Si $l<k$, posons $f_{k,l}(x)=0$ pour x réels.

Si $l\geqslant k$, posons

$$f_{k,l}(x)=\begin{cases} 0 \quad \text{pour} \quad x\leqslant\frac{l-k-1}{2^k} \quad \text{et} \quad x\geqslant\frac{l-k+2}{2^k}, \\ 1 \quad \text{pour} \quad \frac{l-k}{2^k}\leqslant x\leqslant\frac{l-k+1}{2^k} \end{cases}$$

et prolongeons $f_{k,l}(x)$ linéairement dans les intervalles

$$\frac{l-k-1}{2^k}\leqslant x\leqslant\frac{l-k}{2} \quad \text{et} \quad \frac{l-k+1}{2^k}\leqslant x\leqslant\frac{l-k+2}{2^k}.$$

Soient $k_1,k_2,\ldots$ et $l_1,l_2,\ldots$ deux suites infinies croissantes de nombres naturels. Désignons par $E_{k,l}$ l'intervalle

$$\frac{l-k-1}{2^k}\leqslant x\leqslant\frac{l-k+2}{2^k}$$

et posons $E=\overline{\lim_n}\, E_{k_n,l_n}$. On a, pour $n=1,2,\ldots$

$$E\subset E_{k_n,l_n}+E_{k_{n+1},l_{n+1}}+\cdots$$

Comme $m(E_{k_n,l_n})=\frac{3}{2^{k_n}}$ et la suite $k_1,k_2\ldots$ est croissante, on trouve

$$m(E)\leqslant\frac{3}{2^{k_n}}+\frac{3}{2^{k_{n+1}}}+\cdots\leqslant\frac{3}{2^{k_n-1}}\leqslant\frac{3}{2^{n-1}}$$

d'où $m(E)=0$. L'ensemble E est donc de mesure nulle.

Soit x un nombre réel, tel que x non $\epsilon\, E$. Vu la définition de l'ensemble E, on a x non $\epsilon\, E_{k_n,l_n}$ pour n suffisamment grand, soit pour $n\geqslant q$. D'après la définition de la fonction $f_{k,l}(x)$, on a donc $f_{k_n,l_n}(x)=0$ pour $n\geqslant q$, d'où $\lim_{n=\infty} f_{k_n,l_n}(x)=0$. On a par conséquent la formule (6) pour x non $\epsilon\, E$, donc presque partout. La suite double $f_{k,l}(x)$ jouit ainsi de la propriété 1.

Soit maintenant $x\geqslant 0$. Il existe évidemment pour tout k naturel un nombre naturel $l_k\geqslant k$ tel que

$$\frac{l_k-k}{2^k}\leqslant x\leqslant\frac{l_k-k+1}{2^k}.$$

D'après (12), nous avons donc $f_{k,l_k}(x)=1$ pour $k=1,2,\ldots$; vu que $l_k\geqslant k$, il en résulte que l'on n'a pas $\lim_{m,n} f_{m,n}(x)=0$. La suite double $f_{k,l}(x)$ jouit donc de la propriété 2.

Le théorème 3 se trouve ainsi démontré.

An Algebraic Characterization of Quantifiers.

By

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Introduction. In this paper[1]) we shall be concerned with the relation between certain formal systems, in the sense of modern logic, and certain algebraic structures which serve as „models“ for these systems. Perhaps the most elementary and fundamental instance of what we have in mind is the relationship between the classical (two-valued) propositional calculus and the abstract structures known as Boolean algebras, which was explored very early in the literature. More recently a similar relationship has been shown[2]) by Tarski and McKinsey to hold between intuitionistic (propositional) logic and Brouwerian lattices; and also between certain modal logics and closure algebras.

Each of the propositional calculi mentioned above has been extended to a first order functional calculus embodying a theory of quantification for individual variables. The classical functional calculus is widely known; the extension for the intuitionistic logic has been carried out by Heyting[3]); while for modal logics this has recently been done by Barcan[4]), and independently by Carnap[5]).

[1]) This work was begun while the author was a Frank B. Jewett Fellow at Princeton. The author wishes to express his indebtedness to Professor Mostowski not only for suggesting the problem which led to this work, but also for the suggestion that the results be expressed with the present degree of generality.

[2]) J. C. C. McKinsey and Alfred Tarski, *Some theorems about the sentential calculi of Lewis and Heyting*, The Journal of Symbolic Logic, vol. **13** (1948), pp. 1-15.

[3]) A. Heyting, *On weakened quantification*, The Journal of Symbolic Logic, vol. **11** (1946), pp. 119-121.

[4]) Ruth C. Barcan, *A functional calculus of first order based on strict implication*, The Journal of Symbolic Logic, vol. **11** (1946), pp. 1-16.

[5]) Rudolf Carnap, *Modalities and quantification*, The Journal of Symbolic Logic, vol. **11** (1946), pp. 33-64.

It is natural to inquire whether the universal and existential quantifiers which are introduced in these functional calculi can be given an interpretation in terms of the algebraic structures which have been shown to provide a faithful set of models for the corresponding propositional calculi. An affirmative answer was conjectured by Mostowski [6]) for the case of the intuitionistic calculus, and the appropriate interpretation proved to be valid in one direction. The question was left open, however, as to whether the converse relation also held, and this question was the starting point of the present investigation. After a slight modification of Mostowski's question, which is seen to be appropriate, an affirmative answer is given.

It is easy to see that Mostowski's interpretation for quantifiers of the intuitionistic logic has its exact analogue in the case of the other functional calculi. Hence it seemed natural to give an account of this algebraic characterization of quantifiers sufficiently abstract to provide a unified theory covering all of the formal systems to which it applies. Upon investigation it is seen that the classical, intuitionistic, and modal logics all have in common a symbol for implication which, though differing in some of its usages among the several logics, satisfies a certain well-defined set of laws which are valid in each of the calculi. Thus we are led to consider this common sub-calculus as a separate formal system, which following Hilbert and Bernays we call the system of *Positive Implicative Logic*. Furthermore, it is precisely those laws of interaction between the quantifiers and this symbol for implication which the several functional calculi have in common, which are responsible for the validity of Mostowski's interpretation of the quantifiers. This gives us a natural method of extending the system of positive logic to a functional calculus, in terms of which our results can be formulated with adequate generality.

1. The system of positive (implicative) logic H_p. The primitive symbols of this calculus consist of

propositional variables: $p\ q\ r \ldots$

special symbols: $\supset$ ( ).

A finite sequence of primitive symbols is called a *formula*, and certain formulae are designated as *well-formed* according to the following rule:

[6]) Andrzej Mostowski, *Proofs of non-deducibility in intuitionistic functional calculus*, The Journal of Symbolic Logic, vol. **13** (1948), pp. 204-207.

A propositional variable is a well-formed formula (wff); if A and B are wffs so is $(A \supset B)$.

If A, B; C are any wffs the following wffs are called *axioms*:

H 1. $(A \supset (B \supset A))$,

H 2. $(((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))$.

If A, B are any wffs the operation which leads from the pair of wffs A, $(A \supset B)$ to the wff B is called *modus ponens*. If $\mathfrak{X}$ is a set of wffs, $\mathfrak{B}$ a finite sequence of wffs such that each element is either an axiom, an element of $\mathfrak{X}$, or the result of operating on two previous formulae of $\mathfrak{B}$ by modus ponens, and if A is the last member of $\mathfrak{B}$, we say that $\mathfrak{B}$ is a *formal proof* of the wff A on the *assumptions* $\mathfrak{X}$. If such a $\mathfrak{B}$ exists we write $\mathfrak{X} \vdash A$. In case $\mathfrak{X}$ is the empty set we write simply $\vdash A$, and in this case A is called a *formal theorem*.

On the basis of axiom schemata 1 and 2 and the rule of modus ponens it may be shown [7]) that the following, known in the literature as the *Deduction Theorem*, holds for our system of positive logic:

If $\mathfrak{X}, A \vdash B$ *then* $\mathfrak{X} \vdash (A \supset B)$, *where* „$\mathfrak{X}, A$“ *denotes the set obtained from* $\mathfrak{X}$ *by adjoining the wff* A.

It is easily seen that conversely each instance of axiom schemata 1 and 2 may be established as a formal theorem by means of modus ponens and the deduction theorem. Thus the formal theorems of positive logic are precisely those determined by these two rules.

2. Implicative models. These structures are defined to be triplets $\langle X, 0, \dot{-} \rangle$, where X is an arbitrary set, 0 is an element of X, and $\dot{-}$ is a binary operation defined on X, satisfying certain axioms given below. In addition to the primitive operation $\dot{-}$ it is convenient to introduce a relation $\leqslant$ by the

Definition. $x \leqslant y$ for $x \dot{-} y = 0$.

The following axioms hold for all x, y, z of X:

M 1. $x \dot{-} y \leqslant x$,

M 2. $(x \dot{-} z) \dot{-} (y \dot{-} z) \leqslant (x \dot{-} y) \dot{-} z$,

M 3. $0 \leqslant x$,

M 4. *If* $x \leqslant y$ *and* $y \leqslant x$ *then* $x = y$.

[7]) Cf. Alonzo Church, *An Introduction to Mathematical Logic*, Princeton University Press 1944, pp. 9, 45.

In addition to the axioms it is useful to have available a few theorems which are simple consequences of them.

M 5. *If* $x \leqslant 0$ *then* $x = 0$.

Proof: By *M* 3 and *M* 4.

M 6. $0 \dot{-} z = 0$.

Proof: By *M* 1, $0 \dot{-} z \leqslant 0$; then use *M* 5.

M 7. $x \leqslant x$.

Proof: Writing *M* 1 as $(x \dot{-} y) \dot{-} x = 0$ we have, in particular, $(x \dot{-} x) \dot{-} x = 0$ and $(x \dot{-} (x \dot{-} x)) \dot{-} x = 0$. But by *M* 2, $(x \dot{-} x) \dot{-} ((x \dot{-} x) \dot{-} x) \leqslant \leqslant (x \dot{-} x)) \dot{-} x$. Therefore $(x \dot{-} x) \dot{-} 0 \leqslant 0$. By *M* 5 and the definition of $\leqslant$ we thus get $x \dot{-} x \leqslant 0$, and the same argument then gives *M* 7.

M 8. *If* $x \leqslant y$ *and* $y \leqslant z$ *then* $x \leqslant z$.

Proof: Writing our assumptions in the form $x \dot{-} y = 0$ and $y \dot{-} z = 0$, and applying *M* 2, we have $(x \dot{-} z) \dot{-} 0 \leqslant 0 \dot{-} z$. Using *M* 6 and then *M* 5 (and the definition of $\leqslant$) we get $x \dot{-} z \leqslant 0$, from which $x \leqslant z$ follows by *M* 5.

Thus we see that an implicative model is partially ordered by a reflexive relation admitting no cycles, and contains a unique minimal element.

M 9. *If* $x \dot{-} y \leqslant z$ *then* $x \dot{-} z \leqslant y$.

Proof: Writing our hypothesis in the form $(x \dot{-} y) \dot{-} z = 0$ we get $(x \dot{-} z) \dot{-} (y \dot{-} z) \leqslant 0$ by *M* 2. Then *M* 5 gives $x \dot{-} z \leqslant y \dot{-} z$. But $y \dot{-} z \leqslant y$ by *M* 1, hence $x \dot{-} z \leqslant y$ by *M* 8.

M 10. *If* $x \leqslant y$ *then* $z \dot{-} y \leqslant z \dot{-} x$.

Proof: By *M* 2 we have $(z \dot{-} y) \dot{-} (x \dot{-} y) \leqslant (z \dot{-} x) \dot{-} y$. But $x \dot{-} y = 0$ by assumption; and since (by *M* 1) $(z \dot{-} x) \dot{-} y \leqslant z \dot{-} x$, we obtain, using *M* 8, $(z \dot{-} y) \dot{-} 0 \leqslant z \dot{-} x$. Using *M* 9 and then *M* 5, $(z \dot{-} y) \dot{-} (z \dot{-} x) = 0$, which is the desired conclusion.

Given a subset Y of X we define top Y to be an element z of X such that $y \leqslant z$ for every $y \in Y$, while for each element $a \in X$ we have $z \dot{-} a \leqslant z'$ whenever $y \dot{-} a \leqslant z'$ for all $y \in Y$. Of course not every subset Y will have a top, in general. If, in this definition, we consider the case where $a = 0$, we see that if top Y exists it must be identical with sup Y so that when top Y exists it is unique; however, in some models there may be sets Y which possess a top but no sup. If every subset of X has both a top and an inf the model will be called $\dot{-}$*-complete*.

Boolean algebras, Brouwerian lattices, and closure algebras are examples of implicative models. In each of these cases $y = \sup X$ implies $y = \text{top } X$. A Boolean algebra of all subsets of some domain, a Brouwerian lattice of all closed sets of some topological space, or a closure algebra of all sets in some topological space, is an implicative model which is $\dot{-}$-complete.

3. The relationship between implicative models and the system H_p. Let $\langle X, 0, \dot{-} \rangle$ be an arbitrary implicative model and let φ be a mapping of the propositional variables of the system H_p into X. We extend φ to a mapping φ' of the wffs of H_p into X, as follows:

i) $\varphi'(A) = \varphi(A)$ *for every propositional variable* A.

ii) $\varphi'((A \supset B)) = \varphi'(B) \dot{-} \varphi'(A)$ *for all wffs* A, B.

If $\varphi'(A) = 0$ then φ is said to *satisfy* A. If A is satisfied by every φ (with values in an arbitrary model X) it is called *valid*.

Theorem I. *For any wff* A, A *is valid if and only if* A *is a formal theorem.*

We shall not give a proof of this theorem since this will be easily obtainable from our proof of the corresponding theorem about the functional calculus H_f constructed below.

4. The first-order functional calculus H_f. This is an extension of the system H_p. We add the further primitive symbols:

individual variables: $x \; y \; z \ldots$

n-ary function variables: $F_n \; G_n \; H_n \; \ldots \; (n = 1, 2, \ldots)$

special symbols: $\exists$, .

We expand the definition of „wff" by adding the clauses:

$\Phi_n(a_1, \ldots, a_n)$ is a wff, where Φ_n is an n-ary function variable and $a_1, \ldots, a_n$ are individual variables. If A is a wff so are $(a)A$ and $(\exists a)A$, where a is any individual variable.

An occurrence of a variable a is called *bound* if it is within a formula having one of the forms $(a)A$ or $(\exists a)A$; otherwise it is *free*. To the axiom schemata *H* 1 and *H* 2 of H_p we add the following.

H 3. $((\alpha)A_{[\alpha]} \subset A_{[\beta]})$, *where* α, β *are individual variables*, $A_{[\alpha]}$ *is any wff, and* $A_{[\beta]}$ *is obtained from* $A_{[\alpha]}$ *by replacing each free occurrence of* a *by a free occurrence of* β.

H 4. $((\alpha)(A \supset B) \supset (A \supset (\alpha)B))$, *where* a *has no free occurrences in* A.

H 5. $(A_{[\beta]} \supset (\exists \alpha) A_{[\alpha]})$, *where the symbols are used as in H* 3.

H 6. $(((\alpha)(B \supset A) \supset (\exists \alpha) B \supset A))$, *where the symbols are used as in H*4.

Along with modus ponens we consider the operation of *generalization*, which to any individual variable a and wff A associates the wff $(a)A$. And we modify the definition of a formal proof $\mathcal{B}$ from assumptions $\mathfrak{X}$ by inserting the condition that a member of $\mathcal{B}$ may be obtained from a preceding formula of the sequence by generalization on any variable with no free occurrence in the formulae of $\mathfrak{X}$. The description of our use of the symbol „$\vdash$" and the term „formal theorem" remain unchanged from that given for H_p.

The Deduction Theorem as stated for H_p holds also for H_f, the proof being identical to that found in the literature for the classical functional calculus [7]. We shall also make use of the following theorems about H_f.

H 7. $B\supset A\vdash((A\supset C)\supset(B\supset C))$.

Proof:

$B\supset A,\ B\vdash A$ modus ponens.

$B\supset A,\ B,\ A\supset C\vdash C$ modus ponens.

$B\supset A,\ A\supset C\vdash B\supset C$ deduction theorem.

$B\supset A\vdash((A\supset C)\supset B\supset C))$ deduction theorem.

H 8. $C\supset B\vdash((A\supset C)\supset(A\supset B))$.

Proof similar.

5. The relationship between implicative models and the system H_f. Let $\langle X, 0, \dot{-}\rangle$ be an implicative model, and let D be an arbitrary domain whose elements will be called individuals. By a *value-assignment* we mean a function φ which assigns values to the variables of H_f as follows. $\varphi(A)$ is an element of X for each propositional variable A; $\varphi(a)$ is an element of D for each individual variable a; $\varphi(\Phi_n)$ is a function of n arguments ranging over D, with values in X, for each n-ary function variable Φ_n, $n=1,2,\ldots$

Let φ be a value-assignment; then by O_φ we mean the set of all value-assignments ψ which have the same value as φ for each propositional and functional variable. We shall say that $\langle X, 0, \dot{-}, D\rangle$ is an implicative *functional* model in case there exists a value-assignments φ (called an *interpretive* mapping) such that for each $\psi\in O_\varphi$ there exists a mapping ψ' of the wffs of H_f into X satisfying the following conditions.

[7]) Cf. Alonzo Church, op. cit.

(i) $\psi'(A)=\psi(A)=\varphi(A)$ for each propositional variable A. $\psi'(\Phi_n(a_1,\ldots,a_n))$ is the value of the function $\psi(\Phi_n)$ for the n-tuple of arguments $\langle\psi(a_1),\ldots,\psi(a_n)\rangle$, where Φ_n is any n-ary function variable and $a_1,\ldots,a_n$ are any individual variables.

(ii) $\psi'((A\supset B))=\psi'(B)\dot{-}\psi'(A)$ for all wffs A, B.

(iii) $\psi'((a)A)=\text{top } Y$, where A is any wff, a any individual variable, and Y is the subset of X consisting of all elements $\varrho'(A)$ for value-assignments ϱ in O_φ which differ from ψ only in the value which is assigned to a.

(iv) $\psi'((\exists a)A)=\inf Y$.

It is easily seen that if $\langle X, 0, \dot{-}, D\rangle$ is a functional model there is only one ψ' which can be associated with each value-assignment ψ of O_φ in such a way that (i), (ii), (iii) and (iv) are satisfied. Every $\dot{-}$-complete implicative model is a functional model. If $\varphi'(A)=0$ we say that φ *satisfies* A; and as before, we say that A is valid if it is satisfied by every interpretive mapping (into an arbitrary functional model X).

Theorem II. *If A is a formal theorem of H_f then A is valid.*

Proof: Let $\langle X, 0, \dot{-}, D\rangle$ be any functional model and φ a value-assignment (the interpretive mapping) such that to each ψ in O_φ there corresponds a unique ψ' satisfying (i), (ii), (iii) and (iv). If A an instance of one of axiom schemata $H1$, $H2$, then $\psi'(A)=0$ is by (ii), $M1$, $M2$, and the definition of „$\leqslant$". If A is an instance of axiom schema $H3$, say A is $((a)B_{[a]}\supset B_{[\beta]})$, then to show $\psi'(A)=0$ it suffices to show that $\psi'(B_{[\beta]})\leqslant\psi'((a)B_{[a]})$. But this is so because by (iii) $\psi'((a)B_{[a]})$ is the top (and hence sup) of a subset of X of which $\psi'(B_{[\beta]})$ is an element. Similarly $\psi'(A)=0$ when A is an instance of schema $H5$. Now suppose that A is an instance of $H4$, say $((a)(C\supset B)\supset(C\supset(a)B))$, where a has no free occurrences in C. We have to show that $\psi'((a)B)\dot{-}\psi'(C)\leqslant\psi'((a)(C\supset B))$. But this is so because $\psi'((a)B)$ is the top of a set Y for which we have $\psi'((a)(C\supset B))$ as the sup of the set of all elements $y\dot{-}\psi'(C)$, where y is in Y. Finally suppose A to be $((a)(B\supset C)\supset((\exists a)B\supset C))$, an instance of $H6$. We must show that $z\dot{-}\inf Y\leqslant\text{top } W$, where W is the set of all $z\dot{-}y$, $y\in Y$. By $M9$ it suffices to show $z\dot{-}\text{top} W\leqslant\inf Y$ and hence, by definition of inf, $z\dot{-}\text{top } W\leqslant y$ for each $y\in Y$. But this is indeed the case, using $M9$ again, since $z\dot{-}y\leqslant\text{top} W=\text{sup}$ of all elements $z\dot{-}y$.

We have thus shown that $\psi'(A)=0$ for any axiom A. Suppose, next, that A arises by modus ponens from B and $B\supset A$, and we already now that $\psi'(B)=0$ and $\psi'(B\supset A)=\psi'(A)\dot{-}\psi'(B)=0$. Thus we have $\psi'(A)\dot{-}0=0$ which gives us $\psi'(A)=0$ by M 5. Finally suppose that A is $(\alpha)B$ (i. e. arises from B by generalization), and that we already know that $\psi'(B)=0$ for all ψ in O_φ. Then in particular the set of elements $\varrho'(B)$, where ϱ ranges over those functions in O_φ which differ from ψ only in the value assigned to the variable α, will contain 0 as the only element. But $\psi'((\alpha)B)$ is the top of this set. Hence $\psi'(A)=0$.

This completes the proof of Theorem II by induction on the length of the formal proof of A.

Theorem III. *There exists a functional model $\langle X, 0, \dot{-}, D\rangle$, and an interpretive mapping φ (with the values of φ' in X), such that for each wff A which is not a formal theorem there is a ψ in O_φ with $\psi'(A)\neq 0$. Furthermore, both X and D are denumerable* [8]).

Proof. We form a new formal system, H_f^*, whose primitive symbols are those of H_f together with new symbols $u_1, u_2, u_3, \ldots$ which we call *special (individual) variables*. The definition of „wff" is the same as for H_f except that we do not permit the special variables to appear bound; i. e. if A is a wff of H_f^* then $(\alpha)A$ and $(\exists\alpha)A$ are wffs if α is an ordinary individual variable, but not if α is a special variable. The axiom schemata and rules of inference for H_f^* are taken over without change from H_f.

A wff of H_f^* will be called *eligible* if it has no free occurrence of an ordinary individual variable. We define a relation $\simeq$ between eligible formulae, writing $A\simeq B$ in case both $\vdash A\supset B$ and $\vdash B\supset A$. Clearly the relation $\simeq$ is symmetric; it is reflexive since $\vdash A\supset A$ (as one sees at once from the Deduction Theorem); and it is transitive as one sees by using H 7 and modus ponens. Hence the relation $\simeq$ partitions the eligible formulae of H_f^* into disjoint equivalence classes. We write „$[A]$" for the class to which A belongs, and have $[A]=[B]$ if and only if $A\simeq B$. We take the set of all these equivalence classes to be X; and we take as 0 the class consisting of all eligible formal theorems of H_f^*. That these formal theorems form an equivalence class is seen from the fact that $\vdash A$ and $\vdash B$ imply $\vdash A\supset B$ and $\vdash B\supset A$ by H 1 and modus ponens; while $\vdash A$ and $A\simeq B$ imply $\vdash B$ directly by modus ponens.

[8]) I am informed by Prof. Mostowski that a very similar theorem has been found independently by H. Rasiowa.

Now we define an operation $\dot{-}$ on X by the law $[A]\dot{-}[B]=[B\supset A]$. In order to be sure that the operation so defined is really an operation on the classes $[A]$ and $[B]$ and independent of the particular representatives A and B we must show that $A\simeq A'$ and $B\simeq B'$ imply $(B\supset A)\simeq(B'\supset A')$. This is easily done with the aid of H 7, H 8 and modus ponens.

The fact that the operation $\dot{-}$ thus defined on X satisfies M 1 and M 2 is a direct consequence of H 1 and H 2 respectively. M 3 is easily established using H 1 and modus ponens, while M 4 is immediate from the definition of $\simeq$. Thus $\langle X, 0, \dot{-}\rangle$ is indeed an implicative model.

Let D be the set of all special variables $u_1, u_2, \ldots$ We shall show that $\langle X, 0, \dot{-}, D\rangle$ is a functional model by demonstrating that the value-assignment φ defined below is an interpretive mapping. Clearly both X and D are denumerable.

If A is a propositional variable set $\varphi(A)=[A]$. If α is an individual variable set $\varphi(\alpha)=\alpha$ or u_1 according as α is special or ordinary. If Φ_n is an n-ary function variable let $\varphi(\Phi_n)$ be that function whose value for the arguments $\langle u_{i_1}, \ldots, u_{i_n}\rangle$ is $[\Phi_n(u_{i_1}, \ldots, u_{i_n})]$.

Now let ψ be any element of O_φ; i. e. let ψ be any value-assignment which has the same value as φ for each propositional and functional variable. Let ψ' be the mapping (of the wffs of H_f^* into X) such that $\psi'(A)=[A_\psi]$, where A_ψ results from the wff A by replacing each free occurrence of an individual variable α by $\psi(\alpha)$. We shall show for all $\psi\in O_\varphi$ that the functions ψ' so defined satisfy (i), (ii), (iii) and (iv).

(i) If A is a propositional variable then $A_\psi=A$ and $\psi'(A)=[A]=\varphi(A)$. If C is $\Phi_n(\alpha_1, \ldots, \alpha_n)$ then C_ψ is $\Phi_n(\psi(\alpha_1), \ldots, \psi(\alpha_n))$ and $\psi'(C)=[\Phi_n(\psi(\alpha_1), \ldots, \psi(\alpha_n))]=\psi(\Phi_n)(\langle\psi(\alpha_1), \ldots, \psi(\alpha_n)\rangle)$.

(ii) Consider the formula $(A\supset B)$. We have $\psi'((A\supset B))=[(A\supset B)_\psi]=[(A_\psi\supset B_\psi)]=[B_\psi]\dot{-}[A_\psi]=\psi'(B)\dot{-}\psi'(A)$.

(iii) Consider the formula $(\alpha)A$. Let Y be the subset of X consisting of all elements $\varrho'(A)$ such that $\varrho\in O_\varphi$ and ϱ differs from ψ only in the value assigned to α. Then Y is the set of all $[A_\varrho]$.

Now $((\alpha)A)_\psi\supset A_\varrho$ is an axiom by H 3, since A_ϱ is obtained from $((\alpha)A)_\psi$ by dropping the initial quantifier and then replacing each free occurrence of α by $\varrho(\alpha)$ — which is a special variable and so certainly not bound in A. Hence $[A_\varrho]\dot{-}[((\alpha)A)_\psi]=[((\alpha)A)_\psi\supset A_\varrho]=0$ so that $[A_\varrho]\leqslant[((\alpha)A)_\psi]$. That is, $y\leqslant[((\alpha)A)_\psi]$ for all $y\in Y$.

Now let $a=[B]$ be any element of X, and suppose that $z=[C]$ is such that $y \dot{-} a \leqslant z$ for all $y \in Y$. In other words we have $[B \supset A_\varrho] \leqslant [C]$ for each ϱ, from which $\vdash C \supset (B \supset A_\varrho)$ for each ϱ.

But C and B are certain eligible formulae of H_f^* and together have at most a finite number of special variables occurring in them. Choose u_k not occurring in either C or B and let ϱ_1 be the value-assignment such that $\varrho_1(a)=u_k$ (while ϱ_1 has the same value as ψ for all other variables).

We know there exists a formal proof $\mathcal{B}$ of $C \supset (B \supset A_{\varrho_1})$. By systematically replacing each occurrence of u_k by a throughout $\mathcal{B}$ we obtain [9] a formal proof of $C \supset (B \supset A_{\tilde{\psi}})$, where $A_{\tilde{\psi}}$ is like A_ψ except for having a in place of each occurrence of $\psi(a)$. Hence by generalization $\vdash (a)\,(C \supset (B \supset A_{\tilde{\psi}}))$; and by two uses of $H\,4$ and modus ponens $\vdash C \supset (B \supset ((a)A)_\psi)$ (since B and C, being eligible, cannot contain free occurrences of a). From this it follows that $[((a)A)_\psi] \dot{-} [B] \leqslant [C]$; in other words $[((a)A)_\psi] \dot{-} a \leqslant z$.

This completes the proof that $\psi'((a)A)=\text{top } Y$. The proof of (iv) is entirely analogous.

We have thus shown that φ is an interpretive mapping so that $\langle X, 0, \dot{-}, D \rangle$ is a functional model Furthermore, if A is any eligible formula $A_\varphi = A$ so that $\varphi'(A)=[A] \neq 0$ if A is not a formal theorem.

Now suppose that B is not a formal theorem but that it is not eligible it contains free occurrences of the ordinary individual variables $a_1, \ldots, a_n$. It follows that $(a_1)\ldots(a_n)B$ is eligible, and it cannot be a formal theorem else we could get a formal proof of B by n uses of $H\,3$ and modus ponens. Hence $[(a_1)\ldots(a_n)B] \neq 0$.

But $[(a_1)\ldots(a_n)B] = \text{top } Y_1$, where Y_1 is the set of all elements $[((a_2)\ldots(a_n)B)_{\psi_1}]$ such that ψ_1 differs from φ only in the value assigned to a_1. Hence there must be a $y \in Y_1$ such that $y \neq 0$; i. e. a ψ_1 such that $[((a_2)\ldots(a_n)B)_{\psi_1}] \neq 0$. But $[((a_2)\ldots(a_n)B)_{\psi_1}] = \text{top } Y_2$, where Y_2 contains all elements $[((a_3)\ldots(a_n)B)_{\psi_2}]$ such that ψ_2 differs from ψ_1 only in the value assigned to a_2. Hence there must be such a ψ_2 with $[((a_3)\ldots(a_n)B)_{\psi_2}] \neq 0$. Continuing in this way we see that there must be a ψ_n in O_φ such that $\psi_n'(B) = [B_{\psi_n}] \neq 0$. This completes the proof of Theorem III.

[9]) Actually it may be necessary first to change certain of the auxiliary variables appearing in $\mathcal{B}$. This point is encountered in proving that the rule of substitution holds in formal systems where axiom schemata are employed and substitution is not taken as primitive. Cf. The reference mentioned in footnote 7, p. 56.

Theorem IV. *For any wff A of H_f, A is a formal theorem if and only if A is valid.*

Proof: By Theorems II and III.

6. Extensions of the system H_f. We describe here a certain type of extension of H_f such that the classical, intuitionistic, and modal functional calculi can all be obtained by a finite number of such extensions. The extended system, H_f^+, is obtained as follows.

a) We add a new primitive symbol, Λ.

b) We add to the definition of „wff" the clause: „If $A_1, \ldots, A_n$ are wffs so is $\Lambda(A_1, \ldots, A_n)$" (where n is a certain fixed integer associated with Λ).

c) We add to the axiom schemata $H\,1$—$H\,6$ a finite number of further schemata each of the following form. B is some wff of H_f^+ in which the only variables which appear are propositional variables; and the schema stipulates that each formula shall be an axiom which is obtained from B by substituting specified wffs, one for each propositional variable, throughout B.

Further axioms are added as follows:

$$(A_i \supset A_i') \supset ((A_i' \supset A_i) \supset (\Lambda(A_1, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots, A_n) \supset \supset \Lambda(A_1, \ldots, A_{i-1}, A_i', A_{i+1}, \ldots, A_n))).$$

d) The rules of inference remain just modus ponens and generalization.

For any such system H_f^+ we obtain an algebraic model $\langle X, 0, \dot{-}, +, D \rangle$ as follows. Let $\langle X, 0, \dot{-}, D \rangle$ be a functional model for the system H_f. Then take $+$ to be a function of n arguments ranging over X, with values in X, satisfying certain postulates as given below.

There will be one postulate for each axiom schema resulting from a wff B of H_f^+ in the way described in c) above. The postulate may be obtained from B by substituting for each propositional variable a variable ranging over X, replacing „Λ" throughout by „$+$", and writing $D_2 \dot{-} D_1$ in place of each part $D_1 \supset D_2$.

The relationship between the system H_f^+ and the models $\langle X, 0, \dot{-}, +, D \rangle$ will be entirely analogous to the situation as described for the system H_f, the only change being that to the conditions (i), (ii), (iii) and (iv) we add a condition (v) which ψ' must satisfy:

(v) $\psi'(A(A_1,\dots,A_n)) = +(\psi'(A_1),\dots,\psi'(A_n))$ for all wffs $A_1,\dots,A_n$.

From this description it can be seen that the presence of symbols from the propositional calculus (other than „⊃") is *irrelevant* for the characterization of quantifiers which we have given. It is this fact which provides the justification for our having abstracted from these symbols and worked with the system of basic implication.

The Tychonoff Product Theorem Implies the Axiom of Choice.

By

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Recently S. Kakutani has conjectured that the axiom of choice is a consequence of that theorem of Tychonoff[1]) which states that the Cartesian product of compact topological spaces is compact. It is the purpose of this note to show that this conjecture is correct.

1. ***Definitions.*** We first review the pertinent definitions. A topological space is a set X, together with a family $\mathcal{T}$ of subsets (called *open* subsets), the family $\mathcal{T}$ having as members the void set, X, all finite[2]) intersections and arbitrary unions of members of $\mathcal{T}$. If we adjoin the requirement that complements of finite sets be open, the topological space is a Kuratowski closure space[3]). In the proof which we give the topological spaces constructed are closure spaces.

The space is compact (=bicompact) if each covering of X by members of $\mathcal{T}$ has a finite subcovering. (In particular, the void set A with the topology $\{A\}$, is compact). If, for each member a of a set A, X_a is a set, the product $\mathbf{P}_{a\in A}X_a$ is the set of all functions x on A for which, for each $a\in A$, $x_a \in X_a$. If each X_a has a topology we let $\mathcal{S}$ be the family of all subsets of the Cartesian product which, for some set U open in some X_a, are the set of all x with $x_a \in U$. The product is then topologized by calling a set open if it is the union of finite intersections of members of $\mathcal{S}$.

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[1]) Mathematische Annalen, vol. **111** (1935), pp. 762-766.

[2]) In the absence of the axiom of choice it is necessary to define „finite". We agree that a set is finite if it may be ordered so that every non-void subset has both a first and a last element in the ordering. Then the axiom of choice for finite families of sets can be proved. See A. Tarski, Fund. Math. **6** (1924), pp. 49-95, for a full discussion of this and related questions.

[3]) See C. Kuratowski, *Topologie I*, Monogr. Mat. **3** (1933), p. 15.