2. *Proof of the theorem.* We now demonstrate the following statement of the axiom of choice:

If for each $a \in A$, X_a is a non-void set, then the Cartesian product $\mathbf{P}_{a\in A}X_a$ is non-void.

We begin by adjoining a single point, say Δ, to each of the sets X_a: Let $Y_a = X_a \cup \{\Delta\}$. We assign a topology for Y_a by defining the void set and complements of finite sets to be open. It is clear that Y_a, with this topology, is compact.

For each $a \in A$, let Z_a be that subset of $\mathbf{P}_{a\in A}Y_a$ consisting of all points whose a-th coordinate lies in X_a. Surely Z_a is closed in $\mathbf{P}_{a\in A}Y_a$ since X_a is closed in Y_a. Moreover, for any finite subset B of A the intersection $\bigcap_{a\in B}Z_a$ is non-void, for, since each X_a is non-void we may by the finite axiom of choice choose $x_a \in X_a$ for $a \in B$, and set $x_a = \Delta$ for $a \in A-B$. Consequently the family of all sets of the form Z_a, for some $a \in A$, is a family of closed subsets of $\mathbf{P}_{a\in A}Y_a$, with the property that the intersection of any finite subfamily is non-void. Hence, since by the Tychonoff Theorem $\mathbf{P}_{a\in A}Y_a$ is compact, the intersection $\bigcap_{a\in A}Z_a$ is non-void. But this intersection is precisely $\mathbf{P}_{a\in A}X_a$, and the axiom of choice is proved.

3. *Remarks.* It is of some interest to note, in the various proofs of Tychonoff's theorem, the precise lemmas which require the axiom of choice. In each of the proofs which have been published the axiom of choice is used in the proof of two distinct subsidiary propositions. In what is probably the most illuminating proof [4]), that of J. W. Alexander, these results are:

i) Let $\mathcal{S}$ be the family of subsets of a Cartesian product of compact spaces as defined in Section 1. Then every covering of the product by members of $\mathcal{S}$ has a finite subcovering.

ii) Let $\mathcal{R}$ be any family of sets with the property: any subfamily which covers the union $\bigcup_{A\in\mathcal{R}}A$ has itself a finite subfamily which also covers. Then the family $\mathcal{Q}$ of all finite intersections of members of $\mathcal{R}$ enjoys the same property.

Proposition i) implies the axiom of choice. Indeed, the above proof uses only i). However, I am unable to discover whether ii) does or does not imply the choice axiom.

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[4]) Proceedings of the National Academy of Sciences **25** (1939), pp. 296-298.

A Paradoxical Theorem.

By

J. Novák (Praha).

In this paper the following theorem is proved: B being an uncountable closed subset of the set C of all countable ordinal numbers, let $f(x)$ be a single-valued transformation of B onto $A\subset C$ having the property that $f(x)<x$ for all $x \in B$. Then there exists a countable ordinal number $\alpha \in A$ and an uncountable subset $B^*\subset B$ such that $f(x)=\alpha$ for all $x \in B^*$.

This theorem is used in the first instance to prove Theorem 2, which in a special case gives this paradoxical result: We take away one element s_1 from the given infinite countable set A_0, we add a new infinite countable set A_1 to the remainder, from the set $\bigcup_{\lambda<2}A_\lambda - \bigcup_{\lambda<2}s_\lambda$ we take away one element s_2, add a new infinite countable set A_2 and we continue in this way so that from the set $\bigcup_{\lambda<\alpha}A_\lambda - \bigcup_{\lambda<\alpha}s_\lambda$ (unless it is empty) we take away one element s_α and then we add a new infinite countable set A_α. Then there exists a countable ordinal number ϑ such that the set of all given and added elements is the same as the set of elements taken away i. e. $\bigcup_{\lambda<\vartheta}A_\lambda = \bigcup_{\lambda<\vartheta}s_\lambda$.

In the second instance the theorem mentioned above is used to prove Theorems 3 and 4, in which necessary and sufficient conditions are given for ordered continuum with the Souslin property (i. e. every disjoint system of intervals is countable) to be a linear set. One of these conditions is the existence of a rational dyadic partition of the ordered continuum with the Souslin property (Theorem 3) and the second condition is the existence of a closed dyadic partition (Theorem 4).

Theorem 1. *Let*

(1) $$\beta_0<\beta_1<\ldots<\beta_\lambda<\ldots$$

be an increasing sequence of ordinal numbers $\beta_\lambda<\Omega$ such that

(2) $$\lim \beta_\nu = \beta_{\lim \nu}$$

Ω being the first ordinal number of power $\aleph_1$ and $\lim \nu < \Omega$. *Let*

$$\alpha_0, \alpha_1, \ldots, \alpha_\lambda, \ldots \tag{3}$$

be a sequence of ordinal numbers $\alpha_\lambda < \Omega$ with the property

$$\alpha_\lambda < \beta_\lambda \tag{4}$$

for all $\lambda < \Omega$. Then there exists at least one ordinal number α_{λ_0} in the sequence (3) *such that $\alpha_{\lambda^*} = \alpha_{\lambda_0}$ for an uncountable number of λ^*.*

Proof. Suppose the contrary: every ordinal number α_λ appears in the sequence (3) a countable number of times. Let β_{λ_n} be any ordinal number contained in the sequence (1). The set of all ordinal numbers $\xi \leqslant \beta_{\lambda_n}$ being countable, and each of them appearing in (3) (according to our supposition) a countable number of times (or not appearing at all), because Ω is a regular ordinal number, there exists an index $\lambda_{n+1} > \lambda_n$ such that $\beta_{\lambda_n} < \alpha_\lambda$ for all $\lambda \geqslant \lambda_{n+1}$. Using the method of induction we can construct an ordinary sequence of ordinal numbers $\lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$ such that

$$\alpha_{\lambda_0} < \beta_{\lambda_0} < \alpha_{\lambda_1} < \beta_{\lambda_1} < \ldots < \alpha_{\lambda_n} < \beta_{\lambda_n} < \ldots$$

(here the α_{λ_n}-s are ordinal numbers taken from (3) and β_{λ_n}-s from (1) (satisfying the condition (4)) and

$$\beta_{\lambda_n} < \alpha_\lambda \quad \text{for all} \quad \lambda \geqslant \lambda_{n+1} \quad \text{and for every} \quad n = 0, 1, 2, \ldots$$

As $\lim \alpha_{\lambda_n} = \lim \beta_{\lambda_n}$ and $\lim \beta_{\lambda_n} = \beta_\gamma$ (according to (2)), where $\gamma = \lim \lambda_n$ and because $\alpha_\gamma < \beta_\gamma$ (according to (4)), there exists a natural number m such that $\alpha_\gamma < \beta_{\lambda_m}$. On the other hand $\gamma > \lambda_{m+1}$, whence $\beta_{\lambda_m} < \alpha_\gamma$, which contradicts the inequality $\alpha_\gamma < \beta_{\lambda_m}$. The theorem is thus proved.

Notes. Without the supposition (2) the theorem 1 does not hold as the following example shows: $\alpha_\lambda = \lambda$, $\beta_\lambda = \lambda + 1$ for $\lambda < \Omega$. Nor can we assert that the set of all elements contained in (3) must be countable. For example $\alpha_\lambda = \lambda$, $\beta_\lambda = \lambda + 1$ for isolated $\lambda \geqslant 0$. and $\alpha_\lambda = 1$, $\beta_\lambda = \lambda$ for all limit numbers λ.

If we denote by B the set of all β_λ and by A the set of all α_λ, then $f(\beta_\lambda) = \alpha_\lambda$ is a single-valued transformation of B onto A which is *not one-to-one*. There exists an uncountable subset $B^* \subset B$ such that $f(\beta_{\lambda^*}) = \alpha_{\lambda_0}$ for all $\beta_{\lambda^*} \in B^*$.

Theorem 2. *Let Z be a non-void abstract set. Let the following rule for the construction of subsets $N_\gamma \subset Z$ be prescribed: If all countable subsets N_λ, S_λ, A_λ of Z are constructed for all $\lambda < \gamma$, where $N_0 = S_0 = 0 \neq A_0$ and $S_\lambda \neq 0$ for $\lambda > 0$, we put $N_\gamma = \bigcup_{\lambda<\gamma} A_\lambda - \bigcup_{\lambda<\gamma} S_\lambda$ and choose $S_\gamma \neq 0$, $S_\gamma \subset N_\gamma$ supposing $N_\gamma \neq 0$, and then choose the countable set $A_\gamma \subset Z - \bigcup_{\lambda<\gamma} A_\lambda$. Then there exists a countable ordinal number $\vartheta > 1$ with the property $N_\vartheta = 0$, that is*

$$\bigcup_{\lambda<\vartheta} A_\lambda = \bigcup_{\lambda<\vartheta} S_\lambda$$

Proof. Suppose the contrary: that no ϑ with the mentioned property exists. Then $N_\lambda \neq 0$ for $0 < \lambda < \Omega$. First let us consider the sets A_λ. As $A_{\lambda'} \subset Z - \bigcup_{\mu<\lambda'} A_\mu \subset Z - A_\lambda$ for $\lambda < \lambda'$, the sets A_λ are disjoint. Let us denote the elements of the countable set A_λ — as far as $A_\lambda \neq 0$ — by ordinal numbers

$$\{\omega\lambda, \omega\lambda + 1, \ldots, \omega\lambda + n\},$$

where in the case when A_λ is finite, n is a non-negative integer, whereas in the case when A_λ is an infinite set, n runs through all non-negative integers. In this way we get a single-valued correspondence; this correspondence is one-to-one. In fact, let $x \neq y$ be any two different elements of the set $\bigcup_{\lambda<\Omega} A_\lambda$ (which is non-void because $0 \neq A_0 \subset \bigcup_{\lambda<\Omega} A_\lambda$); let $\omega\lambda + m$ and $\omega\lambda' + n$ be two corresponding ordinal numbers. Then either $\lambda \neq \lambda'$, which implies[1]) that $\omega\lambda + m \neq \omega\lambda' + n$, or $\lambda = \lambda'$ and $m \neq n$, which also implies $\omega\lambda + m \neq \omega\lambda' + n$. Therefore without a loss of generality we can identify the elements of the set $\bigcup_{\lambda<\Omega} A_\lambda$ with the corresponding ordinal numbers $\omega\lambda + n$ for those λ for which $A_\lambda \neq 0$.

Let us now consider the sets S_λ. Since, according to our supposition, $S_\lambda \neq 0$ for $0 < \lambda < \Omega$, we can choose a point

$$\alpha_\lambda = \omega\mu + m \in S_\lambda \subset \bigcup_{\lambda<\Omega} A_\lambda$$

for every $\lambda > 0$ and $\lambda < \Omega$. The correspondence $f(\beta_\lambda) = \alpha_\lambda \in S_\lambda$, where $\beta_\lambda = \omega\lambda$ and $0 < \lambda$, is one-to-one. In fact[1]) from $\omega\lambda = \omega\lambda'$ follows $\lambda = \lambda'$, so that $f(\omega\lambda)$ is the only point α_λ; further[1]) $\omega\lambda < \omega\lambda'$ implies $\lambda < \lambda'$. Therefore, as $\alpha_{\lambda'} \in S_{\lambda'} \subset N_{\lambda'} \subset Z - \bigcup_{\mu<\lambda'} S_\mu \subset Z - S_\lambda$ and $\alpha_\lambda \in S_\lambda$, we get $\alpha_\lambda \neq \alpha_{\lambda'}$, i. e. $f(\omega\lambda) \neq f(\omega\lambda')$.

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[1]) F. Hausdorff, *Mengenlehre*, Leipzig 1927, p. 63.

Let $\beta_\lambda=\omega\lambda$, $0<\lambda<\Omega$, be any countable limit ordinal. Then $f(\beta_\lambda)=a_\lambda \epsilon S_\lambda \subset N_\lambda \subset \bigcup_{\mu<\lambda} A_\mu - \bigcup_{\mu<\lambda} S_\mu$; hence $a_\lambda \epsilon A_\mu$ for a suitable index $\mu<\lambda$. Therefore there is a non-negative integer m for which $a_\lambda=\omega\mu+m$. As $\mu<\lambda$, we get $a_\lambda<\beta_\lambda$ for $0<\lambda<\Omega$. Moreover, if we define $a_\lambda=0$ and $\beta_\lambda=1$ for $\lambda=0$, we see the suppositions in Theorem 1 have been fulfilled. Therefore according to the theorem quoted the correspondence $f(\beta_\lambda)=a_\lambda$ is not one-to-one. This is a contradiction.

The theorem 1 will now be applied to the ordered continua. The intervals will play an important part[2]). Let I be an ordered continuum[2]). By *dyadic partition* Δ we mean the following process:

The continuum I will be called the interval of order 0. Let us choose a point a inside I which divides I into two closed intervals I_0 and I_1 of order 1 such that

$$I_0 \cup I_1 = I \quad \text{and} \quad I_0 \cap I_1 = a$$

the first of which is to the left from the other. Having already defined the intervals $I_{i_0 i_1 \dots i_\xi \dots}$ $(\xi<\lambda)$, where $i_\xi=0$ or $=1$, for all orders $\lambda<\alpha$, we shall define the intervals $I_{i_0 i_1 \dots i_\xi \dots}$ $(\xi<\alpha)$ of order α in this way:

1⁰ α is an isolated ordinal. In this case we choose a point inside every interval $I_{i_0 i_1 \dots i_\xi \dots}$ $(\xi<\alpha-1)$ of order $\alpha-1$ which divides this interval into two closed subintervals of order α $I_{i_0 i_1 \dots i_\xi \dots 0}$ and $I_{i_0 i_1 \dots i_\xi \dots 1}$ the first of which is to the left of the other.

2⁰ α is a limit ordinal. Let us form all products $\bigcap_{\lambda<\alpha} I_{i_0 i_1 \dots i_\xi}$ $(\xi<\lambda)$, where $i_\xi=0$ or $=1$. Each product like this is either a closed interval or a point; in the former case we call it *the interval of order* α and denote it by $I_{i_0 i_1 \dots i_\xi \dots}$ $(\xi<\alpha)$.

We continue this construction as long as there is at least one interval of order α. Because all possible intervals of the continuum I are as many as the different pairs of points of I (viz. the cardinal number of the set I), there exists the least ordinal $\delta>0$ such that we get no interval of order δ. This δ will be called the *order of the partition* Δ. Evidently, δ is a limit ordinal of a power which does not exceed the cardinal number of I. The system of all closed intervals of orders $\alpha<\delta$ will be denoted by $\mathfrak{S}_\Delta$ and will be called the *dyadic system of intervals*.

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[2]) The point-set consisting of only one point will neither be counted among the intervals nor among the ordered continua.

The following *statement* can be easily verified:

It is $I_{i_0 i_1 \dots i_\xi \dots}(\xi<\alpha) \subset I_{j_0 j_1 \dots j_\xi \dots}(\xi<\beta)$ *if and only if*:

$$\beta \leqslant \alpha \quad \textit{and} \quad i_\xi = j_\xi \quad \textit{for} \quad \xi<\beta.$$

Using this statement we prove

Lemma 1. *Any two different intervals of the same order* α *are disjoint or they have only one point in common.*

Proof. Let $I_{i_0 i_1 \dots i_\xi \dots}(\xi<\alpha)$ and $I_{j_0 j_1 \dots j_\xi \dots}(\xi<\alpha)$ be two different intervals of the same order α. Then there exists the least index $\beta<\alpha$ such that $i_\xi=j_\xi$ for $\xi<\beta$, whereas $i_\beta \neq j_\beta$. Therefore

$$I_{i_0 i_1 \dots i_\xi \dots}(\xi<\alpha) \subset I_{i_0 i_1 \dots i_\xi \dots i_\beta}; \quad \text{and} \quad I_{j_0 j_1 \dots j_\xi \dots}(\xi<\alpha) \subset I_{i_0 i_1 \dots i_\xi \dots j_\beta}.$$

The common part $I_{i_0 i_1 \dots i_\xi \dots i_\beta} \cap I_{j_0 j_1 \dots j_\xi \dots j_\beta}$ contains only one point; thus the lemma is proved.

Lemma 2. *Let* $\mathfrak{S}_\Delta$ *be a dyadic system of intervals of an ordered continuum* I. *Then the end-points of all intervals in* $\mathfrak{S}_\Delta$ *form a dense subset in* I.

Proof. Suppose, the contrary: there exists an interval $J \subset I$ containing no end-point. Then there exists a decreasing transfinite sequence

$$I \supset I_{i_0} \supset \dots \supset I_{i_0 i_1 \dots i_\xi \dots}(\xi<\alpha) \supset \dots$$

of intervals in $\mathfrak{S}_\Delta$ of all orders $\alpha<\delta$ such that the interval J is contained in the common part of all intervals appearing in the sequence, δ being the order of the dyadic partition Δ. Therefore this common part is an interval in $\mathfrak{S}_\Delta$ of order δ; this contradicts the definition of the order δ.

Definitions. The partition Δ of an ordered continuum I and the corresponding dyadic system $\mathfrak{S}_\Delta$ of closed intervals will be called ***rational***, if for every element $J \epsilon \mathfrak{S}_\Delta$, $J \neq I$ there is an element $J' \neq J$, $J' \epsilon \mathfrak{S}_\Delta$ such that J is contained in J' but not inside J'.

The ordered continuum K is said to posses the *Souslin property* (S), if there is no uncountable disjoint system of intervals in K.

Theorem 3. *The ordered continuum* K *possesing the Souslin property* (S) *contains a countable dense subset if and only if there exists at least one rational dyadic partition of* K.

Proof. The condition is necessary. In fact, every dyadic partition of a linear continuum, whose norms $\delta_n \to 0$ for $n \to \infty$, is a dyadic partition of the order ω. Every dyadic partition like this is evidently rational.

The condition is sufficient. Let Δ be a rational partition of the ordered continuum K. Our task is to prove that the dyadic system $\mathfrak{S}_\Delta$ is countable. Suppose the contrary: that the dyadic system $\mathfrak{S}_\Delta$ is not countable. According to Lemma 1, for every ordinal α, the subsystem $\mathfrak{S}_\alpha \subset \mathfrak{S}_\Delta$ of all intervals of order α is countable, K possessing the Souslin property (S). Therefore there exist intervals in $\mathfrak{S}_\Delta$ of all countable orders $\alpha < \Omega$.

Let us denote by $\mathfrak{T} \subset \mathfrak{S}_\Delta$ the subsystem of intervals of all limit orders $\beta_\lambda = \omega\lambda$, $0 < \lambda < \Omega$, such that for every limit order β_λ there exists only one interval in $\mathfrak{T}$ of order β_λ. Therefore, Δ being a rational partition, it is possible to choose for every limit ordinal β_λ the least ordinal $\alpha_\lambda = f(\beta_\lambda)$ such that the only interval in $\mathfrak{T}$ of order β_λ is contained in — but not inside — an interval in $\mathfrak{S}_\Delta$ of the (least) order α_λ. According to the statement on page 81, evidently $\alpha_\lambda < \beta_\lambda$ for $0 < \lambda < \Omega$. Further $\lim \beta_\nu = \beta_{\lim \nu}$ for $\lim \nu < \Omega$. If we put $\alpha_0 = 0$ and $\beta_0 = 1$, all conditions of Theorem 1 are fulfilled. According to this theorem there exists an index λ_0 such that $f(\beta_{\lambda^*}) = \alpha_{\lambda_0}$ for uncountably many indices λ^*. Two cases are possible:

1° The subsystem of all intervals in $\mathfrak{S}_\Delta$ of order α_{λ_0} is uncountable. But according to the lemma 1 this contradicts the Souslin property (S).

2° The subsystem of all intervals in $\mathfrak{S}_\Delta$ of order α_{λ_0} is countable. In this case there exists an interval $I' \in \mathfrak{S}_\Delta$ of order α_{λ_0} such that uncountably many intervals of the system $\mathfrak{T}$ are contained in — but not inside — I'. All these intervals can be divided into two groups. All intervals with the common left end-point belong to the first group, and all intervals with the common right end-point belong to the second group. At least one of both groups is uncountable. Consequently there exists an uncountable decreasing or increasing sequence of different intervals. This, again, contradicts the Souslin property (S). Thus we have proved that the system $\mathfrak{S}_\Delta$ is countable. According to lemma 2 a countable dense subset is contained in I.

Definition. Let Δ be a dyadic partition of order δ and $\mathfrak{S}_\Delta$ — the corresponding dyadic system of intervals of the ordered continuum I. Let $H \subset I$ be the set of end-points of all intervals in $\mathfrak{S}_\Delta$. For any point $x \in I$ let us denote by the symbol $\varphi(x)$ the least ordinal number such that there is in $\mathfrak{S}_\Delta$ no interval of order $\alpha \geqslant \varphi(x)$ and containing the point x. The relation $\omega \leqslant \varphi(x) \leqslant \delta$ for all $x \in I$ can be easily proved. *The dyadic partition Δ is said to be closed*, if the set of ordinals $\varphi(H) - \delta$ is closed (in the order topology) in the set of all ordinals $\xi < \delta$.

Theorem 4. *The ordered continuum K possessing the Souslin property (S) contains a countable dense subset if and only if there exists at least one closed dyadic partition of K.*

Proof. Every dyadic partition of the linear continuum with norms $\delta_n \to 0$ for $n \to \infty$ is closed; indeed its order is $\delta = \omega$, the set $\varphi(H)$ contains only one ordinal ω, and the empty set $\varphi(H) - \delta$ is a closed set.

Now, let us prove the sufficiency of the condition. Let Δ be a closed dyadic partition and $\mathfrak{S}_\Delta$ the dyadic system of intervals of K. As the ordered continuum K possesses the Souslin property (S) the inequality $\varphi(x) < \Omega$ holds for all $x \in K$; furthermore every subsystem of intervals in $\mathfrak{S}_\Delta$ of any countable order α is countable. Under the assumption that there exists no dense subset in I the order of the partition is $\delta = \Omega$, according to Lemma 2. As $\varphi(x) > \alpha$ for any end-point x of any interval in $\mathfrak{S}_\Delta$ of order α, the set $\varphi(H)$ is uncountable and can be arranged in the following form as an increasing transfinite sequence

$$\beta_0 < \beta_1 < \ldots < \beta_\lambda < \ldots \qquad (0 \leqslant \lambda < \Omega)$$

According to our supposition the set $\varphi(H) - \delta = \varphi(H)$ is closed in the set of all countable ordinals $\xi < \Omega$. Consequently $\lim \beta_\nu = \beta_{\lim \nu}$ for $\lim \nu < \Omega$. To any ordinal $\beta_\lambda \in \varphi(H)$ we let correspond one end-point $x_\lambda \in H$ such that $\varphi(x_\lambda) = \beta_\lambda$. In this way we get a one-to-one correspondence. Let us denote by $f(\beta_\lambda) = \alpha_\lambda$ the only ordinal such that x_λ is the end-point of an interval in $\mathfrak{S}_\Delta$ of the least order α_λ. Evidently $\alpha_\lambda < \beta_\lambda$ for $\lambda < \Omega$. According to Theorem 1 there exists an index λ_0 such that $f(\beta_{\lambda^*}) = \alpha_{\lambda_0}$ for uncountably many indices x_{λ^*}. Therefore there are uncountably many end-points x_{λ^*} of intervals in $\mathfrak{S}_\Delta$ of the order α_{λ_0}, which, according to Lemma 1, contradicts the Souslin property (S).