this syntax. We shall assume here that the definition of $\ulcorner\zeta$ *is an axiom*$\urcorner$ can be formulated in this syntax, i. e. that there exists a formula P of syntax such that ζ is the only free variable of P and

$$Ax_S\zeta\equiv P.$$

If (S) is based on a finite number of axioms $\Phi_1,\dots,\Phi_n$ this assumption is satisfied, for we can then take P simply as $\ulcorner\zeta=\Phi_1\vee\dots\vee\zeta=\Phi_n\urcorner$, but the assumption is also satisfied in many other cases. The following definition can then also be taken from Quine:

$$\ulcorner\zeta \textit{ is a theorem of } (S)\urcorner \quad\text{for}\quad \ulcorner Th_S\,\zeta\urcorner.$$

Further definitions are [10]):

$$\varDelta\vdash_S:\quad \ulcorner\vdash_S\zeta\urcorner \quad\text{for}\quad \ulcorner Th_S\, cl\,\zeta\urcorner.$$
$$\ulcorner\zeta \textit{ occurs in } \eta\urcorner \quad\text{for}\quad \ulcorner\zeta \textit{ in } \eta\urcorner.$$
$$\ulcorner\zeta \textit{ is a part of the expression } \eta\urcorner \quad\text{for}\quad \ulcorner\zeta\, P\,\eta\urcorner.$$
$$\ulcorner\zeta \textit{ begins } \eta\urcorner \quad\text{for}\quad \ulcorner\zeta\, B\,\eta\urcorner.$$

$\ulcorner\zeta'$ *is like* ζ *except for containing occurrences of* η' *at all places where* ζ *contains free occurrences of* $\eta\urcorner$ *for* $\ulcorner(\zeta'\, S^{\eta'}_{\eta}\,\zeta)\urcorner$.

$\ulcorner\zeta$ *is a bound variable of* $\eta\urcorner$:

$$\varDelta BV:\quad \ulcorner(\zeta\, BV\,\eta)\urcorner \quad\text{for}\quad \ulcorner Vbl\,\zeta\cdot(\exists\alpha)\,(\alpha\, BO_\eta\,\zeta)\cdot\sim\zeta F\,\eta\urcorner.$$

$\ulcorner\zeta$ *is an* ε*-term*$\urcorner$:

$$\varDelta eTm:\quad \ulcorner eTm\,\zeta\urcorner \quad\text{for}\quad \ulcorner(\exists\alpha)\,(\exists\beta)\,(Fmla\,\alpha\cdot Vbl\,\beta\cdot\zeta=S_{9+n}{}^\frown\beta{}^\frown\alpha)\urcorner\,{}^{11)}.$$

$\ulcorner\zeta$ *is an atomic formula of* $(S_\varepsilon)\urcorner$:

$$\begin{aligned}\varDelta eFmla_0:\quad & \ulcorner eFmla_0\,\zeta\urcorner \quad\text{for}\quad \ulcorner(\exists\alpha)\,(\exists\beta)\,(\exists\gamma)\,(Vbl\,\alpha\cdot Vbl\,\beta\cdot Vbl\,\gamma\cdot\vee\cdot\\ & Vbl\,\alpha\cdot Vbl\,\beta\cdot eTm\,\gamma\cdot\vee\cdot Vbl\,\alpha\cdot eTm\,\beta\cdot Vbl\,\gamma\cdot\vee\cdot eTm\,\alpha\cdot Vbl\,\beta\cdot\\ & Vbl\,\gamma\cdot\vee\cdot Vbl\,\alpha\cdot eTm\,\beta\cdot eTm\,\gamma\cdot\vee\cdot eTm\,\alpha\cdot Vbl\,\beta\cdot eTm\,\gamma\cdot\vee\cdot\\ & eTm\,\alpha\cdot eTm\,\beta\cdot Vbl\,\gamma\cdot\vee\cdot eTm\,\alpha\cdot eTm\,\beta\cdot eTm\,\gamma : \zeta=S_9{}^\frown\alpha\cdot\vee\cdot\\ & \zeta=S_{10}{}^\frown\alpha{}^\frown\beta\cdot\vee\cdot\zeta=S_{11}{}^\frown\alpha{}^\frown\beta\cdot\vee\cdot\ldots\cdot\vee\cdot\zeta=S_{8+n}{}^\frown\alpha{}^\frown\beta{}^\frown\gamma)\urcorner.\end{aligned}$$

$\ulcorner\zeta$ *is a formula of* $(S_\varepsilon)\urcorner$ and $\ulcorner\zeta$ *is a statement of* $(S_\varepsilon)\urcorner$ (abbrev. $\ulcorner eStat\,\zeta\urcorner$) are then defined in the same way as $\ulcorner\zeta$ *is a formula*$\urcorner$, and $\ulcorner\zeta$ *is a statement*$\urcorner$ except that in the definitions we put $\ulcorner eFmla_0\,\alpha\urcorner$ instead of $\ulcorner Fmla_0\,\alpha\urcorner$.

$\ulcorner$*the formula obtained by substituting* η *for every free occurrence of the only free variable of* $\zeta\urcorner$:

$$\varDelta S':\quad \ulcorner S^\eta\,\zeta\urcorner \quad\text{for}\quad \ulcorner(\iota\delta)\,(\exists\alpha)\,((\beta)\,(\alpha F\,\zeta\cdot\beta F\zeta\cdot\supset\cdot\alpha=\beta)\cdot\delta S^\eta_\alpha\,\zeta)\urcorner.$$

---

[10]) Definitions of expressions not explicitly defined here are carried over from Quine [7], Ch. VII.

[11]) By $\ulcorner Fmla\,\zeta\urcorner$ is of course meant $\ulcorner\zeta$ *is a formula of* $(S)\urcorner$. The definition $\varDelta$ *eFmla* is based on the hypothesis that the definition of $\ulcorner Fmla\,\zeta\urcorner$ precedes it.

In the following, Φ, Ψ, X with or without accents or subscripts will represent statements or matrices, Φ, Ψ, X names of statements, α, β, γ, μ, ν etc. variables and α, β, γ names of variables. If R is a formula of the metalanguage, $\ulcorner\models R\urcorner$ shall mean ⌜the closure of R is a meta-theorem⌝[12]). The axioms on which this formal syntax is based will be the following:

A. Axioms of Quantification Theory in the Syntax:

Let R and P be formulas of the metalanguage.

M 100: *If R is tautologous,* $\models R$.

M 101: $\models\ulcorner(\mu)\,(\nu)R\supset(\nu)\,(\mu)R\urcorner$.

M 102: $\models\ulcorner(\mu)\,(R\supset P)\supset\cdot(\mu)R\supset(\mu)P\urcorner$.

M 103: *If μ is not free in R,* $\models\ulcorner R\supset(\mu)R\urcorner$.

M 104: *If R' is like R except for containing free occurrences of ν wherever R contains free occurrences of μ, then* $\models\ulcorner(\mu)R\supset R'\urcorner$.

M 105: *If $\ulcorner R\supset P\urcorner$ and R are meta-theorems, so is P.*

From these follow metatheorems corresponding to the usual theorems of quantification theory.

B. Axiom of Identity:

M 1: *If R is like P except for containing free occurrences of ν in place of some free occurrences of μ,* $\models\ulcorner\mu=\nu\cdot\supset\cdot P\equiv R\urcorner$.

C. Axioms of Existence:[13])

M 2: $\models\ulcorner E!\,S_1\urcorner$

M 3: $\models\ulcorner E!\,S_2\urcorner$

. . . .

$M_{(10+n)}$: $\models\ulcorner E!\,S_{9+n}\urcorner$

$M_{(11+n)}$: $\models\ulcorner E!\,\mu\cdot E!\nu\cdot\supset\cdot E!(\mu^\frown\nu)\urcorner$

$M_{(12+n)}$: $\models\ulcorner\mu^\frown\nu\neq S_1\cdot\ldots\cdot\mu^\frown\nu\neq S_{9+n}\cdot S_1\neq S_2\cdot S_1\neq S_3\cdot\ldots\cdot S_{8+n}\neq S_{9+n}\cdot \mu^\frown\nu\neq\mu\cdot\mu^\frown\nu\neq\nu\cdot\gamma^\frown\mu^\frown\nu\neq\mu\urcorner$.

D. Induction.

In order to state this axiom without the use of set theory in the syntax it is necessary to set up a model for the natural numbers in the syntax. That this can be done in a syntax such as the one used here was shown by Quine[14]) who gives a definition of ⌜*α is a natural number*⌝ (abbrev. $\ulcorner Nn\ \alpha\urcorner$) in terms of concatenation.

[12]) A meta-meta-mathematical definition of „*R is a formula of syntax*“ can of course easily be given, but it is omitted here, since the meaning of the phrase is evident. A similar remark applies to the term „closure“.

[13]) Cf. Tarski [11].

[14]) Cf. Quine [8].

Writing $\ulcorner R_\alpha\urcorner$ to denote a formula of the meta-language in which the natural number α occurs as a free variable, the induction postulate reads:

$M_{(13+n)}$: *If R_μ, $R_{\alpha+1}$ and R_1 are like R_α except for containing free occurrences of μ, $\alpha+1$ and 1 resp. wherever R_α contains free occurrences of α, then*

$$\models\ulcorner(\alpha)\,(Nn\ \alpha\cdot R_\alpha\cdot\supset R_{\alpha+1})\cdot R_1\cdot Nn\ \mu\cdot\supset R_\mu)\urcorner.$$

Induction is needed as an axiom since it is used in the derivation of many meta-theorems[15]).

D. Axiom of Consistency.

$M_{(14+n)}$: $\models\ulcorner(\exists\,\Phi)\,(Stat\ \Phi\cdot\sim Th_S\,\Phi)\urcorner$.

This last axiom states that not all statements are theorems of (S), which is equivalent to the assertion that (S) is consistent.

The predicate „T“ of statements of (S_ε) which plays such an essential rôle in the construction of the model will now be defined.

The statements of the object language of (S_ε) can be written down in order, for instance arranged lexicographically. „T“ is then inductively defined as follows[16]):

(i) *If Φ_1 is the first element of this sequence of statements,*

$$T\Phi_1\equiv\sim Th_{S_\varepsilon}(\sim\Phi_1).$$

(ii) *If $\Phi_1,\ldots,\Phi_k$ are all the statements preceding Φ_n for which $T\Phi_i$, $(1\leqslant i\leqslant k)$, then*

$$T\,\Phi_n\cdot\equiv\cdot\sim Th_{S_\varepsilon}(\Phi_1\cdot\ldots\cdot\Phi_k\cdot\supset\cdot\sim\Phi_n).$$

This can be expressed formally in syntax without using set-theoretic notions, but a number of preparatory definitions are necessary:

⌜*ζ is a single sign*⌝:

$$\Delta\ SSg\!:\quad \ulcorner SSg\ \zeta\urcorner\quad\text{for}\quad \ulcorner\zeta=S_1\cdot\vee\cdot\ldots\cdot\vee\cdot\zeta=S_{11+n}\urcorner.$$

[15]) Cf. Quine [7], p. 90, proof of *111, p. 91, proof of *112, p. 99, proof of *121 and many others.

[16]) Cf. Tarski [10], p. 394, Satz I. 56 (due to Lindenbaum). See also note 3 on p. 109.

The following definitions are based on the notion of „*framed ingredients*" due to Quine[17]. If ζ is an expression (not necessarily a formula) of the meta-language, then η is a framed ingredient of ζ (abbrev. $\ulcorner \eta\ Ing\ \zeta \urcorner$) if it is a part of ζ preceded and followed by the expression $S_6{}^\frown S_7$ (i. e. a pair of empty parentheses), but does not itself have the expression $S_6{}^\frown S_7$ as part. By this device, we can, instead of referring to the elements μ_i of the sequence of expressions $\mu_1, \ldots, \mu_n$ speak of the framed ingredients of the expression $S_6{}^\frown S_7{}^\frown \mu_1{}^\frown S_6{}^\frown S_7{}^\frown \mu_2{}^\frown S_6{}^\frown S_7 \ldots \mu_n{}^\frown S_6{}^\frown S_7$. In contrast to the notions of „sequence" and „elementhood", $\ulcorner \eta\ Ing\ \zeta \urcorner$ is definable in a syntax such as the one presented here. Framed ingredients are used repeatedly in the following definitions:

$\ulcorner$*the framed ingredient* η *of* ζ *immediately precedes the framed ingredient* η' *of* $\zeta\urcorner$:

Δ *ImPr*: $\ulcorner \eta\ ImPr_\zeta\ \eta' \urcorner$ for $\ulcorner(\eta\ Ing \zeta \cdot \eta'\ Ing\ \zeta \cdot \eta^\frown S_6{}^\frown S_7{}^\frown \eta' P \zeta)\urcorner$

$\ulcorner\zeta$ *is the first framed ingredient of* $\eta\urcorner$:

Δ *FIng*: $\ulcorner \zeta\ FIng\ \eta \urcorner$ for $\ulcorner \zeta\ Ing\ \eta \cdot (\mu)\ (\mu\ Ing\ \eta \cdot \mu \neq \zeta : \supset \cdot \zeta\ Pr_\eta\ \mu)\urcorner$.

$\ulcorner\zeta$ *is the last framed ingredient of* $\eta\urcorner$:

Δ *LIng*: $\ulcorner \zeta\ LIng\ \eta \urcorner$ for $\ulcorner \zeta\ Ing\ \eta \cdot (\mu)\ (\mu\ Ing\ \eta \cdot \mu \neq \zeta : \supset \cdot \mu\ Pr_\eta\ \zeta)\urcorner$.

$\ulcorner\zeta$ *is the next but one ingredient after* η *in* $\delta\urcorner$:

Δ *NIng*: $\ulcorner \zeta\ NIng_\delta\ \eta \urcorner$ for $\ulcorner \zeta\ Ing\ \delta \cdot \eta\ Ing\ \delta \cdot (\exists \mu)\ (\mu\ Ing\ \delta \cdot \cdot S_6{}^\frown S_7{}^\frown \mu^\frown S_6{}^\frown S_7{}^\frown \eta P \delta)$.

ζ is shorter than η if there is a sequence δ such that (i) the first and last elements of δ begin η, (ii) if μ is an element of δ which begins η, then the next but one element following μ also begins η, but contains one more sign, (iii) if μ is an element of δ which begins ζ, then the next but one element following μ also begins ζ, but contains one more sign, and (iv) if μ is the second element of the sequence, then μ is the first sign beginning ζ, i. e.:

$\ulcorner\zeta$ *is shorter than* $\eta\urcorner$:

Δ *sh*: $\ulcorner \zeta\ sh\ \eta \urcorner$ for $\ulcorner(\exists \delta)\ ((\mu)\ (\mu\ FIng\ \delta \cdot \vee \cdot \mu\ LIng\ \delta \cdot \supset \cdot \mu B \eta) \cdot (\mu)\ (\mu\ Ing\ \delta \cdot \mu B \eta \cdot \supset \cdot (\gamma)\ (\gamma\ NIng_\delta\ \mu \cdot \supset \cdot \gamma B \eta \cdot (\exists \beta)\ (SSg\ \beta \cdot \gamma = \mu^\frown \beta))) \cdot (\mu)\ (\mu\ Ing\ \delta \cdot \mu B \zeta \cdot \supset \cdot (\gamma)\ (\gamma\ NIng_\delta\ \mu \cdot \supset \cdot \gamma B \zeta \cdot (\exists \beta)\ (SSg\ \beta \cdot \gamma = \mu^\frown \beta))) \cdot (\mu)\ (\gamma)\ (\mu\ Ing\ \delta \cdot \gamma\ FIng\ \delta \cdot \gamma\ ImPr_\delta \mu \cdot \supset \cdot \gamma B \zeta \cdot SSg\ \gamma))\urcorner$.

[17]) Cf. Quine [7], p. 296. This notion is there defined in a syntax which also contains no set theory and is virtually identical with the one presented here.

$\ulcorner\gamma$ *precedes* δ *in the finite alphabet of* $n+11$ *signs*$\urcorner$:

Δ *AP*: $\ulcorner \gamma AP\ \delta)\urcorner$ for $\ulcorner(\gamma = S_1 \cdot (\delta = S_2 \cdot \vee \cdot \delta = S_3 \cdot \vee \cdot \ldots \cdot \vee \cdot \delta = S_{11+n})) \vee (\gamma = S_2 \cdot (\delta = S_3 \cdot \vee \cdot \delta = S_4 \cdot \vee \cdot \ldots \cdot \vee \cdot \delta = S_{11+n})) \vee (\ldots) \vee (\gamma = S_{10+n} \cdot \delta = S_{11+n})\urcorner$.

$\ulcorner$*the statement* ζ *precedes the statement* $\eta\urcorner$:

Δ *Prec*: $\ulcorner \zeta\ Prec\ \eta \urcorner$ for $\ulcorner eStat\,\zeta \cdot eStat\,\eta \cdot \zeta sh\ \eta \cdot \vee (\sim \zeta\ sh\ \eta \cdot \sim \eta\ sh\ \zeta \cdot (\exists \mu)\ (\exists \gamma)\ (\exists \gamma')\ (\exists \delta)\ (\exists \delta')\ (\delta AP\ \delta' \cdot (\zeta = \mu^\frown \delta^\frown \gamma \cdot \eta = \mu^\frown \delta'^\frown \gamma' \cdot \vee \cdot \zeta = \mu^\frown \delta \cdot \eta = \mu^\frown \delta' \cdot \vee \cdot \zeta = \delta^\frown \gamma \cdot \eta = \delta'^\frown \gamma')))\urcorner$.

ζ will be the conjunction of all the framed ingredients of η if it is the last element in the sequence:

$$\eta_1, \quad \eta_2, \quad \eta_1 \cdot \eta_2, \quad \eta_3, \quad \eta_1 \cdot \eta_2 \cdot \eta_3, \ldots, \eta_1 \cdot \eta_2 \cdot \ldots \cdot \eta_n$$

where η_i is the i^{th} framed ingredient of η; i. e. if there is a sequence γ such that (i) the first element of γ is also the first element of η, (ii) the second element of γ is the second element of η, (iii) if μ is any element of γ, then either (a) μ is the conjunction of its 2 predecessors and the element preceding μ is the immediate predecessor in η of the element following μ, or (b) μ is an element of η and the element following μ is the conjunction of μ and predecessor and the element next but one after μ in γ is the element of η which follows μ, (iv) ζ is the last element of this sequence:

$\ulcorner\zeta$ *is the conjunction of the framed ingredients of* $\eta\urcorner$:

Δ *Conj*: $\ulcorner \zeta\ Conj\ \eta \urcorner$ for $\ulcorner(\exists \gamma)\ ((\mu)\ (\mu\ FIng\ \gamma \cdot \equiv \cdot \mu\ FIng\ \eta) \cdot (\beta)\ ((\mu)\ (\mu\ FIng\ \gamma \cdot \supset \cdot \mu\ ImPr_\gamma\ \beta) \cdot \equiv \cdot (\delta)\ (\delta\ FIng\ \eta \cdot \supset \cdot \delta\ ImPr_\eta\ \beta)) \cdot (\mu)\ (\mu\ Ing\ \gamma \cdot \supset : \cdot (\beta)\ (\delta)\ (\theta) (\delta\ Ing\ \eta \cdot \beta\ ImPr_\gamma\ \delta \cdot \delta\ ImPr_\gamma\ \mu \cdot \mu\ ImPr_\gamma\ \theta \cdot \supset \cdot \mu = (\beta \cdot \delta) \cdot \delta\ ImPr_\eta\ \theta) : \vee : (\beta)\ (\delta)\ (\theta)\ (\delta\ ImPr_\gamma\ \mu \cdot \mu\ ImPr_\gamma\ \theta \cdot \mu\ Ing\ \eta \cdot \theta\ ImPr_\gamma\ \beta : \supset \cdot \mu\ ImPr_\eta\ \beta \cdot \theta = (\mu \cdot \delta))) \cdot \zeta\ LIng\ \gamma)\urcorner$.

Then $T\zeta$ if there is a sequence γ such that (i) every element μ of γ is a statement and if β precedes μ in γ, then β precedes μ in lexicographic order, (ii) if μ is the first statement in this ordering, then μ is an element of γ if and only if the negation of μ is not a theorem, (iii) if μ is any other element of the sequence and δ is the sequence of elements of γ preceding μ, then $\ulcorner Conj\ \delta \cdot \supset \cdot \sim \mu \urcorner$ is not a theorem, but if β is any statement preceding μ in lexicographic order and not an element of γ, then $\ulcorner Conj\ \delta \cdot \beta \cdot \supset \cdot \sim \mu \urcorner$ is a theorem, (iv) ζ is the last element of that sequence:

ΔT: $\ulcorner T\zeta \urcorner$ for $\ulcorner (\exists\gamma)\,((\mu)\,(\mu\, Ing\, \gamma \cdot \supset \cdot Stat\, \mu : (\beta)\,(\beta\, Pr_\gamma\, \mu \cdot \supset \cdot \beta\, Prec\, \mu)) \cdot (\mu)\,(eStat\, \mu \cdot (\beta)\,(eStat\, \beta \cdot \supset \cdot \mu\, Prec\, \beta \vee \mu = \beta) \cdot \supset : \mu\, Ing\, \gamma \cdot \equiv \cdot \sim Ths_\varepsilon \sim \mu) \cdot (\mu)\,(\mu\, Ing\, \gamma \cdot \sim \mu\, FIng\, \gamma \cdot \supset : (\delta)\,(\delta P \gamma \cdot \sim \mu P \delta \cdot \delta B \gamma \cdot \supset \sim Ths_\varepsilon\, (Conj\, \delta \cdot \supset \cdot \sim \mu)) \cdot (\beta)\,(\beta\, Prec\, \mu \cdot eStat\, \beta \cdot \sim \beta\, Pr_\gamma\, \mu \cdot \supset \cdot (\exists\eta)\,(\eta P \gamma \cdot \sim \mu P \gamma \cdot \eta B \gamma \cdot Ths_\varepsilon\, (Conj\, \eta \cdot \delta \cdot \supset \cdot \sim \mu)) \cdot \zeta\, LIng\, \gamma\,) \urcorner$.

An immediate consequence of this definition is that whenever Φ is a theorem, then $T\Phi$, i. e.

(I) $\models \ulcorner (\Phi)\,(Ths_\varepsilon\, \Phi \cdot \supset \cdot T\Phi) \urcorner$.

Also

(II) $\models \ulcorner (\Phi)\,(\sim T(\sim \Phi) \cdot \equiv \cdot T\Phi) \urcorner$

and

(III) $\models \ulcorner (\Phi)\,(\Psi)\,(T(\Phi \cdot \Psi) \cdot \equiv \cdot T\Phi \cdot T\Psi) \urcorner$.

Proof of (II). Φ is shorter than $\sim\Phi$, so that if $T\Phi$, Φ is contained in the conjunction Ψ of all statements X such that TX and X precedes $\sim\Phi$. Hence $(\Psi \supset \sim(\sim\Phi))$ is a theorem, so that $\sim(T\sim\Phi)$. Conversely, if $\sim T(\sim\Phi)$, then $Ths_\varepsilon(\Psi \supset \sim(\sim\Phi))$, i. e. $Ths_\varepsilon(\Psi \supset \Phi)$ and since Φ precedes $\sim\Phi$, $\sim Ths_\varepsilon(\Psi' \supset \sim\Phi)$, where Ψ' is the conjunction of all statements X′ such that TX′ and X′ precedes Φ. Hence $T\Phi$.

Proof of (III). Let Ψ_1, Ψ_2, Ψ_3 be the conjuncts of all statements X such that TX and X precedes Φ, Ψ, $(\Phi \cdot \Psi)$ resp. If $T(\Phi \cdot \Psi)$ but $\sim T\Phi$, then $Ths_\varepsilon(\Psi_1 \supset \sim\Phi)$. Φ however precedes $(\Phi \cdot \Psi)$, so that every statement in Ψ_1 is also in Ψ_3, i. e. $Ths_\varepsilon(\Psi_3 \supset \Psi_1)$. Hence $Ths_\varepsilon(\Psi_3 \supset \sim\Phi)$, and $Ths_\varepsilon(\Psi_3 \supset \sim(\Phi \cdot \Psi))$, and therefore $\sim T(\Phi \cdot \Psi)$. This shows that if $T(\Phi \cdot \Psi)$, then $T\Phi$. Similarly one can show that if $T(\Phi \cdot \Psi)$, then $T\Psi$. If therefore $T(\Phi \cdot \Psi)$, then $T\Phi$ and $T\Psi$. Conversely, if $T\Phi \cdot T\Psi$ but $\sim T(\Phi \cdot \Psi)$, then $Ths_\varepsilon(\Psi_3 \cdot \supset \cdot \sim(\Phi \cdot \Psi))$. But Φ and Ψ are components of the conjunction Ψ_3, so that then $Ths_\varepsilon(\Psi_3 \cdot \Phi \cdot \Psi \cdot \supset \cdot \sim(\Phi \cdot \Psi))$, which is a contradiction. Hence $T\Phi$ and $T\Psi$ together imply $T(\Phi \cdot \Psi)$.

These proofs depend on the manner in which the statements are ordered, but it is easy to see that analogous proofs hold for any ordering of the statements. They also depend essentially on the hypothesis that (S) is consistent: For they both use the hypothesis that if a statement of (S_ε) is provable, then its denial is not provable, which is equivalent to assuming (S_ε) consistent, and therefore that (S) is consistent.

(I), (II) and (III) show that the class of statements Φ such that $T\Phi$ is an extension of the class of all theorems of (S_ε) to a complete class. Although $\ulcorner T\Phi \urcorner$ is here defined for statements of (S_ε), a similar definition could be made for the statements of other consistent systems.

Corollaries of (I), (II) and (III) are:

(IV) $\models \ulcorner (\Phi)\,(eStat\, \Phi \cdot \supset \cdot T\Phi \vee T(\sim\Phi)) \urcorner$

(V) $\models \ulcorner (eStat\, \Phi \cdot \supset \cdot \sim (T\Phi \cdot T(\sim\Phi)) \urcorner$

(VI) $\models \ulcorner T(\Phi \vee \Psi) \cdot \equiv \cdot T\Phi \vee T\Psi \urcorner$

(VII) $\models Ths(\Phi \supset \Psi) \cdot T\Phi : \supset \cdot T\Psi \urcorner$.

§ 3. Models.

Many definitions of what is meant by a model of a system have been given. Most of these however involve semantic ideas such as satisfaction and truth, which I have tried to avoid here [18]). In terms of strictly syntactic notions (using this narrow sense of the word „syntax“), a great many definitions are still possible though they are not as satisfactory. Two such definitions are suggested here.

Definition 3.1. A set of statements (S_M) forms a *pseudo-model* for the system (S), if to every statement Φ of (S) there corresponds a unique statement Φ_M of (S_M) such that (i) if Φ corresponds to Φ_M and Ψ to Ψ_M, then $(\Phi \downarrow \Psi)$ corresponds to $(\Phi_M \downarrow \Psi_M)$, (ii) if $Ths\,\Phi$, then $Ths_M\,\Phi_M$.

This is rather a liberal definition, but sufficient for investigations of consistency. For if (S) is inconsistent, then there exists a statement Φ of (S) such that both Φ and $\sim\Phi$ are theorems of (S). But condition (i) of the above definition assures us that if Φ corresponds to Φ_M, then $\sim\Phi$ corresponds to $\sim\Phi_M$, and Φ and $\sim\Phi$ are both theorems, so that by (ii), Φ_M and $\sim\Phi_M$ must both be theorems of (S_M). Hence (S_M) is also inconsistent. Existence of a consistent model in this sense therefore assures the consistency of (S).

Definition 3.2. A set of statements (S_M) forms a *real model* of (S), if (i) (S_M) is a pseudo-model of (S), and (ii) if from $Ths\,\Phi$ follows $Ths\,\Psi$ in (S), then from $Ths_M\,\Phi_M$ follows $Ths_M\,\Psi_M$ in (S_M).

A more conventional definition which however depends on the notion of truth would be the following [18]):

[18]) Cf. for instance Kemeny [6] or Tarski [12].

Definition 3.3. A set of statements (S_M) forms a *model* of (S) if (i) to every statement Φ of (S) there corresponds a unique statement Φ_M of (S_M), (ii) if Φ is an axiom of (S), then Φ_M is true in (S_M), (iii) if Φ can be deduced from $\Phi_1, \ldots, \Phi_n$ and if $\Phi_{1,M}, \ldots, \Phi_{n,M}$ are true in (S_M), then Φ_M is true in (S_M).

Here a model is defined to be a set of statements, and therefore a fortiori a denumerable set. Usually, a model is defined to be a set of elements satisfying certain conditions, whereas here a model is taken to be the set of statements about such a set of elements. By a *denumerable model* is therefore meant a set of statements about a denumerable set of elements.

Definition 3.2 is in a certain sense more restrictive than definition 3.3, for it not only requires all axioms and theorems of the model to be true, but also that they be theorems of the model. The model of (S') to be constructed will satisfy this condition.

If (S) and (S_M) have a common syntax, then both of these definitions can easily be seen to be definable in that syntax, for all the concepts used in these definitions are syntactically definable and the definitions of § 2 carry over (mutatis mutandis). The definitions are however stated here in an informal meta-language, for we do not want to exclude the possibility that the object language of (S_M) should be the meta-language of (S), or perhaps even the meta-meta-language of (S), etc., nor that they might be two quite unrelated languages. In the case here, the language of (S_M) will be part of the syntax of (S), but by arithmetization this could of course be made part of the object language of (S), provided (S) contains arithmetic. Even if (S) does not contain arithmetic, (S_M) can still be represented in the object language of a system which does contain arithmetic.

§ 4. Construction of the model.

From now on, let us use the variables Φ_ε, Ψ_ε, X_ε for ε-terms. With help of the Hilbert-Bernays theory of ε-terms, the following two theorems can be proved:

(4.1) $$\models \ulcorner Th_S (\exists \alpha)\, \Phi \cdot \supset \cdot (\exists X_\varepsilon)\, (Th_{S_\varepsilon} S_\alpha^{X_\varepsilon}\, \Phi) \urcorner$$

(4.2) $$\models \ulcorner Th_S (\alpha)\, \Phi \cdot \supset \cdot (X_\varepsilon)\, (Th_{S_\varepsilon} S_\alpha^{X_\varepsilon}\, \Phi) \urcorner.$$

Proof of (4.1). Since (S_ε) contains all the axioms and statements of (S),

(i) $$\models \ulcorner Th_S (\exists a)\, \Phi \cdot \supset \cdot Th_{S_\varepsilon} (\exists a)\, \Phi \urcorner.$$

The fundamental condition on ε-terms can be stated as

(ii) $$\models \ulcorner Th_{S_\varepsilon} ((\exists \alpha)\, \Phi \cdot \supset \cdot S_\alpha^{\varepsilon\alpha\Phi}\, \Phi) \urcorner.$$

Combining (i) and (ii), we obtain (4.1).

Proof of (4.2). By (i),

(iii) $$\models \ulcorner Th_S (\alpha)\, \Phi \cdot \supset \cdot Th_{S_\varepsilon} (\alpha)\, \Phi \urcorner.$$

From one of the theorems of quantification theory follows that if $Th_{S_\varepsilon}(\alpha)\,\Phi$ and α' is an element in the range of α, then $Th_{S_\varepsilon} S_\alpha^{\alpha'}\, \Phi$, i. e.

(iv) $$\models \ulcorner Th_{S_\varepsilon} (\alpha)\, \Phi \cdot \supset \cdot (\alpha')\, (Th_{S_\varepsilon} (\exists \alpha)\, (\alpha = \alpha') \cdot \supset \cdot Th_{S_\varepsilon} S_\alpha^{\alpha'}\, \Phi) \urcorner.$$

But one of the fundamental theorems on ε-terms states that

(v) $$(X_\varepsilon)\, (Th_{S_\varepsilon} (\exists\, \alpha)\, (\alpha = X_\varepsilon)),$$

so that on combining (iii), (iv) and (v),

$$\models \ulcorner Th_S (\alpha) \Phi \cdot \supset \cdot (X_\varepsilon)\, (Th_{S_\varepsilon} S_\alpha^{X_\varepsilon} \Phi) \urcorner.$$

The following lemma, though at present irrelevant, will prove useful (cf. step 3 below):

Lemma: *If $\ulcorner (Q) \urcorner$ is a string of quantifiers, R, S are statements of the meta-language and $\models \ulcorner (R \supset S) \urcorner$, then $\models \ulcorner ((Q) R \supset (Q) S) \urcorner$.*

Proof. By induction on the number of quantifiers in $\ulcorner (Q) \urcorner$, using the following two theorems of quantification theory [19]):

Theorem A: *If $\models \ulcorner (R \supset S) \urcorner$, then $\models \ulcorner (\mu) R \supset (\mu) S \urcorner$.*

Theorem B: *If $\models$ $(R \supset S)$, then $\models \ulcorner (\exists \mu) R \supset (\exists \mu) S \urcorner$.*

Transformation of Axioms and Theorems:

Each step in the construction of the sentence Φ_M of the model corresponding to the sentence Φ of (S) will first be described and then carried out taking Φ as

$$(\alpha)\, (\exists\, \beta) (\alpha = \beta \cdot \vee \cdot \sim (\alpha = \alpha)).$$

[19]) Cf. Quine [7], *102, p. 88, *149, p. 107.

It is here assumed that Φ is a theorem of (S) and that (S) has identity as a primitive connective.

Step 1: Replace the sentence under consideration by an equivalent one in praenex form. This is always possible[20]. Since the sentence Φ here is already in praenex form, step 1 may be omitted in this particular case.

Step 2: Use (4.1) and (4.2) repeatedly until the part of the axiom following „Th_S" contains no explicit quantifiers[21], i. e.

$$Th_S(\alpha)(\exists\beta)(\alpha=\beta\cdot\vee\cdot\sim(\alpha=\alpha))$$

becomes

$$(\mathrm{X}_e)\,(Th_{S_e}(\exists\beta)(\mathrm{X}_e=\beta\cdot\vee\cdot\sim(\mathrm{X}_e=\mathrm{X}_e))),$$

and finally

$$(1)\qquad (\mathrm{X}_e)(\exists\Psi_e)(Th_{S_e}(\mathrm{X}_e=\Psi_e\cdot\vee\cdot\sim(\mathrm{X}_e=\mathrm{X}_e))).$$

Step 3: Combining the lemma and the fact that $\models\ulcorner Th_{S_e}\Phi\supset T\Phi\urcorner$, replace „$Th_{S_e}$" by „$T$" in the statement obtained in step 2. For example, from (1) and

$$\models\ulcorner(Th_{S_e}(\mathrm{X}_e=\Psi_e\cdot\vee\cdot\sim(\mathrm{X}_e=\mathrm{X}_e))\supset T(\mathrm{X}_e=\Psi_e\cdot\vee\cdot\sim(\mathrm{X}_e=\mathrm{X}_e)))\urcorner$$

follows

$$(2)\qquad \models\ulcorner(\mathrm{X}_e)(\exists\Psi_e)(T(\mathrm{X}_e=\Psi_e\cdot\vee\cdot\sim(\mathrm{X}_e=\mathrm{X}_e)))\urcorner.$$

Step 4: Using the distributive properties of „T" (cf. 2, (II) and (III)), distribute it until the expression „governed" by „T" contains no explicit truth-functional connectives, e. g. (2) becomes

$$\models\ulcorner(\mathrm{X}_e)(\exists\Psi_e)(T(\mathrm{X}_e=\Psi_e)\cdot\vee\cdot T(\sim(\mathrm{X}_e=\mathrm{X}_e)))\urcorner$$

and then

$$(3)\qquad \models\ulcorner(\mathrm{X}_e)(\exists\Psi_e)(T(\mathrm{X}_e=\Psi_e)\cdot\vee\cdot\sim T(\mathrm{X}_e=\mathrm{X}_e)).$$

Step 5: This step is really superfluous, but in order to make the statements obtained in step 4 look more like the original and exhibit their relationship more clearly, write

$$\ulcorner(\mathrm{X}=_M\Psi)\urcorner \quad\text{for}\quad \ulcorner T(\mathrm{X}=\Psi,\urcorner,$$
$$\ulcorner(\Delta_M\mathrm{X})\urcorner \quad\text{for}\quad \ulcorner T(\Delta\mathrm{X})\urcorner,$$

[20]) Cf. Hilbert-Ackermann [4], p. 67.

[21]) After the first quantifiers has been taken out, the following versions of (4.1) and (4.2) are used:

$$4.1':\qquad \models\ulcorner Th_{S_e}(\exists\alpha)\Phi\cdot\supset\cdot(\exists\mathrm{X}_e)(Th_{S_e}S_{\alpha}^{\mathrm{X}_e}\Phi)$$

$$4.2':\qquad \models\ulcorner Th_{S_e}(\alpha)\Phi\cdot\supset\cdot(\mathrm{X}_e)\,(Th_{S_e}S_{\alpha}^{\mathrm{X}_e}\Phi).$$

These follow immediately from (ii), (iv) and (v).

and so on for the other primitive connectives and substitute then in the statement obtained in step 4. Thus (3) becomes

$$(4)\qquad \models\ulcorner(\mathrm{X}_e)(\exists\Psi_e)(\mathrm{X}_e=_M\Psi_e\cdot\vee\cdot\sim(\mathrm{X}_e=_M\mathrm{X}_e))\urcorner.$$

The statement so obtained will be the $\ulcorner\Phi_M\urcorner$ corresponding to Φ. Note that although Φ is a theorem of the object language, $\ulcorner\Phi_M\urcorner$ is a sentence of the meta-language, and in fact a meta-theorem. A model (S_M) of (S) is then formed by all the statements $\ulcorner\Phi_M\urcorner$ which can be obtained by such transformations from theorems of (S). To see this, we set up a correspondence between statements of (S) and of (S_M) as follows: If Φ is a statement of (S) containing the variables a, b ... and the connectives $\Delta,\ldots,{}^*$, then the corresponding statement $\ulcorner\Phi_M\urcorner$ of (S_M) will have in their place $\mathrm{X}'_e,\mathrm{X}''_e,\ldots$ and $\Delta_M,\ldots,{}^*_M$ resp., distinct variables being replaced by distinct variables. Truth functional connectives and parentheses remain unchanged. (S_M) may be shown to be a model of (S) by the methods used below to show that (S'_M) is a model of (S').

(S_M) must now be enlarged to a model (S'_M) of (S'). To do this, we consider axiom- and theorem-schemata which can be written in the form

$$\models\ulcorner(\Phi_1)(\ldots)(\exists\Phi_k)\,Th_S((\alpha_1)(\ldots)(\exists\alpha_n)\Psi)\urcorner,$$

where $\ulcorner(\Phi_1)(\ldots)(\exists\Phi_k)\urcorner$ represents a string of quantifiers, possibly existential and universal mixed, $\Phi_1,\ldots,\Phi_k$ are matrices each containing one free variable, Ψ is a formula of (S) which may contain $\Phi_1,\ldots,\Phi_k$ as parts and in which $\alpha_1,\ldots,\alpha_k$ occur as free variables but in which no others are free, and $(\alpha_1)(\ldots)(\exists\alpha_n)$ is a string of quantifiers (again possibly universal and existential mixed).

We shall from now on assume that all matrices $\Phi,\Psi,\ldots$ contain exactly one free variable.

An example of this type of schema is given by

$$(5)\qquad \models\ulcorner(\Phi)(\exists\Psi)\,(Th_S(\alpha)(S^{\alpha}\Phi\cdot\equiv\cdot\sim S^{\alpha}\Psi))\urcorner.$$

The result of transforming (5) is

$$(6)\qquad \models\ulcorner(\Phi)(\exists\Psi)(\mathrm{X}_e)\,(T(S^{\mathrm{X}_e}\Phi)\cdot\equiv\cdot\sim T(S^{\mathrm{X}_e}\Psi))\urcorner$$

or, writing $\ulcorner\Phi\circ_M\mathrm{X}_e\urcorner$ for $\ulcorner T(S^{\mathrm{X}_e}\Phi)\urcorner$:

$$(7)\qquad \models\ulcorner(\Phi)(\exists\Psi)(\mathrm{X}_e)(\Phi\circ_M\mathrm{X}_e\cdot\equiv\cdot\sim\Psi\circ_M\mathrm{X}_e)\urcorner.$$

The axiom schemata of Zermelo's system are of this form[22], as are many theorem schemata.

The statements of (S'_M), the model of (S') will then be taken to be all the statements of the syntax of (S) which can be obtained by such transformations from theorems and schemata of (S).

We now set up a correspondence between statements of (S') and of (S'_M) as follows: If Φ is a statement of (S') containing the variables $\mathrm{A}, \mathrm{B}, \ldots, \mathrm{a}, \mathrm{b}, \ldots$ and the connectives „Δ“, ..., „*“, „$\circ$“, „$=$“, then the corresponding statement Φ_M of (S'_M) will have in their place $\Psi_1, \Psi_2, \ldots, \mathrm{X}'_e, \mathrm{X}''_e, \ldots$, and „$\Delta_M$“, ..., „$*_M$“, „$\circ_M$“, „$=_M$“ resp., distinct variables of Φ being replaced by distinct variables. Truth-functional connectives and parentheses remain unchanged. „$\circ_M$“ and „$=_M$“ have not yet been defined in general. The necessary definitions and preparatory definitions follow. We write:

$$\ulcorner \Phi \circ_M X_e \urcorner \quad \text{for} \quad \ulcorner T(\Phi \circ X_e) \urcorner$$
$$\ulcorner \Phi = \Psi \urcorner \quad \text{for} \quad \ulcorner (\alpha)(S^\alpha \Phi \equiv S^\alpha \Psi) \urcorner$$
$$\ulcorner \Phi = \mathrm{X}_e \urcorner \quad \text{for} \quad \ulcorner (\alpha)(S^\alpha \Phi \equiv \cdot\, \alpha = \mathrm{X}_e) \urcorner$$
$$\ulcorner \Phi =_M \Psi \urcorner \quad \text{for} \quad \ulcorner T(\Phi = \Psi) \urcorner$$
$$\ulcorner \Phi =_M \mathrm{X} \urcorner \quad \text{for} \quad \ulcorner T(\Phi = \mathrm{X}_e) \urcorner.$$

One can easily show in the syntax of (S) that

(8) $$\models \ulcorner (\exists \Phi)(Th_S(\alpha)(\Phi \circ \alpha \equiv S^\alpha \Psi)) \urcorner$$

simply by taking Φ as Ψ and applying M 104, writing $\Phi \circ \alpha$ for $S^\alpha \Phi$.

Transforming (8), imagining Ψ written out in full gives:

(9) $$\models \ulcorner (\exists \Phi)(\mathrm{X}_e)(\Phi \circ_M \mathrm{X}_e \equiv S^{\mathrm{X}_e} \Psi_M) \urcorner,$$

where the quantifiers of Ψ_M will all be restricted to ε-terms.

For example, the sentence of (S'_M) corresponding to the sentence

$$(\mathrm{A})(\exists \mathrm{B})(\mathrm{a})(\mathrm{A} \circ \mathrm{a} \equiv \cdot \sim \mathrm{B} \circ \mathrm{a})$$

would thus be

$$(\Phi)(\exists \Psi)(\mathrm{X}_e)(\Phi \circ_M \mathrm{X}_e \equiv \cdot \sim \Psi \circ_M \mathrm{X}_e).$$

[22]) Cf. Wang [13]. Z 6 on p. 151 can be written

$$\models \ulcorner (\Phi)(Ax_S(\mu)((\alpha)(\beta)(\gamma)((S^{\langle\alpha\beta\rangle}\Phi \cdot S^{\langle\alpha\gamma\rangle}\Phi) \supset \cdot\, \beta = \gamma)(\exists \delta)(\alpha)(\alpha \varepsilon \delta \equiv (\exists \beta)(\beta \varepsilon \mu \cdot S^{\langle\beta\alpha\rangle}\Phi))) \urcorner.$$

Evidently steps 1, ..., 5 apply to axioms as well as theorems, since all axioms are theorems. The correspondent to the transforms of this schema can easily be seen to be N 6 (Gödel's C 4).

Thus, except for alphabetic variation, each statement of (S') corresponds to a unique statement of (S'_M). Moreover, theorems and axioms of (S) are transformed into meta-theorems, that is, if Φ is a theorem of (S') and if its correspondent Φ_M is obtainable by transforming some theorem of (S), then Φ_M is a meta-theorem. It is however still necessary to show that the correspondents of all axioms and theorems of (S') can be obtained in this manner from axioms and schemata of (S) or (S_ε).

The axiom schema (1.1) (cf. § 1) of (S') corresponds to the schema of statements of (S'_M) of the same type as (9) above, and each statement of this form can be derived from a corresponding one of (S) of the form of (8). The axioms of identity of (S') correspond to

(10) $$\models \ulcorner (\Phi =_M \Psi) \cdot \equiv \cdot (\mathrm{X}_e)(\Phi \circ_M \mathrm{X}_e \equiv \Psi \circ_M \mathrm{X}_e) \urcorner$$

(11) $$\models \ulcorner (\mathrm{X}_e =_M \mathrm{X}'_e \cdot \supset \cdot \Phi \circ_M \mathrm{X}_e \equiv \Phi \circ_M \mathrm{X}'_e) \urcorner$$

respectively, which derive from the schemata

$$\models \ulcorner Th_S(\Phi = \Psi \cdot \equiv \cdot (\alpha)(S^\alpha \Phi \equiv S^\alpha \Psi) \urcorner$$

and

$$\models \ulcorner Th_S(\alpha)(\beta)(\alpha = \beta \cdot \supset \cdot S^\alpha \Phi \equiv S^\beta \Phi) \urcorner.$$

The correspondent of (1.4) however is not derived from a schema of (S), but from one of (S_ε), for in the syntax of (S_ε) we certainly have

$$\models \ulcorner (\mathrm{X}_e)(\exists \Phi)(Th_{S_\varepsilon}(\mathrm{X}_e = \Phi)) \urcorner,$$

simply by taking Φ as $(\alpha = \mathrm{X}_e)$ and applying M 104.

All the correspondents of axioms and schemata of (S') are thus seen to be metatheorems. (S'_M) will then be a real model of (S') if the rules of deduction of (S') and (S'_M) also correspond. But this can easily be seen to be the case, since in both systems the rules of deduction are based on quantification theory, as for instance determined by Quine's *100, ..., *105[23]) and their paraphrases in syntax M 100, ..., M 105 (cf. § 2). (S'_M) is therefore a real model of (S').

The statements of (S'_M) which contain only quantifiers restricted to ε-terms will correspond to statements of (S), and it is evident that theorems of (S) will, after transformation according

[23]) Cf. for instance Quine [7], Ch. II, for a development of first order functional calculus from these axioms.

to steps 1,...,5 correspond to metatheorems of (S'_M). Every theorem of (S) will have such a correspondent, since every theorem can be thus transformed. (S'_M) will therefore contain a subset of statements forming a model (S_M) of (S).

It may be interesting to note that the range of the variables in the model comprises in effect the „nameable" classes of (S_ε), if by a „nameable" class we mean one definable by a matrix of (S_ε). In other words, the nameable classes of (S_ε) form the universe of a model of (S').

§ 5. Existence of models and relative consistency.

In the statement of the following theorems we shall write „(S)" for a system satisfying the conditions of § 1, i. e. containing a finite number of connectives (among which must be „=" and the usual truth-functional connectives), and an infinite sequence of variables formalized within quantification and identity theory, and which is such that the definition of ⌜ζ *is an axiom of* (S)⌝ can be formalized in the syntax of (S). (The word „syntax" is again used in the sense of § 2 and denotes a meta-theory without set theory and semantic notions such as „truth", but containing the hypothesis „(S) *is consistent*"). (S_ε) will be the system obtained by enlarging (S) so as to include the Hilbert ε-operator, (S') the system obtained by enlarging (S) by adding a new type or variables and a new connective satisfying (1.1), (1.2), (1.3) and (1.4). Most of the following proofs will only be sketched.

Theorem I. *If the syntax of (S) is consistent, then (S') has a real consistent model in the syntax of (S_ε).*

Proof. The method described above shows how to construct such a model.

Theorem II. *If the syntax of (S) is consistent, then (S') has a denumerable real consistent model.*

Proof. For this syntax contains only a denumerable number of expressions.

Corollary II.1. *(S) has a denumerable real model.*

Theorem III. *If arithmetic of natural numbers and (S) are consistent, then (S') is consistent and has a denumerable consistent real model which is constructible in arithmetic under the arithmetical assumption equivalent to „(S) is consistent".*

Proof. The syntax of (S) without $M_{(14+n)}$ is equivalent to arithmetic based on the 5 Peano axioms [24]. In the syntax of (S) we can show that (S_ε) is consistent [25]. But within (S_ε) we can build a model of (S'), so that (S') must be consistent. The entire syntax of (S) is then equivalent to the arithmetic system obtained by adding the arithmetical equivalent of „(S) *is consistent*" to the 5 Peano axioms. By using Gödel arithmetization [26], this entire construction could have been carried out in arithmetic.

Corollary III.1. *If (S) contains arithmetic and (S) is consistent, then (S') is consistent.*

Corollary III.2. *If the Zermelo-Fraenkel system of set-theory is consistent, then so is the von Neumann-Bernays-Gödel system.*

Proof. Take (S) and (S') of Cor. III.1 as Zermelo-Fraenkel and von Neumann-Bernays-Gödel set theory resp. [27]).

Lemma I. *If (S) is complete, so is (S_ε).*

Proof. For the statements of (S_ε) are deductively equivalent to the statements of (S) [28]).

Definition 5.1. If two models of a system (S) are in the same language, they are called *distinct*, if there is a statement which is a theorem of one and not of the other.

Theorem IV. *If (S) is complete and consistent, then the real model of (S') obtained by the method of § 4 is uniquely determined.*

Proof. The only step in the construction which is arbitrary is the manner of ordering statements previous to defining „T". If (S) is complete, „T" is independent of the ordering of the statements, for this ordering determines only for which undecidable statements Φ „$T\Phi$" will hold.

Theorem V. *If (S) is incomplete, but has only a finite number N of distinct complete extensions, then there are N distinct models of (S) obtainable by the method of § 4.*

[24]) Cf. Tarski [12].
[25]) Cf. Hilbert-Bernays [5], § 1.
[26]) Cf. Gödel [3].
[27]) The formulation of these two systems by Wang [13] shows that the relation of (S) to (S') is the one required by the conditions of § 1.
[28]) Cf. Hilbert-Bernays [5], § 1.

Proof. Using Lemma I, it is easily seen that if (S) has this property, so does (S_ε). Let $(S_1), \ldots, (S_N)$ be the N complete extensions of (S_ε). For each of these there exists a unique predicate „T_i“ $(i=1,\ldots,N)$ and all these predicates are distinct (i. e. if $i \neq j$, then there is at least one statement Φ such that $T_i\Phi$ but not $T_j\Phi$), for the extensions $(S_1), \ldots, (S_N)$ are distinct. There are therefore at least N models. But for any such predicate „T“ the class of all statements such that $T\Phi$ is a complete extension of the class of all theorems of (S_ε), and therefore must coincide with the theorems of one of the systems (S_i). There are therefore exactly N different predicates „T“ and N different models of (S).

Theorem VI. *If (S) is incomplete but if there is a denumerable number of distinct consistent complete extensions of (S) then (S) has a denumerable number of real models obtainable by the method of § 4* [29].

Proof. For there is a one-to-one correspondence between models of (S) and distinct predicates „T“ and also between distinct predicates „T“ and complete extensions of (S).

Theorem VII. *If (S) is essentially incomplete, then there exists at least a continuum of denumerable models of (S)* (but only a denumerable number of these are actually definable in syntax (or arithmetic), since it is a denumerable system).

Proof: Assume (S) is essentially incomplete, and Φ_1 the first undecidable statement of (S) for some ordering of the statements of (S). Let (S, Φ_1) be the system obtained by adjoining Φ_1 to (S), $(S, \sim\Phi_1)$ the system obtained by adjoining $\sim\Phi_1$ to (S). These two new systems are also both essentially incomplete. Let Φ_2 be the next statement undecidable in (S, Φ_1), Φ_3 the next statement undecidable in $(S, \sim\Phi_1)$ [30]. Then the systems (S, Φ_1, Φ_2), $(S, \Phi_1, \sim\Phi_2)$, $(S, \sim\Phi_1, \Phi_3)$, $(S, \sim\Phi_1, \sim\Phi_3)$ are also all essentially undecidable. Continuing this procedure, we shall have 2^n different essentially undecidable systems after the n^{th} stage. But (S) was essentially incompletable, so $\aleph_0$ such steps are possible, resulting in a continuum of different complete extensions of (S), and therefore a continuum of models. But of course only a denumerable number of these models are constructible, if by „constructible“ we mean here that the required predicate can be defined in syntax.

[29]) Such a situation was described by Tarski [11]. In this case, all the required predicates „T“ can easily be defined in syntax. The same theorem will hold if there exists only a finite number of infinite extensions of (S).

[30]) Φ_2 and Φ_3 are not necessarily identical.

Notes added during proof. 1 (to p. 88). In the case of truth-functional connectives „§“ and „∘“ it was not possible to print the symbols in two different types. It is hoped that this will not prove too confusing in § 2 and § 4.

2 (to p. 92). Because of typographical difficulties, the corresponding truth-functional connectives which form statements from statements are however similarly shaped.

3 (to p. 95). Henkin [3a] uses essentially the same predicate. Henkin's paper gives another proof of the Skolem-Löwenheim theorem, which is however different from the one given here.

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Some Impredicative Definitions in the Axiomatic Set-Theory.

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Let (S) denote the Zermelo-Fraenkel set-theory based on the following axioms

(A_1) $(x_1, x_2)\,[(x_3)\,(x_3 \epsilon x_1 \equiv x_3 \epsilon x_2) \supset x_1 = x_2]$,

(A_2) $(x_1, x_2)\,(\exists x_3)\,(x_4)\,[x_4 \epsilon x_3 \equiv (x_4 = x_1 \vee x_4 = x_2)]$,

(A_3) $(x_1)\,(\exists x_2)\,(x_3)\,[x_3 \epsilon x_2 \equiv (x_4)\,(x_4 \epsilon x_3 \supset x_4 \epsilon x_1)]$,

(A_4) $(x_1)\,(\exists x_2)\,(x_3)\,[x_3 \epsilon x_2 \equiv (\exists x_4)\,(x_3 \epsilon x_4 \cdot x_4 \epsilon x_1)]$,

(A_5) $(\exists x_1)\,(\exists x_2)\,(x_2 \epsilon x_1 \cdot (x_2)\,\{x_2 \epsilon x_1 \supset (\exists x_3)\,[x_2 \neq x_3 \cdot x_3 \epsilon x_1$
$\cdot (x_4)\,(x_4 \epsilon x_2 \supset x_4 \epsilon x_3)]\})$,

(A_6) $(x_k)\,(x_{k_1}, \ldots, x_{k_p})\,\{(x_l)\,[x_l \epsilon x_k \supset (\exists x_m)\,(x_n)\,(\Phi \equiv x_n = x_m)] \supset$
$\supset (\exists x_q)\,(x_n)\,[x_n \epsilon x_q \equiv (\exists x_l)\,(x_l \epsilon x_k \cdot \Phi)]\}$,

(A_7) $(x_{k_1}, \ldots, x_{k_p})\,\{(\exists x_k)\Phi \supset (\exists x_k)\,[\Phi \cdot (x_l)\,(x_l \epsilon x_k \supset \sim \Phi')]\}$ [1]).

(A_6) and (A_7) are axiom schemata. The letter Φ in (A_6) replaces any expression (with free variables $x_l, x_n, x_{k_1}, \ldots, x_{k_p}$, and x_k [2]) built up according to the following rules: If i and j are integers, then $x_i \epsilon x_j$ and $x_i = x_j$ are formulas; if Θ is a formula and j an integer, then $(\exists x_j)\Theta$ is a formula; if Θ and Z are formulas, then so is $\Theta | Z$ [3]). We assume that x_q is not free in Φ.

The letter Φ in (A_7) replaces a formula with free variables $x_k, x_{k_1}, \ldots, x_{k_p}$ and Φ' replaces the formula resulting from Φ by substitution of the letter x_l for x_k on every place where x_k is free in Φ. It is supposed that x_l is not bound in Φ.

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[1]) (A_1) is the axiom of extensionality, (A_2) — the pair-axiom, (A_3) — the powerset axiom, (A_4) — the sum-set axiom, (A_5) — the axiom of infinity, (A_6) — the axiom of replacement, and (A_7) — the restrictive axiom (the „Axiom der Fundierung" of Zermelo).

[2]) x_k must not necessarily be a free variable of Φ.

[3]) Other logical connectives can be defined by the stroke | in the well-known manner.