On a en vertu de (8.4), (8.5) et du lemme 2 sur les accroissements finis

$$|x_\nu - x_0| = |g(\sigma(x_\nu)) - g(\sigma(x_0))| \leqslant \delta |\sigma(x_\nu) - \sigma(x_0)|$$

donc $\dfrac{|\sigma(x_\nu)-\sigma(x_0)|}{|x_\nu - x_0|} \geqslant \dfrac{1}{\delta}$. Cette relation rapprochée de (8.9) donne à la limite l'inégalité (8.6).

À la fonction $\sigma(x)$, envisagée dans la sphère Sph $(a, \beta R)$, on pourra appliquer le théorème 3. Il existera donc une fonction $h(y)$ continue dans la sphère Sph $\left(b, \frac{\beta}{\delta} R\right)$, c.-à-d. dans la sphère (8.7), telle que

$$\sigma(h(y)) = y \quad \text{pour} \quad y \in \mathrm{Sph}\left(b, \frac{\beta}{\delta} R\right). \tag{8.10}$$

En posant $x = h(y)$ dans (8.4) on trouvera

$$h(y) = g(\sigma(h(y))) = g(y) \quad \text{lorsque} \quad y \in \mathrm{Sph}\left(b, \frac{\beta}{\delta} R\right)$$

donc en vertu de (8.10)

$$\sigma(g(y)) = y \quad \text{dans} \quad \mathrm{Sph}\left(b, \frac{\beta}{\delta} R\right). \tag{8.11}$$

Soient $y_1 \neq y_2$ deux points de Sph $\left(b, \frac{\beta}{\delta} R\right)$. Afin de prouver que $g(y)$, envisagée dans Sph $\left(b, \frac{\beta}{\delta} R\right)$ est inversible il suffit de prouver que $g(y_1) \neq g(y_2)$.

Supposons, pour la démonstration par l'impossible, que $g(y_1) = g(y_2)$. En raison de (8.11) on aura

$$y_1 = \sigma(g(y_1)) = \sigma(g(y_2)) = y_2$$

contrairement à l'hypothèse que $y_1 \neq y_2$.

Afin de prouver que l'image de (8.7) par l'intermédiaire de g englobe la sphère (8.8), il suffit d'appliquer le théorème 3 (en y posant $\frac{\beta}{\delta} \cdot R$ au lieu de R) et de remarquer que la fonction $g(y)$ étant envisagée dans Sph $\left(b, \frac{\beta}{\delta} R\right)$, la fonction $\sigma(x)$ vérifie la relation $x = g(\sigma(x))$ dans la sphère (8.8).

Cartesian Products of Boolean Algebras.

By

Roman Sikorski (Warszawa).

The definition of cartesian products of fields[1] of sets presents no difficulty.

For every $\tau \in T$[2] let $\mathbf{X}_\tau$ be a field of subsets of a set $\mathfrak{X}_\tau$. The *cartesian product* $\mathbf{P}^{\alpha}_{\tau \in T} \mathbf{X}_\tau$ of all fields $\mathbf{X}_\tau$ is the least field (of subsets of $\mathbf{P}_{\tau \in T} \mathfrak{X}_\tau$)[3] which contains all sets $\mathbf{P}_{\tau \in T} X_\tau$ where $X_\tau \in \mathbf{X}_\tau$ and the inequality $X_\tau \neq \mathfrak{X}_\tau$ holds only for a finite number of elements $\tau \in T$[4]. The *cartesian σ-product* $\mathbf{P}^{\beta}_{\tau \in T} \mathbf{X}_\tau$ of all fields $\mathbf{X}_\tau$ is the least σ-field (of subsets of $\mathbf{P}_{\tau \in T} \mathfrak{X}_\tau$) which contains the field $\mathbf{P}^{\alpha}_{\tau \in T} \mathbf{X}_\tau$[5].

The following two theorems[6] hold for the so-defined cartesian products:

0.1. *If for every* $\tau \in T$, $\mathbf{X}_\tau$ *and* $\mathbf{Y}_\tau$ *are isomorphic*[7] *fields of sets, then the cartesian products* $\mathbf{P}^{\alpha}_{\tau \in T} \mathbf{X}_\tau$ *and* $\mathbf{P}^{\alpha}_{\tau \in T} \mathbf{Y}_\tau$ *are also isomorphic.*

0.2. *If, for every* $\tau \in T$, $\mathbf{X}_\tau$ *and* $\mathbf{Y}_\tau$ *are isomorphic σ-fields*[8] *of sets, then the cartesian σ-products* $\mathbf{P}^{\beta}_{\tau \in T} \mathbf{X}_\tau$ *and* $\mathbf{P}^{\beta}_{\tau \in T} \mathbf{Y}_\tau$ *are also isomorphic.*

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1) A class $\mathbf{X}$ of subsets of a set $\mathfrak{X}$ is called a *field* if $X_1, X_2 \in \mathbf{X}$ implies $X_1 + X_2 \in \mathbf{X}$ and $\mathfrak{X} - X_1 \in \mathbf{X}$. A field $\mathbf{X}$ is called a *σ-field* if $X_n \in \mathbf{X}$ $(n = 1, 2, \ldots)$ implies $X_1 + X_2 + X_3 + \ldots \in \mathbf{X}$.

2) T denotes always a fixed non-empty set.

3) $\mathbf{P}_{\tau \in T} X_\tau$ will denote always the set-theoretical cartesian product of sets X_τ $(\tau \in T)$.

4) For instance, if $\mathbf{X}_\tau$ is the field of all both open and closed subsets of a bicompact space $\mathfrak{X}_\tau$, then $\mathbf{P}^{\alpha}_{\tau \in T} \mathbf{X}_\tau$ is the field of all both open and closed subsets of the bicompact space $\mathbf{P}_{\tau \in T} \mathfrak{X}_\tau$.

5) For instance, if $\overline{\overline{T}} \leqslant \aleph_0$ and if $\mathbf{X}_\tau$ is the σ-field of all Borel subsets of a metric space $\mathfrak{X}_\tau$, then $\mathbf{P}^{\beta}_{\tau \in T} \mathbf{X}_\tau$ is the σ-field of all Borel subsets of the metric space $\mathbf{P}_{\tau \in T} \mathfrak{X}_\tau$.

6) Theorems 0.1 and 0.2 follow immediately from theorems II and 3 (i) in my paper [5].

7) The definition of isomorphisms and homomorphisms is given on p. 31.

8) The condition that $\mathbf{X}_\tau$ and $\mathbf{Y}_\tau$ are σ-fields is essential.

The purpose of this paper is to generalize the notion of the cartesian product and the σ-product of fields to the case of arbitrary Boolean algebras $\boldsymbol{A}_\tau$ $(\tau \in T)$.

The definition of the cartesian product and the σ-product of Boolean algebras $\boldsymbol{A}_\tau$ should satisfy the following two conditions:

1) If, for every $\tau \in T$, $\boldsymbol{A}_\tau$ is isomorphic to a field (σ-field) $\boldsymbol{X}_\tau$ of sets, then the cartesian product (σ-product) of all $\boldsymbol{A}_\tau$ should be isomorphic to $\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{X}_\tau$ $(\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau)$.

2) The cartesian product (σ-product) of Boolean algebras should possess main properties of the cartesian product (σ-product) of fields of sets. The properties under consideration are the following [9]:

0.3 [10]. *For every $\tau \in T$ let h_τ be a homomorphism (σ-homomorphism) of a field (σ-field) $\boldsymbol{X}_\tau$ in a field (σ-field) $\boldsymbol{Y}$. Then there is a homomorphism* [7] *(σ-homomorphism) h of $\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{X}_\tau$ $(\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau)$ in $\boldsymbol{Y}$ such that $h(\mathbf{P}_{\tau\in T}X_\tau)=\prod_{\tau\in T}h_\tau(X_\tau)$ for every set $\mathbf{P}_{\tau\in T}X_\tau \in \mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{X}_\tau$.*

0.4 [11]. *For every $\tau \in T$, let μ_τ be a normalized measure (σ-measure) on a field (σ-field) $\boldsymbol{X}_\tau$. Then there is a measure (σ-measure) μ on $\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{X}_\tau$ $(\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau)$ such that $\mu(\mathbf{P}_{\tau\in T}X_\tau)=\prod_{\tau\in T}\mu_\tau(X_\tau)$* [12] *for every set $\mathbf{P}_{\tau\in T}X_\tau \in \mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{X}_\tau$.*

The definition of the cartesian product of Boolean algebras $\boldsymbol{A}_\tau$ $(\tau \in T)$ presents no difficulty. As Stone has proved, every Boolean algebra $\boldsymbol{A}_\tau$ is isomorphic to a field of sets $\boldsymbol{X}_\tau$. We may define [13] the cartesian product of all algebras $\boldsymbol{A}_\tau$ as the Boolean algebra $\boldsymbol{A}=\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{X}_\tau$. In fact, the so-defined product $\boldsymbol{A}$ does not depend on the choice of the isomorphic fields $\boldsymbol{X}_\tau$ on account of 0.1, and it satisfies the conditions 1) and 2) (see theorems 6.1 and 14.1).

There arrise some difficulties concerning the definition of the cartesian σ-products of Boolean algebras. We can not say as before that the cartesian σ-product is the least σ-complete Boolean algebra which contains the cartesian product $\boldsymbol{A}$ of all $\boldsymbol{A}_\tau$. In fact, for

[9]) That is, the cartesian (σ)-product should satisfy theorems 0.3 and 0.4 where the term „(σ)-field" is replaced by „(σ-complete) Boolean algebra".

[10]) This theorem follows immediately from theorems I and 3 (i) in my paper [5].

[11]) See Łomnicki and Ulam [1] p. 245 and 252 and Andersen and Jessen [1], p. 22.

[12]) Almost all factors in this product are equal to 1.

[13]) Another equivalent definition of the cartesian product of Boolean algebras has been given by Kappos [1], pp. 53-58.

a given Boolean algebra $\boldsymbol{A}$ there are, in general, many non-isomorphic σ-complete Boolean algebras $\boldsymbol{B}$ such that $\boldsymbol{A}$ is a subalgebra [14] and a σ-generator [15] of $\boldsymbol{B}$. One can, however, distinguish among all such Boolean algebras $\boldsymbol{B}$ a σ-complete Boolean algebra $\boldsymbol{A}^b$ which is „the least" in an absolute sense. Let namely $\boldsymbol{A}^c$ be MacNeille's minimal extension [16] of $\boldsymbol{A}$, $\boldsymbol{A}\subset\boldsymbol{A}^c$; $\boldsymbol{A}^b$ is the least σ-subalgebra of $\boldsymbol{A}^c$ which contains $\boldsymbol{A}$.

The σ-complete Boolean algebra $\boldsymbol{A}^b$, where $\boldsymbol{A}$ is the cartesian product of all $\boldsymbol{A}_\tau$, is called the *minimal σ-product* of all $\boldsymbol{A}_\tau$ $(\tau \in T)$ and is denoted by $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$. The minimal σ-product, however, fulfils neither of the conditions 1) and 2).

Marczewski's concept of independent fields [17] of sets suggests another definition of the cartesian σ-product of Boolean algebras $\boldsymbol{A}_\tau$ $(\tau \in T)$. Let us assume for simplicity that all $\boldsymbol{A}_\tau$ are σ-complete.

Let $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ be a family of subalgebras of a Boolean algebra $\boldsymbol{B}$. We shall say that the subalgebras $\boldsymbol{B}_\tau$ are *independent in* $\boldsymbol{B}$ if [18]

$$(*)\qquad (\boldsymbol{B})\prod_n B_n \neq 0$$

for every finite sequence $B_n \in \boldsymbol{B}_{\tau_n}$ such that $B_n \neq 0$ and, $\tau_i \neq \tau_j$ for $i \neq j$.

If the condition (*) holds for every finite or enumerable sequence [19] B_n (satisfying the above hypotheses), the subalgebras $\boldsymbol{B}_\tau$ are said to be *σ-independent in* $\boldsymbol{B}$.

The following theorem [20] shows the connexion between the cartesian multiplication and independent subalgebras:

[14]) The definition of subalgebras and σ-subalgebras is given on p. 30.

[15]) That is, the smallest σ-subalgebra of $\boldsymbol{B}$, which contains $\boldsymbol{B}$, is $\boldsymbol{A}$ itself. See the definition on p. 31.

[16]) MacNeille [1], p. 437. The definition and fundamental properties of minimal extensions will be given in § 3.

[17]) See Marczewski [1], pp. 125-126.

[18]) $(\boldsymbol{B})\prod_n B_n$ denotes the Boolean product (common part) of all B_n in the Boolean algebra $\boldsymbol{B}$. See p. 29-30.

[19]) In the case of an enumerable sequence $B_n \in \boldsymbol{B}$ the inequality $(\boldsymbol{B})\prod_n B_n \neq 0$ means: if the element $(\boldsymbol{B})\prod_n B_n$ exists, it differs from 0.

[20]) Proved in my paper [5], Theorem III. An analogous theorem holds for independent fields and the cartesian product $\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{X}_\tau$.

0.5. *Let* $\{\boldsymbol{Y}_\tau\}_{\tau\in T}$ *be a family of σ-subfields*[21]) *of a σ-field of sets $\boldsymbol{Y}$, such that*

a) $\boldsymbol{Y}$ *is the least σ-field containing all the σ-fields* $\boldsymbol{Y}_\tau$;

b) *the σ-subfields $\boldsymbol{Y}_\tau$ are σ-independent;*

If, for every $\tau\in T$, $\boldsymbol{X}_\tau$ is a σ-field isomorphic to $\boldsymbol{Y}_\tau$, then the cartesian σ-product $\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau$ is isomorphic to $\boldsymbol{Y}$.

On the other hand, the cartesian σ-product $\boldsymbol{Y}=\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau$ of σ-fields $\boldsymbol{X}_\tau$ contains a family of σ-subfields $\boldsymbol{Y}_\tau$ which satisfy a) and b) and is isomorphic to $\boldsymbol{X}_\tau$ respectively ($\boldsymbol{Y}_\tau$ are the so-called cylinder fields).

This fact suggests to refer as a *cartesian σ-product of all $\boldsymbol{A}_\tau$ ($\tau\in T$)* to every σ-complete Boolean algebra $\boldsymbol{B}$ which contains a family $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ of σ-subalgebras such that

(i) the class $\sum_{\tau\in T}\boldsymbol{B}_\tau$ is a σ-generator of $\boldsymbol{B}$;

(ii) the σ-subalgebras $\boldsymbol{B}_\tau$ are σ-independent in $\boldsymbol{B}$;

(iii) for every $\tau\in T$ there is an isomorphism h_τ of $\boldsymbol{A}_\tau$ on $\boldsymbol{B}_\tau$.

It can be proved that the class $\mathfrak{L}^*$ of all such algebras $\boldsymbol{B}$ is not empty. In general, $\mathfrak{L}^*$ contains many non-isomorphic Boolean algebras[22]) $\boldsymbol{B}$.

The class $\mathfrak{L}^*$ can be partly ordered. Let $\boldsymbol{B}^0\in\mathfrak{L}^*$ and let $\boldsymbol{B}^0_\tau$ and h^0_τ have an analogous meaning. We shall write $\boldsymbol{B}^0\leqslant\boldsymbol{B}$ if the isomorphisms $h^0_\tau h_\tau^{-1}$ (of $\boldsymbol{B}_\tau$ on $\boldsymbol{B}^0_\tau$) can be extended to a σ-homomorphism of $\boldsymbol{B}$ in $\boldsymbol{B}^0$.

If at the same time $\boldsymbol{B}^0\leqslant\boldsymbol{B}$ and $\boldsymbol{B}\leqslant\boldsymbol{B}^0$, then $\boldsymbol{B}$ and $\boldsymbol{B}^0$ are isomorphic. In this case the algebras $\boldsymbol{B}$ and $\boldsymbol{B}^0$ will be identified.

The following two elements should be distinguished in $\mathfrak{L}^*$: the greatest element of $\mathfrak{L}^*$, called the *maximal σ-product* and denoted by $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$, and a minimal element of $\mathfrak{L}^*$, called the *minimal σ^*-product* and denoted by $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$.

The maximal σ-product satisfies property 2) but it does not possess, in general, property 1). The minimal σ^*-product possesses property 1) but it does not possess, in general, property 2). I know no natural definition of the cartesian σ-product of Boolean algebras which satisfies both properties 1) and 2).

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[21]) $\boldsymbol{Y}_0$ is a *σ-subfield* of a σ-field $\boldsymbol{Y}$ of subsets of $\mathcal{Y}$ if $\boldsymbol{Y}_0$ is also a σ-field of subsets of $\mathcal{Y}$ and $\boldsymbol{Y}_0\subset\boldsymbol{Y}$.

[22]) This holds also in the case where all $\boldsymbol{A}_\tau$ are σ-fields. On account of 0.2 and 0.5 the class $\mathfrak{L}^*$ contains then exactly one σ-field. Other algebras $\boldsymbol{B}\in\mathfrak{L}^*$ are then not isomorphic to a σ-field of sets. See § 12.

Consequently I shall consider in this paper three different σ-products $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$, $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$, and $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$. The definition of $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$ can be formulated in the same way as that of $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$ or $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$. In fact, the class $\mathfrak{L}$ of all σ-complete Boolean algebras $\boldsymbol{B}$ which satisfies (i), (iii) and

(ii′) the σ-subalgebras $\boldsymbol{B}_\tau$ are independent in $\boldsymbol{B}$

is also a partly ordered set with the same ordering relation, and $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$ is a minimal element of $\mathfrak{L}$ ($\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$ is also the greatest element of $\mathfrak{L}$). $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$ coincides with $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ if and only if T is finite.

The restricting hypothesis that all $\boldsymbol{A}_\tau$ are σ-complete may be omitted. In this paper the definitions of $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$, $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$ and $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ are formulated for arbitrary Boolean algebras $\boldsymbol{A}_\tau$. It seems to me natural to assume in the general case that the subalgebras $\boldsymbol{B}_\tau$ (see the definition of $\mathfrak{L}$) are σ-regular[23]), that is, all enumerable sum and products of elements $B\in\boldsymbol{B}_\tau$ coincide in $\boldsymbol{B}_\tau$ and in $\boldsymbol{B}$.

A σ-complete Boolean algebra $\boldsymbol{B}$ is called a *σ-extension*[24]) of a Boolean algebra $\boldsymbol{A}$, if $\boldsymbol{B}$ contains a σ-regular subalgebra $\boldsymbol{B}_0$ which is a σ-generator of $\boldsymbol{B}$ and an isomorph of $\boldsymbol{A}$. The study of σ-extensions of Boolean algebras is closely related to the study of cartesian σ-products of Boolean algebras; therefore it constitutes a part of this paper (§§ 2, 3, and 13). In general, a Boolean algebra has many non-isomorphic σ-extensions.

1. Definitions and lemmas.

A Boolean algebra is a set $\boldsymbol{A}$ of elements (denoted by $A, B, \ldots$) with two operations: multiplication $A\cdot B$ and complementation A' satisfying the well known axioms. The element $(A'\cdot B')'$ is denoted by $A+B$. We write $A\subset B$ if $A\cdot B=A$. The relation $\subset$ orders partly the set $\boldsymbol{A}$. The symbol 0 denotes the least element of $\boldsymbol{A}$, that is, $0\subset A$ for every $A\in\boldsymbol{A}$. Consequently $0'$ is the greatest element of $\boldsymbol{A}$.

Let $\{A_u\}_{u\in U}$ be a family of elements of $\boldsymbol{A}$, distinct or not. The symbol[25]) $(\boldsymbol{A})\prod_{u\in U}A_u$ will denote the greatest element $A\in\boldsymbol{A}$

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[23]) See the definition on p. 30.

[24]) See p. 33.

[25]) The value of infinite Boolean products depends on the considered Boolean algebra $\boldsymbol{A}$. In fact, if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $A_u\in\boldsymbol{B}$, then $(\boldsymbol{B})\prod_{u\in U}A_u\subset(\boldsymbol{A})\prod_{u\in U}A_u$ whenever these products exist but the converse inclusion does not hold in general. Therefore we shall always write before „Π“ the symbol denoting the considered Boolean algebra.

If X_u are sets, the symbols $\prod_{u\in U}X_u$ and $\sum_{u\in U}X_u$ will denote always the set theoretical product and sum of all sets X_u.

(called the *product* of all A_u in $\boldsymbol{A}$) such that $A \subset A_u$ for every $u \in U$, whenever it exists. The meaning of the symbol $(\boldsymbol{A})\prod_{n=1}^{\infty} A_n$ is clear. In particular, the condition $(\boldsymbol{A})\prod_{u \in U} A_u = 0$ means that there is no element $A \neq 0$ ($A \in \boldsymbol{A}$) such that $A \subset A_u$ for every $u \in U$.

If the product $(\boldsymbol{A})\prod_{u \in U} A_u$ exists for any family $\{A_u\}_{u \in U}$, then the Boolean algebra $\boldsymbol{A}$ is called *complete*. If it exists for any enumerable family $\{A_u\}_{u \in U}$, $\boldsymbol{A}$ is said to be *σ-complete*.

A class $\boldsymbol{B} \subset \boldsymbol{A}$ is called a *subalgebra* of $\boldsymbol{A}$ if $A \cdot B \in \boldsymbol{B}$ and $A' \in \boldsymbol{B}$ for arbitrary $A, B \in \boldsymbol{B}$. A subalgebra is also a Boolean algebra with the same operations A' and $A \cdot B$.

A subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is called a *regular subalgebra* of $\boldsymbol{A}$, if all infinite products in $\boldsymbol{B}$ and $\boldsymbol{A}$ coincide, that is, if for any family $\{B_u\}_{u \in U}$ of elements of $\boldsymbol{B}$

(i) the condition $(\boldsymbol{B})\prod_{u \in U} B_u = B \in \boldsymbol{B}$ implies $(\boldsymbol{A})\prod_{u \in U} B_u = B$.

If (i) holds for any enumerable family $\{A_u\}_{u \in U}$, then $\boldsymbol{B}$ is said to be a *σ-regular subalgebra* of $\boldsymbol{A}$.

In the above definition of a regular or σ-regular subalgebra the proposition (i) may be replaced by:

(i') the condition $(\boldsymbol{B})\prod_{u \in U} B_u = 0$ implies $(\boldsymbol{A})\prod_{u \in U} B_u = 0$.

1.1 *If $\boldsymbol{C}$ is a regular (σ-regular) subalgebra of $\boldsymbol{B}$, and $\boldsymbol{B}$ is a regular (σ-regular) subalgebra of $\boldsymbol{A}$, then $\boldsymbol{C}$ is a regular (σ-regular) subalgebra of $\boldsymbol{A}$.*

A subalgebra $\boldsymbol{B}$ of a σ-complete Boolean algebra $\boldsymbol{A}$ is called a *σ-subalgebra* of $\boldsymbol{A}$ if $(\boldsymbol{A})\prod_{n=1}^{\infty} B_n \in \boldsymbol{B}$ for any sequence $B_n \in \boldsymbol{B}$.

1.2 *Every σ-subalgebra $\boldsymbol{B}$ of a σ-complete Boolean algebra $\boldsymbol{A}$ is a σ-regular subalgebra of $\boldsymbol{A}$.*

A set $\boldsymbol{K}$ of elements of a Boolean algebra $\boldsymbol{A}$ is said to be *dense* in $\boldsymbol{A}$ if for every $A \in \boldsymbol{A}$, $A \neq 0$, there is an element $A_0 \in \boldsymbol{K}$ such that $0 \neq A_0 \subset A$.

1.3. *Let $\boldsymbol{B}$ be a dense subalgebra of a Boolean algebra $\boldsymbol{A}$. Then:*

a) *$\boldsymbol{B}$ is a regular subalgebra of $\boldsymbol{A}$;*

b) *every element $A \in \boldsymbol{A}$ is the product of all elements $B \in \boldsymbol{B}$ such that $A \subset B$.*

Let $\{B_u\}_{u \in U}$ be a family of elements of $\boldsymbol{B}$ such that the product $(\boldsymbol{B})\prod_{u \in U} B_u = B \in \boldsymbol{B}$ exists. Suppose an element $A \in \boldsymbol{A}$ satisfies the inclusion $A \subset B_u$ for every $u \in U$. If $AB' \neq 0$ there is an element $B_0 \in \boldsymbol{B}$ such that $0 \neq B_0 \subset AB'$. Consequently $B \neq B + B_0 \subset B_u$ for every $u \in U$, and $B + B_0 \in \boldsymbol{B}$, in contradiction with the hypothesis that $B = (\boldsymbol{B})\prod_{u \in U} B_u$. Thus we infer that $A \subset B$, which proves that $B = (\boldsymbol{A})\prod_{u \in U} B_u$, that is, $\boldsymbol{B}$ is a regular subalgebra of $\boldsymbol{A}$.

Let $A \in \boldsymbol{A}$, $A_1 \in \boldsymbol{A}$ and suppose $A \subset A_1 \subset B$ for every element $B \in \boldsymbol{B}$ such that $A \subset B$. If $A_1 \cdot A' \neq 0$, there is an element $B_0 \in \boldsymbol{B}$ such that $0 \neq B_0 \subset A_1 \cdot A'$. We have $B_0' \in \boldsymbol{B}$, $A \subset B_0'$ and $A_1 \not\subset B_0'$, which gives a contradiction. Thus we infer that $A_1 = A$ which proves b).

A set $\boldsymbol{K}$ of elements of a Boolean algebra $\boldsymbol{A}$ is said to be a *generator* of $\boldsymbol{A}$ if the smallest subalgebra of $\boldsymbol{A}$ containing $\boldsymbol{K}$ is the algebra $\boldsymbol{A}$ itself. If $\boldsymbol{A}$ is σ-complete, a set $\boldsymbol{K} \subset \boldsymbol{A}$ is said to be a *σ-generator* of $\boldsymbol{A}$ provided the smallest σ-subalgebra of $\boldsymbol{A}$ containing $\boldsymbol{K}$ is the algebra $\boldsymbol{A}$ itself.

A mapping h of a Boolean algebra $\boldsymbol{A}$ in another Boolean algebra $\boldsymbol{B}$ is called a *homomorphism* of $\boldsymbol{A}$ in $\boldsymbol{B}$, if for $A, B \in \boldsymbol{A}$

$$h(A \cdot B) = h(A) \cdot h(B) \quad \text{and} \quad h(A') = h(A)'.$$

A homomorphism h is said to be a *σ-homomorphism* of $\boldsymbol{A}$ in $\boldsymbol{B}$ if

$$(\boldsymbol{A})\prod_{n=1}^{\infty} A_n = 0 \quad \text{implies} \quad (\boldsymbol{B})\prod_{n=1}^{\infty} h(A_n) = 0.$$

A homomorphism h is one-one if and only if $h(A) = 0$ implies $A = 0$. A one-one homomorphism is called an *isomorphism*. If there exists an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$, the algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic*, in symbols $\boldsymbol{A} \approx \boldsymbol{B}$.

1.4. *Let $\boldsymbol{K}$ and $\boldsymbol{L}$ are generators (σ-generators) of two (σ-complete) Boolean algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively, and let f be a one-one mapping of $\boldsymbol{K}$ on $\boldsymbol{L}$. If f can be extended to a homomorphism (σ-homomorphism) h of $\boldsymbol{A}$ in $\boldsymbol{B}$ and if f^{-1} can be extended to a homomorphism (σ-homomorphism) g of $\boldsymbol{B}$ in $\boldsymbol{A}$, then h is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$ and $g = h^{-1}$.*

We have

(ii) $$gh(A) = A \quad \text{and} \quad hg(B) = B$$

for every $A \in \boldsymbol{K}$ and $B \in \boldsymbol{L}$. Since $\boldsymbol{K}$ and $\boldsymbol{L}$ are generators (σ-generators) of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively, the formulas (ii) hold also for arbitrary $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$, which proves lemma 1.4.

A finite non-negative function μ defined on a Boolean algebra $\boldsymbol{A}$ is called a *measure* on $\boldsymbol{A}$ if $\mu(A+B)=\mu(A)+\mu(B)$ for $A,B \in \boldsymbol{A}$, $A\cdot B=0$. A measure μ is called

two-valued, if it assumes exactly two values 0 and 1;

normalized, if $\mu(0')=1$;

a *σ-measure* on $\boldsymbol{A}$, if $(\boldsymbol{A})\prod_{n=1}^{\infty} A_n=0$ implies $\lim_{n=\infty}\mu(A_n)=0$ for every decreasing sequence $A_n \in \boldsymbol{A}$.

A class $\boldsymbol{I}$ of sets is called a *σ-ideal* if the conditions $X, X_n \in \boldsymbol{I}$ $(n=1,2,...)$ and $Y\subset X$ imply $\sum_{n=1}^{\infty} X_n \in \boldsymbol{I}$ and $Y \in \boldsymbol{I}$.

Suppose $\boldsymbol{I}$ is a σ-ideal. For every set X the symbol $X/\boldsymbol{I}$ will denote the class of all sets which can be represented in the form $X+Y_1-Y_2$ where $Y_1, Y_2 \in \boldsymbol{I}$. If $\boldsymbol{X}$ is a class of sets, then $\boldsymbol{X}/\boldsymbol{I}$ will denote the collection of all $X/\boldsymbol{I}$ where $X \in \boldsymbol{X}$.

If $\boldsymbol{X}$ is a field of sets, then $\boldsymbol{X}/\boldsymbol{I}$ is a Boolean algebra with the following definition of Boolean operations:

$$(X_1/\boldsymbol{I})\cdot(X_2/\boldsymbol{I})=X_1X_2/\boldsymbol{I}, \qquad (X/\boldsymbol{I})'=X'/\boldsymbol{I}.$$

If $\boldsymbol{X}$ is a σ-field of sets, then the Boolean algebra $\boldsymbol{X}/\boldsymbol{I}$ is σ-complete and

$$(\boldsymbol{X}/\boldsymbol{I})\prod_{n=1}^{\infty}(X_n/\boldsymbol{I})=\prod_{n=1}^{\infty}X_n/\boldsymbol{I}.$$

The least σ-field containing a given field of sets $\boldsymbol{X}$ will be denoted by $\boldsymbol{X}^{\sigma}$.

1.5. *Let $\boldsymbol{X}$ and $\boldsymbol{I}$ be respectively a field and a σ-ideal of sets. Then $\boldsymbol{X}/\boldsymbol{I}$ is a subalgebra and a σ-generator of $\boldsymbol{X}^{\sigma}/\boldsymbol{I}$. If no set $X \in \boldsymbol{X}$ $(X\neq 0)$ belongs to $\boldsymbol{I}$, then the mapping $X\rightarrow X/\boldsymbol{I}$ is an isomorphism of $\boldsymbol{X}$ on $\boldsymbol{X}/\boldsymbol{I}$.*

2. σ-extensions. Let $\boldsymbol{A}$ be a Boolean algebra. The symbol $\mathcal{S}(\boldsymbol{A})$ will denote the set of all prime ideals of $\boldsymbol{A}$. For every $A \in \boldsymbol{A}$, $\mathfrak{s}(A)$ will denote the set of all prime ideals of $\boldsymbol{A}$ which do not contain the element A. The class of all sets $\mathfrak{s}(A)$ will be denoted by $\boldsymbol{S}(\boldsymbol{A})$. $\boldsymbol{S}(\boldsymbol{A})$ is a field of subsets of $\mathcal{S}(\boldsymbol{A})$ and $\mathfrak{s}(A)$ is an isomorphism[26]) of $\boldsymbol{A}$ on $\boldsymbol{S}(\boldsymbol{A})$. According to the notation from § 1, $\boldsymbol{S}^{\sigma}(\boldsymbol{A})$ is the least σ-field (of subsets of $\mathcal{S}(\boldsymbol{A})$) which contains $\boldsymbol{S}(\boldsymbol{A})$.

[26]) Stone [1], p. 98 and 106.

We shall always consider the set $\mathcal{S}(\boldsymbol{A})$ as a topological space with $\boldsymbol{S}(\boldsymbol{A})$ as the class of neighborhoods. The space $\mathcal{S}(\boldsymbol{A})$ is totally disconnected, bicompact and normal[27]), and $\boldsymbol{S}(\boldsymbol{A})$ is the field of all sets which are both open and closed in $\mathcal{S}(\boldsymbol{A})$.

$\mathfrak{J}(\boldsymbol{A})$ will denote the σ-ideal of all subsets of first category in $\mathcal{S}(\boldsymbol{A})$. The symbol $\boldsymbol{J}(\boldsymbol{A})$ will denote the least σ-ideal containing all sets $\prod_{n=1}^{\infty}\mathfrak{s}(A_n)$ where $A_n \in \boldsymbol{A}$ be any sequence such that $(\boldsymbol{A})\prod_{n=1}^{\infty}A_n=0$. Obviously, $\boldsymbol{J}(\boldsymbol{A})$ is the class of all subsets of sets $\sum_{m=1}^{\infty}S_m$ where $S_m=\prod_{n=1}^{\infty}\mathfrak{s}(A_n^m)$ and $(\boldsymbol{A})\prod_{n=1}^{\infty}A_n^m=0$.

2.1. $\boldsymbol{J}(\boldsymbol{A})\subset\mathfrak{J}(\boldsymbol{A})$.

In fact, if $(\boldsymbol{A})\prod_{n=1}^{\infty}A_n=0$, the closed set $\prod_{n=1}^{\infty}\mathfrak{s}(A_n)$ contains no open non-empty subset, that is, it is nowhere dense.

A σ-complete Boolean algebra $\boldsymbol{B}$ is called a *σ-extension* of a Boolean algebra $\boldsymbol{A}$ if $\boldsymbol{B}$ contains a σ-regular subalgebra $\boldsymbol{B}_0$ which is an isomorph of $\boldsymbol{A}$ and a σ-generator of $\boldsymbol{B}$.

2.2. *If $\boldsymbol{I}$ is a σ-ideal (of subsets of $\mathcal{S}(\boldsymbol{A})$) such that*

a) *no open non-empty subset of $\mathcal{S}(\boldsymbol{A})$ belongs to $\boldsymbol{I}$;*

b) $\boldsymbol{J}(\boldsymbol{A})\subset\boldsymbol{I}$;

then $\boldsymbol{S}^{\sigma}(\boldsymbol{A})/\boldsymbol{I}$ is a σ-extension of $\boldsymbol{A}$. The mapping $g(A)=\mathfrak{s}(A)/\boldsymbol{I}$ (for $A \in \boldsymbol{A}$) is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{S}(\boldsymbol{A})/\boldsymbol{I}$.

On account of 1.5, it is sufficient to show that $\boldsymbol{S}(\boldsymbol{A})/\boldsymbol{I}$ is a σ-regular subalgebra of $\boldsymbol{S}^{\sigma}(\boldsymbol{B})/\boldsymbol{I}$, or that the mapping g is a σ-homomorphism of $\boldsymbol{A}$ in $\boldsymbol{S}^{\sigma}(\boldsymbol{A})/\boldsymbol{I}$. This follows from b) since if $(\boldsymbol{A})\prod_{n=1}^{\infty}A_n=0$, then $\prod_{n=1}^{\infty}\mathfrak{s}(A_n)$ belongs to $\boldsymbol{I}$.

In particular, by 2.1, since no open non-empty subset of a normal bicompact space is of first category[28]):

2.3. *The Boolean algebras $\boldsymbol{S}^{\sigma}(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A})$ and $\boldsymbol{S}^{\sigma}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$ are σ-extensions of $\boldsymbol{A}$.*

Every Boolean algebra isomorphic to $\boldsymbol{S}^{\sigma}(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A})$ will be called a *maximal σ-extension*[29]) of $\boldsymbol{A}$. Every Boolean algebra isomorphic to $\boldsymbol{S}^{\sigma}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$ will be called a *minimal σ-extension*[30]) of $\boldsymbol{A}$.

[27]) Stone [2], p. 378.

[28]) See Sikorski [1], p. 256.

[29]) An invariant characterization of maximal σ-extension will be given in § 13, th. 13.4.

[30]) An invariant characterization of minimal σ-products will be given in § 3, th. 3.7 and 3.8.

2.4. *Let $\boldsymbol{A}$ and $\boldsymbol{C}$ be two Boolean algebras. Every σ-homomorphism h of $\boldsymbol{S}(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A})$ in $\boldsymbol{S}(\boldsymbol{C})/\boldsymbol{J}(\boldsymbol{C})$ is induced* [31]) *by a mapping φ, that is, there is a mapping φ of $\mathcal{S}(\boldsymbol{C})$ in $\mathcal{S}(\boldsymbol{A})$ such that*

$$h(S/\boldsymbol{J}(\boldsymbol{A}))=\varphi^{-1}(S)/\boldsymbol{J}(\boldsymbol{C}) \quad \text{for} \quad S\in \boldsymbol{S}(\boldsymbol{A}).$$

The proof of theorem 2.4 is analogous to that of theorem (*) in my paper [3].

We note that [32])

2.5. *If $\boldsymbol{A}$ is a σ-complete Boolean algebra, then*

$$\boldsymbol{A}\approx \boldsymbol{S}^{\sigma}(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A})\approx \boldsymbol{S}^{\sigma}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A}).$$

3. Minimal extensions. In this section we shall explain the connexion between minimal σ-extensions and MacNeille's minimal extensions of Boolean algebras. We shall give also an invariant characterization of minimal σ-extensions.

For every Boolean algebra $\boldsymbol{A}$, the symbol $\mathfrak{B}(\boldsymbol{A})$ will denote the class of all subsets of the space $\mathcal{S}(\boldsymbol{A})$ which possess the property of Baire [33]).

3.1. *$\boldsymbol{S}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$ is a dense subalgebra of the complete Boolean algebra $\mathfrak{B}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$ and of the σ-complete Boolean algebra $\boldsymbol{S}^{\sigma}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$.*

This follows from the following more general theorem:

3.2. *Let $\boldsymbol{K}$ be a class of open neighborhoods of a topological* [34]) *space $\mathfrak{X}$ such that every open subset of $\mathfrak{X}$ is the sum of some neighborhoods belonging to $\boldsymbol{K}$ (that is, $\boldsymbol{K}$ determines the topology in $\mathfrak{X}$). Let $\boldsymbol{I}$ be the ideal of all sets of first category in $\mathfrak{X}$, let $\boldsymbol{X}$ be the smallest σ-field containing $\boldsymbol{K}$, and let $\boldsymbol{Y}$ be the σ-field of all sets possessing the property of Baire. Then $\boldsymbol{K}/\boldsymbol{I}$ is dense in the σ-complete Boolean algebra $\boldsymbol{X}/\boldsymbol{I}$ and in the complete* [35]) *Boolean algebra $\boldsymbol{Y}/\boldsymbol{I}$.*

It is sufficient to prove that $\boldsymbol{K}/\boldsymbol{I}$ is dense in $\boldsymbol{Y}/\boldsymbol{I}$, i. e., if $Y\in\boldsymbol{Y}$ and Y non $\in\boldsymbol{I}$, then there is a set $X\in\boldsymbol{K}$ such that X non $\in \boldsymbol{I}$ and $X-Y\in\boldsymbol{I}$. We have $Y=G-P+R$ where G is open and $P,R\in\boldsymbol{I}$.

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[31]) See Sikorski [2], p. 7.

[32]) See Loomis [1], p. 757, Sikorski [1], p. 256, and Sikorski [3], p. 245.

[33]) A subset X of a topological space $\mathfrak{X}$ possesses the property of Baire if $X=G-P+R$ where G is open, and P and R are of first category.

[34]) That is, $\mathfrak{X}$ satisfies the well known axioms of Kuratowski. See Kuratowski [1], p. 20.

[35]) See Birkhoff [1], p. 178.

Consequently G non $\in\boldsymbol{I}$. On account of Banach's theorem [36]) on sets of first category, there is a neighborhood $X\in\boldsymbol{K}$ such that X non $\in\boldsymbol{I}$ and $X\subset G$. The set X fulfils the required conditions.

3.3. *Let $\boldsymbol{A}_0$ be a dense subalgebra of a Boolean algebra $\boldsymbol{A}$ and let h_0 be an isomorphism of $\boldsymbol{A}_0$ in a complete Boolean algebra $\boldsymbol{B}$. The isomorphism h_0 can be extended to an isomorphism h of $\boldsymbol{A}$ in $\boldsymbol{B}$.*

The isomorphism h_0 can be extended [37]) to a homomorphism h of $\boldsymbol{A}$ in $\boldsymbol{B}$. Let $A\in\boldsymbol{A}$, $A\neq 0$, and let $A_0\in\boldsymbol{A}_0$ be an element such that $0\neq A_0\subset A$. Then $0\neq h_0(A_0)=h(A_0)\subset h(A)$. Consequently $h(A)\neq 0$, which proves that h is an isomorphism.

3.4. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two complete Boolean algebra and let h_0 be an isomorphism of a subalgebra $\boldsymbol{A}_0\subset\boldsymbol{A}$ on a dense subalgebra $\boldsymbol{B}_0\subset\boldsymbol{B}$. Every homomorphism h of $\boldsymbol{A}$ in $\boldsymbol{B}$ which is an extension of h_0 maps $\boldsymbol{A}$ on $\boldsymbol{B}$.*

Let $B\in\boldsymbol{B}$ and let $A\in\boldsymbol{A}$ be the Boolean product of all elements $h_0^{-1}(B_2)$ where $B_2\in\boldsymbol{B}_0$ and $B\subset B_2$. If $B_1,B_2\in\boldsymbol{B}_0$ and $B_1\subset B\subset B_2$, then

$$h_0^{-1}(B_1)\subset A\subset h_0^{-1}(B_2).$$

Consequently

$$B_1=h(h_0^{-1}(B_1))\subset h(A)\subset h(h_0^{-1}(B_2))=B_2.$$

Since $\boldsymbol{B}_0$ is dense in $\boldsymbol{B}$, B is the Boolean product of all $B_2\in\boldsymbol{B}_0$, $B\subset B_2$ (see 1.3 b)), and B is the Boolean sum [38]) of all $B_1\in\boldsymbol{B}_0$, $B_1\subset B_2$. Thus $B=h(A)$ which proves the theorem.

3.5. *Let $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ be dense subalgebras of complete Boolean algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. Every isomorphism of $\boldsymbol{A}_0$ on $\boldsymbol{B}_0$ can be extended to an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$.*

This follows immediately from 1.2 and 1.3.

A complete Boolean algebra $\boldsymbol{B}$ is called, according to MacNeille [16]), a *minimal extension* of a Boolean algebra $\boldsymbol{A}$ if

(i) $\boldsymbol{B}$ contains a subalgebra $\boldsymbol{B}_0$ isomorphic to $\boldsymbol{A}$;

(ii) every isomorphism h_0 of $\boldsymbol{B}_0$ in a complete Boolean algebra $\boldsymbol{C}$ can be extended to an isomorphism h of $\boldsymbol{B}$ in $\boldsymbol{C}$.

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[36]) See Kuratowski [1], p. 49.

[37]) See Sikorski [4], p. 332.

[38]) The definition of the infinite Boolean sum is dual to that of the product. The mentioned property follows from 1.3 b) and de Morgan's formulas.

3.6. *The four following conditions are equivalent (for given Boolean algebras* $\boldsymbol{A}$ *and* $\boldsymbol{B}$*):*

a) $\boldsymbol{B}$ *is a minimal extension of* $\boldsymbol{A}$;

b) $\boldsymbol{B}$ *is isomorphic to* $\mathfrak{B}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$;

c) $\boldsymbol{B}$ *is complete and contains a dense subalgebra* $\boldsymbol{B}_0$ *isomorphic to* $\boldsymbol{A}$;

d) $\boldsymbol{B}$ *is complete and contains a regular subalgebra* $\boldsymbol{B}_0$ *such that* $\boldsymbol{B}_0 \approx \boldsymbol{A}$ *and the smallest complete subalgebra of* $\boldsymbol{B}$ *containing* $\boldsymbol{B}_0$ *is* $\boldsymbol{B}$ *itself.*

Consequently *all minimal extensions of* $\boldsymbol{A}$ *are isomorphic* [39].

a) implies b). Suppose $\boldsymbol{B}$ is a minimal extension of $\boldsymbol{A}$, and let $\boldsymbol{B}_0$ satisfy (i) and (ii). Let h_0 be an isomorphism of $\boldsymbol{B}_0$ on $\boldsymbol{S}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$. By a) the isomorphism h_0 can be extended to an isomorphism h of $\boldsymbol{B}$ in $\mathfrak{B}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$ since the least Boolean algebra is complete. By 3.1 and 3.4, the isomorphism h maps $\boldsymbol{B}$ on $\mathfrak{B}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$, q. e. d.

b) implies c). This follows from 3.1.

c) implies a). This follows from 3.3.

c) implies d). This follows from 1.3.

d) implies a). It follows from the proved equivalence a)≡b) that every Boolean algebra $\boldsymbol{A}$ possesses a minimal extension $\boldsymbol{C}$ (e. g. $\mathfrak{B}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$ is a minimal extension of $\boldsymbol{A}$). We may suppose that $\boldsymbol{A}\subset\boldsymbol{C}$. As we have already proved, $\boldsymbol{A}$ is dense in $\boldsymbol{C}$.

Suppose the condition d) is satisfied. Then there is an isomorphism h of $\boldsymbol{C}$ in $\boldsymbol{B}$ which maps $\boldsymbol{A}$ on $\boldsymbol{B}_0$. On account of d) it is sufficient to prove that the subalgebra $\boldsymbol{B}_1 = h(\boldsymbol{C})$ is a regular subalgebra of $\boldsymbol{B}$ (that is, $\boldsymbol{B}=\boldsymbol{B}_1$).

Suppose $(\boldsymbol{B}_1)\prod_{u\in U} B_u = 0$ where $B_u \in \boldsymbol{B}_1$, that is $B_u = h(C_u)$ where $C_u \in \boldsymbol{C}$. We have also $(\boldsymbol{C})\prod_{u\in U} C_u = 0$ since h is an isomorphism. $\boldsymbol{A}$ being dense in $\boldsymbol{C}$, for every $u\in U$ there exists a set $\{A_v\}_{v\in V_u}$ of elements of $\boldsymbol{A}$ such that $(\boldsymbol{C})\prod_{v\in V_u} A_v = C_u$ (see 1.3 b)). We may assume that $V_{u'}V_{u''}=0$ for $u'\neq u''$. Let $V=\sum_{u\in U} V_u$. We have

$$(\boldsymbol{C})\prod_{v\in V} A_v = (\boldsymbol{C})\prod_{u\in U} C_u = 0 \quad\text{and}\quad B_u = (\boldsymbol{B}_1)\prod_{v\in V_u} h(A_v) \subset (\boldsymbol{B})\prod_{v\in V_u} h(A_v)\ ^{40}).$$

[39]) Mac Neille [1].

[40]) See footnote [25]).

Consequently $(\boldsymbol{A})\prod_{v\in V} A_v = 0$ and $(\boldsymbol{B}_0)\prod_{v\in V} h(A_v) = 0$. Since $\boldsymbol{B}_0$ is a regular subalgebr of $\boldsymbol{B}$, we infer that $(\boldsymbol{B})\prod_{v\in V} h(A_v) = 0$. Finally

$$(\boldsymbol{B})\prod_{u\in U} B_u = (\boldsymbol{B})\prod_{u\in U}\Big((\boldsymbol{B}_1)\prod_{v\in V_u} h(A_v)\Big) \subset (\boldsymbol{B})\prod_{u\in U}\Big((\boldsymbol{B})\prod_{v\in V_u} h(A_v)\Big) = (\boldsymbol{B})\prod_{v\in V} h(A_v) = 0$$

which proves that $\boldsymbol{B}_1$ is a regular subalgebra of $\boldsymbol{B}$, q. e. d.

3.7. *The three following conditions are equivalent (for given Boolean algebras* $\boldsymbol{A}$ *and* $\boldsymbol{B}$*):*

a) $\boldsymbol{B}$ *is a minimal σ-extension of* $\boldsymbol{A}$;

b) $\boldsymbol{B}$ *is σ-complete and contains a dense subalgebra* $\boldsymbol{B}_0$ *which is an isomorph of* $\boldsymbol{A}$ *and a σ-generator of* $\boldsymbol{B}$;

c) $\boldsymbol{B}$ *is σ-complete and contains a regular subalgebra* $\boldsymbol{B}_0$ *which is an isomorph of* $\boldsymbol{A}$ *and a σ-generator of* $\boldsymbol{B}$.

a) implies b). This follows from 3.1 and 1.5.

b) implies c). This follows from 1.3.

c) implies a). Suppose the condition c) is satisfied. Let $\boldsymbol{C}$ be a minimal extension of $\boldsymbol{B}$. We may assume that $\boldsymbol{B}\subset\boldsymbol{C}$, that is, $\boldsymbol{B}$ is a regular subalgebra of $\boldsymbol{C}$ and the smallest complete subalgebra of $\boldsymbol{C}$ which contains $\boldsymbol{B}$, is $\boldsymbol{C}$ itself (see 3.6 d)). On account of 3.6 and 1.1, $\boldsymbol{C}$ is a minimal extension of $\boldsymbol{B}_0$. Let h be an isomorphism of $\boldsymbol{C}$ on $\mathfrak{B}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$ which maps $\boldsymbol{B}_0$ on $\boldsymbol{S}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$. Since $\boldsymbol{B}_0$ is a σ-generator of $\boldsymbol{B}$, we infer by 1.5 that h maps $\boldsymbol{B}$ on $\boldsymbol{S}^{\sigma}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$, q. e. d.

3.8. *Let* $\boldsymbol{C}$ *be a minimal extension of a Boolean algebra* $\boldsymbol{A}$, $\boldsymbol{A}\subset\boldsymbol{C}$, *and let* $\boldsymbol{B}$ *be the smallest σ-subalgebra of* $\boldsymbol{C}$ *which contains* $\boldsymbol{A}$. *Then* $\boldsymbol{B}$ *is a minimal σ-extension of* $\boldsymbol{A}$.

This follows from 3.6 and 3.7.

3.9. *A minimal σ-extension* $\boldsymbol{B}$ *of a Boolean algebra* $\boldsymbol{A}$ *is isomorphic to a σ-field of sets if and only if*

(m) *for every* $A\in\boldsymbol{A}$, $A\neq 0$, *there is a two-valued σ-measure* μ *on* $\boldsymbol{A}$ *such that* $\mu(A)=1$.

We may assume $\boldsymbol{A}\subset\boldsymbol{B}$, that is, $\boldsymbol{A}$ is a dense regular subalgebra of $\boldsymbol{B}$ and a σ-generator of $\boldsymbol{B}$ (see 3.7). Suppose the condition (m) satisfied. Let $0\neq B\in\boldsymbol{B}$. There is an element $A\in\boldsymbol{A}$ such that $0\neq A\subset B$. By (m) there is a two-valued σ-measure μ on $\boldsymbol{A}$ such that $\mu(A)=1$. By Carathéodory's exterior measure method, the measure μ can be extended to a two-valued σ-measure ν on $\boldsymbol{B}$. Obviously $\nu(B)=1$. The element B being arbitrary, we infer that the σ-complete Boolean algebra $\boldsymbol{B}$ is isomorphic to a σ-field of sets [41]).

On the other hand, if $\boldsymbol{B}$ is isomorphic to a σ-field of sets, for every $A \in \boldsymbol{A}$, $A \neq 0$, there is a two-valued σ-measure μ on $\boldsymbol{B}$ such that $\mu(A)=1$ [41]). The measure μ restricted to elements of $\boldsymbol{A}$ satisfies the condition (m).

We note else that a minimal extension of a Boolean algebra $\boldsymbol{A}$ is isomorphic to a completely additive field of sets if and only if $\boldsymbol{A}$ is atomic. This follows [42]) immediately from 3.6 c).

4. The spaces $\mathcal{S}$ and $\mathcal{S}^*$. In the rest of this paper we shall consider a fixed family $\{\boldsymbol{A}_\tau\}_{\tau\in T}$ of Boolean algebras [39]), different or not.

For the simplicity we admit the following notations:

$$\mathcal{S}_\tau=\mathcal{S}(\boldsymbol{A}_\tau),\quad \boldsymbol{S}_\tau=\boldsymbol{S}(\boldsymbol{A}_\tau),\quad \boldsymbol{J}_\tau=\boldsymbol{J}(\boldsymbol{A}_\tau),\quad \mathfrak{J}_\tau=\mathfrak{J}(\boldsymbol{A}_\tau),$$
$$\boldsymbol{S}=\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{S}_\tau,\quad \boldsymbol{S}^{\sigma}=(\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{S}_\tau)^{\sigma}=\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{S}_\tau.$$

$\mathfrak{s}_\tau$ will denote always Stone's isomorphism of $\boldsymbol{A}_\tau$ on $\boldsymbol{S}_\tau$, defined in § 2.

If $X\subset\mathcal{S}_\tau$, then $\pi_\tau(X)$ will denote the set of all points of $\mathbf{P}_{\tau\in T}\mathcal{S}_\tau$ whose τ-th coordinate belongs to X. If $\boldsymbol{X}_\tau$ is a class of subsets of $\mathcal{S}_\tau$, then $\tilde{\boldsymbol{X}}_\tau$ will denote the class of all sets $\pi_\tau(X)$ where $X\in\boldsymbol{X}_\tau$. E. g. $\tilde{\boldsymbol{S}}_\tau$ and $\tilde{\boldsymbol{S}}^{\sigma}_\tau$ are fields (of subsets of $\mathbf{P}_{\tau\in T}\mathcal{S}_\tau$) isomorphic to $\boldsymbol{S}_\tau$ and $\boldsymbol{S}^{\sigma}_\tau$ respectively; $\tilde{\boldsymbol{J}}_\tau$ and $\tilde{\mathfrak{J}}_\tau$ are σ-ideals of subsets of $\mathbf{P}_{\tau\in T}\mathcal{S}_\tau$.

The smallest σ-ideal containing all the ideals $\tilde{\boldsymbol{J}}_\tau$ $(\tau\in T)$ will be denoted by $\boldsymbol{J}$.

We shall consider the cartesian product $\mathbf{P}_{\tau\in T}\mathcal{S}_\tau$ of spaces $\mathcal{S}_\tau$ as a topological space with two different closure operations.

First we shall consider the set $\mathbf{P}_{\tau\in T}\mathcal{S}_\tau$ as the usual topological product of the topological spaces $\mathcal{S}_\tau$. This space will be denoted by $\mathcal{S}$. Neighborhoods in $\mathcal{S}$ are sets $\mathbf{P}_{\tau\in T}X_\tau$ where $X_\tau\in\boldsymbol{S}_\tau$ and the inequality $X_\tau\neq\mathcal{S}_\tau$ holds only for a finite number of elements $\tau\in T$. $\mathcal{S}$ is a bicompact totally disconnected Hausdorff space.

The σ-ideal of all sets of first category in the space $\mathcal{S}$ will be denoted by $\mathfrak{J}$.

[41]) See Sikorski [1], p. 250, th. 1.4.
[42]) See Sikorski [1], p. 249, th. 1.1.

Beside the above-mentioned usual topology in $\mathbf{P}_{\tau\in T}\mathcal{S}_\tau$ we shall also consider another topology in this space. Neighborhoods in this topology are sets $\mathbf{P}_{\tau\in T}X_\tau$ where $X_\tau\in\boldsymbol{S}_\tau$ and the inequality $X_\tau\neq\mathcal{S}_\tau$ holds only for an at most enumerable number of elements $\tau\in T$. The space $\mathbf{P}_{\tau\in T}\mathcal{S}_\tau$ with the so defined topology will be denoted by $\mathcal{S}^*$. $\mathcal{S}^*$ is also a totally disconnected Hausdorff space. The spaces $\mathcal{S}$ and $\mathcal{S}^*$ are identical if and only if T is finite (provided each $\mathcal{S}_\tau$ has at most two elements).

The class of all neighborhoods of $\mathcal{S}^*$, above defined, will be denoted by $\boldsymbol{S}^*_0$. The symbol $\boldsymbol{S}^*$ will denote the least field containing $\boldsymbol{S}^*_0$. The σ-ideal of all sets of first category in the space $\mathcal{S}^*$ will be denoted by $\mathfrak{J}^*$.

4.1. *No open non-empty subset of $\mathcal{S}$ is of first category in $\mathcal{S}$.*

This follows from the fact that $\mathcal{S}$ is a bicompact normal space [28]).

4.2. *No open non-empty subset of $\mathcal{S}^*$ is of first category in $\mathcal{S}^*$.*

Let $G\neq 0$ be open in $\mathcal{S}^*$ and let N_n be a sequence of closed nowhere dense subsets of $\mathcal{S}^*$. Since $G-N_1\neq 0$, there is a neighborhood $G_1=\mathbf{P}_{\tau\in T}X^1_\tau$ such that $0\neq G_1\subset G-N_1$. By induction we define easily a sequence of neighborhoods $G_n=\mathbf{P}_{\tau\in T}X^n_\tau$ such that $0\neq G_n\subset G_{n-1}-N_n$. We have $\prod_{n=1}^{\infty}G_n=\mathbf{P}_{\tau\in T}(\prod_{n=1}^{\infty}X^n_\tau)$. Since $X^1_\tau, X^2_\tau, \ldots$ is a decreasing sequence of closed subsets of the bicompact space $\mathcal{S}_\tau$, we infer $\prod_{n=1}^{\infty}X^n_\tau\neq 0$. Consequently $0\neq\prod_{n=1}^{\infty}G_n\subset G-\sum_{n=1}^{\infty}N_n$, which proves the theorem.

4.3. *If a set $X\subset\mathcal{S}_\tau$ is of first category in $\mathcal{S}_\tau$, then $\pi_\tau(X)$ is of first category in $\mathcal{S}$ and in $\mathcal{S}^*$.*

Consequently $\tilde{\boldsymbol{J}}_\tau\subset\tilde{\mathfrak{J}}_\tau\subset\mathfrak{J}$, $\tilde{\boldsymbol{J}}_\tau\subset\tilde{\mathfrak{J}}_\tau\subset\mathfrak{J}^*$, $\boldsymbol{J}\subset\mathfrak{J}$ and $\boldsymbol{J}\subset\mathfrak{J}^*$.

The easy proof is omitted.

5. Maximal products. Elementary properties. The Boolean algebra $\boldsymbol{S}/\boldsymbol{J}$ is called the (cartesian) *product* of all the Boolean algebras $\boldsymbol{A}_\tau$ $(\tau\in T)$ and denoted by $\mathbf{P}^{a}_{\tau\in T}\boldsymbol{A}_\tau$. The Boolean algebra $\boldsymbol{S}^{\sigma}/\boldsymbol{J}$ is called the (cartesian) *maximal σ-product* of the Boolean algebras $\boldsymbol{A}_\tau$ $(\tau\in T)$ and denoted by $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$. By definition

(a) $\mathbf{P}^{a}_{\tau\in T}\boldsymbol{A}_\tau=\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{S}_\tau/\boldsymbol{J}$, (b) $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau=\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{S}_\tau/\boldsymbol{J}$.

5.1. *$\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$ is a σ-complete Boolean algebra. $\mathbf{P}^{a}_{\tau\in T}\boldsymbol{A}_\tau$ is a subalgebra of $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$.*

In general, $\mathbf{P}^{a}_{\tau\in T}\boldsymbol{A}_\tau$ is not a σ-regular subalgebra of $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$.

5.2. *If, for every* $\tau \in T$, $\boldsymbol{A}_\tau$ *is isomorphic to a Boolean algebra* $\boldsymbol{B}_\tau$, *then* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ *is isomorphic to* $\mathbf{P}^a_{\tau\in T}\boldsymbol{B}_\tau$, *and* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$ *is isomorphic to* $\mathbf{P}^b_{\tau\in T}\boldsymbol{B}_\tau$.

This follows from the fact that $\mathcal{S}(\boldsymbol{A}_\tau)$ is then homeomorphic to $\mathcal{S}(\boldsymbol{B}_\tau)$.

5.3. $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau \approx \mathbf{P}^a_{\tau\in T}\boldsymbol{S}_\tau$.

More exactly

The mapping $S \to S/\boldsymbol{J}$ *(for* $S \in \boldsymbol{S}$*) is an isomorphism of* $\boldsymbol{S} = \mathbf{P}^a_{\tau\in T}\boldsymbol{S}_\tau$ *on* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$.

This results from 1.5, 4.1 and 4.3.

5.4. *If, for every* $\tau \in T$, $\boldsymbol{A}_\tau$ *is isomorphic to a field of sets* $\boldsymbol{X}_\tau$, *then* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau \approx \mathbf{P}^a_{\tau\in T}\boldsymbol{X}_\tau$.

This follows from 5.3 and 0.1.

5.5. *For every* $\tau \in T$, $\boldsymbol{A}_\tau \approx \boldsymbol{S}_\tau/\boldsymbol{J}_\tau \approx \widetilde{\boldsymbol{S}}_\tau/\boldsymbol{J}$ *and* $\boldsymbol{S}^\sigma_\tau/\boldsymbol{J}_\tau \approx \widetilde{\boldsymbol{S}}^\sigma_\tau/\boldsymbol{J}$. *If* $\boldsymbol{A}_\tau$ *is* σ*-complete, then* $\boldsymbol{A}_\tau \approx \boldsymbol{S}^\sigma_\tau/\boldsymbol{J}_\tau \approx \widetilde{\boldsymbol{S}}^\sigma_\tau/\boldsymbol{J}$.

More exactly:

The mapping $A \to \pi_\tau(\mathfrak{s}_\tau(A))/\boldsymbol{J}$ *(for* $A \in \boldsymbol{A}_\tau$*) is an isomorphism of* $\boldsymbol{A}_\tau$ *on* $\widetilde{\boldsymbol{S}}_\tau/\boldsymbol{J}$. *The mapping* $S/\boldsymbol{J}_\tau \to \pi_\tau(S)/\boldsymbol{J}$ *(for* $S \in \boldsymbol{S}^\sigma_\tau$*) is an isomorphism of* $\boldsymbol{S}^\sigma_\tau/\boldsymbol{J}_\tau$ *(the maximal extension of* $\boldsymbol{A}$*) on* $\widetilde{\boldsymbol{S}}^\sigma_\tau/\boldsymbol{J}$, *which transforms* $\boldsymbol{S}_\tau/\boldsymbol{J}_\tau$ *on* $\widetilde{\boldsymbol{S}}_\tau/\boldsymbol{J}$.

The easy proof based on 4.3 is omitted.

5.6. $\widetilde{\boldsymbol{S}}^\sigma_\tau/\boldsymbol{J}$ *is a* σ*-subalgebra of* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$. $\widetilde{\boldsymbol{S}}_\tau/\boldsymbol{J}$ *is a* σ*-regular subalgebra of* $\widetilde{\boldsymbol{S}}^\sigma_\tau/\boldsymbol{J}$, *thus of* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$ *also.*

The first remark is obvious. The second follows from 5.5, 1.1, 1.2, and 2.3.

5.7. *The class* $\sum_{\tau\in T}\widetilde{\boldsymbol{S}}_\tau/\boldsymbol{J}$ *(i. e. the class of all elements of Boolean algebras* $\widetilde{\boldsymbol{S}}_\tau/\boldsymbol{J}$, $\tau \in T$*) is a generator of* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ *and a* σ*-generator of* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$.

This is obvious.

5.8. *The subalgebras* $\boldsymbol{S}_\tau/\boldsymbol{J}$ $(\tau \in T)$ *are independent in* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$, *and* σ*-independent in* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$.

We shall prove only the second remark. The proof of the first is similar.

Let $A_n \in \widetilde{\boldsymbol{S}}_{\tau_n}/\boldsymbol{J}$ $(\tau_i \neq \tau_j$ for $i \neq j)$ be a finite or enumerable sequence. By definition $A_n = \pi_{\tau_n}(S_n)/\boldsymbol{J}$ where $S_n \in \boldsymbol{S}_{\tau_n}$. We have

$$(\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau)\prod_n A_n = (\prod_n \pi_{\tau_n}(S_n))/\boldsymbol{J} = (\mathbf{P}_{\tau\in T}S_\tau)/\boldsymbol{J}$$

where $S_{\tau_n} = S_n$ for $n = 1, 2, \ldots$, and $S_\tau = \mathcal{S}_\tau$ for all remaining τ. The set $\mathbf{P}_{\tau\in T}S_\tau$ is open in the space $\mathcal{S}^*$. Hence $\mathbf{P}_{\tau\in T}S_\tau$ non $\in \boldsymbol{J}$ on account of 4.2 and 4.3. Consequently $(\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau)\prod_n A_n \neq 0$, q. e. d.

5.9. *If* $\overline{\overline{T}} = 1$, *that is, if* $\{\boldsymbol{A}_\tau\}_{\tau\in T}$ *contains only one Boolean algebra* $\boldsymbol{A}$, *then* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau = \boldsymbol{S}(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A})$ *and* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau = \boldsymbol{S}^\sigma(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A})$.

This follows immediately from the definition of the maximal σ-product.

6. Characteristic properties of maximal products.

Let $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ be a family of subalgebras of a Boolean algebra $\boldsymbol{B}$ and let $\boldsymbol{B}_0$ be the smallest subalgebra of $\boldsymbol{B}$ containing all $\boldsymbol{B}_\tau$, $\tau \in T$. We shall say that the family $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ possesses the property (E) if, for every Boolean algebra $\boldsymbol{C}$ and for every family of homomorphisms h_τ $(\tau \in T)$ of $\boldsymbol{B}_\tau$ in $\boldsymbol{C}$, there is a homomorphism h of $\boldsymbol{B}_0$ in $\boldsymbol{C}$ which is a common extension of all the homomorphisms h_τ, $\tau \in T$.

Let $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ be a family of subalgebras of a σ-complete Boolean algebra $\boldsymbol{B}$ and let $\boldsymbol{B}_0$ be the smallest σ-subalgebra of $\boldsymbol{B}$ which contains all $\boldsymbol{B}_\tau$, $\tau \in T$. We shall say that the family $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ possesses the property (E_σ) if, for every σ-complete Boolean algebra $\boldsymbol{C}$ and for every family of σ-homomorphisms h_τ of $\boldsymbol{B}_\tau$ in $\boldsymbol{C}$, there is a σ-homomorphism h of $\boldsymbol{B}_0$ in $\boldsymbol{C}$ which is a common extension of all the σ-homomorphisms h_τ, $\tau \in T$.

6.1. *The family* $\{\widetilde{\boldsymbol{S}}_\tau/\boldsymbol{J}\}_{\tau\in T}$ *of subalgebras of* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ *possesses the property* (E).

This follows immediately from 5.8 and the fact that every family of independent subalgebras possesses the property (E)[43].

6.2. *In order that a Boolean algebra* $\boldsymbol{B}$ *be isomorphic to* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ *it is necessary and sufficient that there be a family* $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ *of subalgebras of* $\boldsymbol{B}$ *such that*

a) $\boldsymbol{A}_\tau \approx \boldsymbol{B}_\tau$ *for every* $\tau \in T$;

b) *the family* $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ *possesses the property* (E);

c) *the set* $\sum_{\tau\in T}\boldsymbol{B}_\tau$ *is a generator of* $\boldsymbol{B}$.

The necessity follows from 5.5, 6.1, and 5.7.

Suppose the conditions a-c) are fulfiled. Let h_τ be an isomorphism of $\boldsymbol{S}_\tau/\boldsymbol{J}$ on $\boldsymbol{B}_\tau$. By 6.1 and 5.7 the isomorphism h_τ can be extended to a homomorphism h of $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ in B. By b) and c),

[43]) See Sikorski [6], Theorem III.

the converse isomorphisms h_τ^{-1} can be extended to an homomorphism g of $\boldsymbol{B}$ in $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$. On account of 1.4, h is an isomorphism of $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ on $\boldsymbol{B}$.

Theorem 6.2 can be also formulated in the following way:

6.2'. *In order that a Boolean algebra $\boldsymbol{B}$ be isomorphic to $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ it is necessary and sufficient that there be a family $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ of subalgebras of $\boldsymbol{B}$ which satisfies the conditions* a) *and* c) *of* 6.2 *and the following condition:*

b') *the subalgebras $\boldsymbol{B}_\tau$ are independent in $\boldsymbol{B}$.*

In fact, b') implies b) [44].

6.3. *The family $\{\tilde{S}^\sigma_\tau/\boldsymbol{J}\}_{\tau\in T}$ of subalgebras of $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$ possesses the property (E_σ).*

It is sufficient to prove the following lemma (see 2.5):

Let $\boldsymbol{C}$ be a σ-complete Boolean algebra, and, for every $\tau\in T$, let h_τ be a σ-homomorphism of $\tilde{S}_\tau/\boldsymbol{J}$ in $\boldsymbol{S}^\sigma(\boldsymbol{C})/\boldsymbol{J}(\boldsymbol{C})$. The σ-homomorphisms $\{h_\tau\}$ can be extended to a σ-homomorphism h of $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$ in $\boldsymbol{S}^\sigma(\boldsymbol{C})/\boldsymbol{J}(\boldsymbol{C})$.

The formula

(i) $$\bar{h}_\tau(S/\boldsymbol{J}_\tau)=h(\pi_\tau(S)/\boldsymbol{J})\quad\text{for}\quad S\in\boldsymbol{S}_\tau$$

defines a σ-homomorphism $\bar{h}_\tau$ of $\boldsymbol{S}_\tau/\boldsymbol{J}_\tau$ in $\mathcal{S}(\boldsymbol{C})/\boldsymbol{J}(\boldsymbol{C})$ on account of 5.5. By 2.4 the σ-homomorphism $\bar{h}_\tau$ is induced by a mapping φ_τ of $\mathcal{S}(\boldsymbol{C})$ in $\mathcal{S}(\boldsymbol{A}_\tau)=\mathcal{S}_\tau$, that is

(ii) $$\bar{h}_\tau(S/\boldsymbol{J}_\tau)=\varphi_\tau^{-1}(S)/\boldsymbol{J}(\boldsymbol{C})\quad\text{for}\quad S\in\boldsymbol{S}_\tau.$$

The condition (ii) implies that

(iii) $$\varphi_\tau^{-1}(S)\in\boldsymbol{J}(\boldsymbol{C})\quad\text{for}\quad S\in\boldsymbol{J}_\tau.$$

since $\bar{h}(S/\boldsymbol{J}_\tau)=h(0)=0$ for $S\in\boldsymbol{J}_\tau$.

Consider the mapping $\varphi(c)=\{\varphi_\tau(c)\}$ of $\mathcal{S}(\boldsymbol{C})$ in $\mathcal{S}=\mathbf{P}_{\tau\in T}\mathcal{S}_\tau$. If $S\in\boldsymbol{J}_\tau$, then by (iii)

$$\varphi^{-1}(\pi_\tau(S))=\varphi_\tau^{-1}(S)\in\boldsymbol{J}(\boldsymbol{C}).$$

Consequently

(iv) $$\varphi^{-1}(S)\in\boldsymbol{J}(\boldsymbol{C})\quad\text{for every}\quad S\in\boldsymbol{J}.$$

[44]) See Sikorski [6], Theorem III. One can prove that b) and b') are equivalent.

It follows from (iv) that the formula

$$h(S/\boldsymbol{J})=\varphi^{-1}(S)/\boldsymbol{J}(\boldsymbol{C})\quad\text{for}\quad S\in\boldsymbol{S}^\sigma$$

defines a σ-homomorphism h of $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$ in $\boldsymbol{S}(\boldsymbol{C})/\boldsymbol{J}(\boldsymbol{C})$. By (i) and (ii), if $S\in\boldsymbol{S}_\tau$, then

$$h(\pi_\tau(S)/\boldsymbol{J})=\varphi^{-1}(\pi_\tau(S))/\boldsymbol{J}(\boldsymbol{C})=\varphi_\tau^{-1}(S)/\boldsymbol{J}(\boldsymbol{C})=\bar{h}_\tau(S/\boldsymbol{J}_\tau)=h_\tau(\pi_\tau(S)/\boldsymbol{J}),$$

which proves that h is a common extension of all h_τ.

6.4. *In order that a σ-complete Boolean algebra $\boldsymbol{B}$ be isomorphic to the maximal σ-product $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$, it is necessary and sufficient that there be a family $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ of σ-regular subalgebras of $\boldsymbol{B}$ such that*

a) *$\boldsymbol{B}_\tau\approx\boldsymbol{A}_\tau$ for $\tau\in T$;*

b) *the family $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ possesses the property (E_σ);*

c) *the set $\sum_{\tau\in T}\boldsymbol{B}_\tau$ is a σ-generator of $\boldsymbol{B}$.*

The necessity follows from 5.6, 5.5, 6.3, and 5.7.

The proof of the sufficiency is analogous to that of 6.2.

The condition b) cannot be replaced by the hypothesis that the subalgebras $\boldsymbol{B}_\tau$ are σ-independent in $\boldsymbol{B}$.

7. Commutativity and associativity of maximal products. It is obvious that

7.1. *If $t(\tau)$ is a one-one mapping of T on T, then $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_{t(\tau)}$ and $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_{t(\tau)}$.*

The maximal cartesian products are thus completely commutative. They are also completely associative. This follows from the following theorem:

7.2. *Let T be the sum of mutually disjoint non-empty sets T_u $(u\in U)$. Then $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^a_{u\in U}(\mathbf{P}^a_{\tau\in T_u}\boldsymbol{A}_\tau)$ and $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^b_{u\in U}(\mathbf{P}^b_{\tau\in T_u}\boldsymbol{A}_\tau)$.*

We shall prove theorem 7.2 only in case of maximal σ-products. The proof of the first part of 7.2 is analogous [45]).

On account of 6.4 the maximal σ-product $\boldsymbol{B}=\mathbf{P}^b_{u\in U}(\mathbf{P}^b_{\tau\in T_u}\boldsymbol{A}_\tau)$ contains a family of σ-regular subalgebras $\boldsymbol{B}'_u$ such that

a') $\boldsymbol{B}'_u\approx\mathbf{P}^b_{\tau\in T_u}\boldsymbol{A}_\tau$;

b') the family $\{\boldsymbol{B}'_u\}_{u\in U}$ possesses the property (E_σ);

c') the set $\sum_{u\in U}\boldsymbol{B}'_u$ is a σ-generator of $\boldsymbol{B}$.

[45]) Another proof of this fact follows from 5.4.

By a′) and 6.4, every algebra $\mathbf{B}'_u$ contains a family of σ-regular subalgebras $\{\mathbf{B}_\tau\}_{\tau\in T_u}$ such that

a″) $\mathbf{B}_\tau \approx \mathbf{A}_\tau$;

b″) the family $\{\mathbf{B}_\tau\}_{\tau\in T_u}$ possesses the property (E_σ);

c″) the set $\sum_{\tau\in T_u}\mathbf{B}_\tau$ is a σ-generator of $\mathbf{B}'_u$.

By 1.1, $\mathbf{B}_\tau$ is a σ-regular subalgebra of $\mathbf{B}$. The conditions a′-a″), b′-b″), and c′-c″) imply respectively the conditions a), b), and c) of theorem 6.4. Therefore $\mathbf{B}\approx \mathbf{P}^b_{\tau\in T}\mathbf{A}_\tau$, q. e. d.

By 5.9 and 7.2 (where $U=T$ and $T_u=(u)$) we obtain

7.3. *If, for every $\tau\in T$, $\mathbf{B}_\tau$ is a maximal σ-extension of $\mathbf{A}_\tau$, then $\mathbf{P}^b_{\tau\in T}\mathbf{B}_\tau\approx\mathbf{P}^b_{\tau\in T}\mathbf{A}_\tau$.*

8. The structure of independent subalgebras. If $\{\mathbf{B}_\tau\}_{\tau\in T}$ and $\{\mathbf{C}_\tau\}_{\tau\in T}$ are two families of subalgebras of Boolean algebras $\mathbf{B}$ and $\mathbf{C}$ respectively, such that

α) $\mathbf{B}_\tau\approx\mathbf{C}_\tau$ for every $\tau\in T$;

β) $\mathbf{B}_\tau\,(\tau\in T)$ are independent in $\mathbf{B}$; $\mathbf{C}_\tau\,(\tau\in T)$ are independent in $\mathbf{C}$;

γ) $\sum_{\tau\in T}\mathbf{B}_\tau$ is a generator of $\mathbf{B}$; $\sum_{\tau\in T}\mathbf{C}_\tau$ is a generator of $\mathbf{C}$;

then $\mathbf{B}\approx\mathbf{C}$. This fact follows easily from 1.4 and Theorem III in my paper [6] [46]).

On the other hand, if $\{\mathbf{B}_\tau\}_{\tau\in T}$ and $\{\mathbf{C}_\tau\}_{\tau\in T}$ are two families of subalgebras of σ-complete Boolean algebras $\mathbf{B}$ and $\mathbf{C}$ respectively, such that

α) $\mathbf{B}_\tau\approx\mathbf{C}_\tau$ for every $\tau\in T$;

β) $\mathbf{B}_\tau\,(\tau\in \mathbf{T})$ are σ-independent in $\mathbf{B}$; $\mathbf{C}_\tau\,(\tau\in T)$ are σ-independent in $\mathbf{C}$;

γ) $\sum_{\tau\in T}\mathbf{B}_\tau$ is a σ-generator of $\mathbf{B}$; $\sum_{\tau\in T}\mathbf{C}_\tau$ is a σ-generator of $\mathbf{C}$;

then, in general, $\mathbf{B}$ is not isomorphic to $\mathbf{C}$. The following theorem explains this fact.

8.1. *Let $\{\mathbf{B}_\tau\}_{\tau\in T}$ be a family of σ-regular subalgebras of a σ-complete Boolean algebra $\mathbf{B}$ such that $\sum_{\tau\in T}\mathbf{B}_\tau$ is a σ-generator of $\mathbf{B}$. Suppose $\mathbf{B}_\tau\approx\mathbf{A}_\tau$ for every $\tau\in\mathbf{T}$. Then:*

a) *$\mathbf{B}$ is isomorphic to the Boolean algebra $\mathbf{S}^\sigma/\mathbf{I}$ where $\mathbf{I}$ is a σ-ideal such that $J\subset\mathbf{I}$;*

b) *the subalgebras $\mathbf{B}_\tau$ are independent in $\mathbf{B}$ if and only if no set $G\neq 0$ open in the space $\mathcal{S}$ belongs to $\mathbf{I}$;*

c) *the subalgebras $\mathbf{B}_\tau$ are σ-independent in $\mathbf{B}$ if and only if no set $G\neq 0$ open in the space $\mathcal{S}^*$ belongs to $\mathbf{I}$.*

[46]) Or: from 6.2′.

In particular:

b′) *If $\mathbf{J}\subset\mathbf{I}\subset\mathfrak{J}$, then the subalgebras $\mathbf{B}_\tau$ are independent in $\mathbf{B}$; if $\mathbf{I}\neq\mathfrak{J}\subset\mathbf{I}$, then the subalgebras $\mathbf{B}_\tau$ are not independent.*

c′) *If $\mathbf{J}\subset\mathbf{I}\subset\mathfrak{J}^*$, then the subalgebras $\mathbf{B}_\tau$ are σ-independent in $\mathbf{B}$; if $\mathbf{I}\neq\mathfrak{J}^*\subset\mathbf{I}$, then the subalgebras $\mathbf{B}_\tau$ are not σ-independent.*

Let h_τ be an isomorphism of $\mathbf{A}_\tau$ on $\mathbf{B}_\tau$. The formula

$$\overline{h}_\tau(\pi_\tau(\mathfrak{s}_\tau(A)))=h_\tau(A)\quad\text{for}\quad A\in\mathbf{A}_\tau$$

defines an isomorphism $\overline{h}_\tau$ of $\widetilde{\mathbf{S}}_\tau/\mathbf{J}$ on $\mathbf{B}_\tau$. Since $\mathbf{B}_\tau$ is a σ-regular subalgebra of $\mathbf{B}$, we infer that h is a σ-homomorphism of $\widetilde{\mathbf{S}}^\sigma_\tau/\mathbf{J}$ in $\mathbf{B}$. By theorem 6.3 the σ-homomorphisms $\overline{h}_\tau$ $(\tau\in T)$ can be extended to a σ-homomorphism $\overline{h}$ of $\mathbf{P}^b_{\tau\in T}\mathbf{A}_\tau=\mathbf{S}^\sigma/\mathbf{I}$ in $\mathbf{B}$. Since $\sum_{\tau\in T}\mathbf{B}_\tau$ is a σ-generator of $\mathbf{B}$, the homomorphism $\overline{h}$ maps $\mathbf{S}^\sigma/\mathbf{J}$ on $\mathbf{B}$.

The required ideal $\mathbf{I}$ is the least σ-ideal which contains all $S\in\mathbf{S}^\sigma$ such that $\overline{h}(S/\mathbf{J})=0$. In fact, the formula

$$h(S/\mathbf{I})=\overline{h}(S/\mathbf{J})\quad\text{for}\quad S\in\mathbf{S}$$

defines an isomorphism h of $\mathbf{S}^\sigma/\mathbf{I}$ on $\mathbf{B}$. By definition, $\mathbf{J}\subset\mathbf{I}$, which proves a).

The subalgebras $\mathbf{B}_\tau$ are independent if and only if the subalgebras $\widetilde{\mathbf{S}}_\tau/\mathbf{I}$ are independent in $\mathbf{S}^\sigma/\mathbf{I}$, that is, if no neighborhood $\mathbf{P}_{\tau\in T}X_\tau\neq 0$ of the space $\mathcal{S}$ belongs to $\mathbf{I}$. This proves b).

The proof of c) is analogous to that of b). Instead of the space $\mathcal{S}$ one must consider the space $\mathcal{S}^*$.

b′) follows from b), theorem 4.1, and the fact that every set $S\in\mathbf{S}^\sigma$ possesses the property of Baire in the space $\mathcal{S}$.

c′) follows from c), theorem 4.2, and the fact that every set $S\in\mathbf{S}^\sigma$ possesses the property of Baire in the space $\mathcal{S}^*$.

Let $\mathfrak{L}_0$ denote the class of all systems $\langle\mathbf{B},\{\mathbf{B}_\tau\},\{h_\tau\}\rangle$ [47]) where

(i) $\mathbf{B}$ is a σ-complete Boolean algebra;

(ii) for every $\tau\in T$, $\mathbf{B}_\tau$ is a σ-regular subalgebra of $\mathbf{B}$, and $\mathbf{A}_\tau\approx\mathbf{B}_\tau$;

(iii) for every $\tau\in T$, h_τ is an isomorphism of $\mathbf{A}_\tau$ on $\mathbf{B}_\tau$;

(iv) the set $\sum_{\tau\in T}\mathbf{B}_\tau$ is a σ-generator of $\mathbf{B}$.

$\mathfrak{L}$ will denote the class of all $\langle\mathbf{B},\{\mathbf{B}_\tau\},\{h_\tau\}\rangle\in\mathfrak{L}_0$ such that the subalgebras $\mathbf{B}_\tau$ are independent in $\mathbf{B}$. $\mathfrak{L}^*$ will denote the class of all $\langle\mathbf{B},\{\mathbf{B}_\tau\},\{h_\tau\}\rangle\in\mathfrak{L}_0$ such that the subalgebras $\mathbf{B}_\tau$ are σ-independent in $\mathbf{B}$. Obviously $\mathfrak{L}^*\subset\mathfrak{L}\subset\mathfrak{L}_0$.

[47]) τ runs always over the set T.

Let $\langle \boldsymbol{B}^i, \{\boldsymbol{B}^i_\tau\}, \{h^i\}\rangle \in \mathfrak{L}_0$ for $i=1,2$. We shall write

$$\langle \boldsymbol{B}^1, \{\boldsymbol{B}^1_\tau\}, \{h^1_\tau\}\rangle \leqslant \langle \boldsymbol{B}^2, \{\boldsymbol{B}^2_\tau\}, \{h^2_\tau\}\rangle$$

if the isomorphisms [48] $h^1_\tau (h^2_\tau)^{-1}$ can be extended to a σ-homomorphism of $\boldsymbol{B}^2$ in $\boldsymbol{B}^1$.

By theorem 1.4, if both $\langle \boldsymbol{B}^1, \{\boldsymbol{B}^1_\tau\}, \{h^1_\tau\}\rangle \leqslant \langle \boldsymbol{B}^2, \{\boldsymbol{B}^2_\tau\}, \{h^2_\tau\}\rangle$ and $\langle \boldsymbol{B}^2, \{\boldsymbol{B}^2_\tau\}, \{h^2_\tau\}\rangle \leqslant \langle \boldsymbol{B}^1, \{\boldsymbol{B}^1_\tau\}, \{h^1_\tau\}\rangle$, then there is an isomorphism of $\boldsymbol{B}^2$ on $\boldsymbol{B}^1$ which transforms $\boldsymbol{B}^2_\tau$ on $\boldsymbol{B}^1_\tau$. Thus we may identify such two systems.

After this identification, the relation $\leqslant$ orders partly the sets $\mathfrak{L}_0$, $\mathfrak{L}$, and $\mathfrak{L}^*$. It is easy to show that $\langle \boldsymbol{B}^1, \{\boldsymbol{B}^1_\tau\}, \{h^1_\tau\}\rangle \leqslant \leqslant \langle \boldsymbol{B}^2, \{\boldsymbol{B}^2_\tau\}, \{h^2_\tau\}\rangle$ if and only if $\boldsymbol{I}^2\boldsymbol{S}^\sigma \subset \boldsymbol{I}^1\boldsymbol{S}^\sigma$ where $\boldsymbol{I}^i$ denotes the σ-ideal constructed in theorem 8.1 for the system $\langle \boldsymbol{B}^i, \{\boldsymbol{B}^i_\tau\}, \{h^i_\tau\}\rangle$.

It follows from theorem 6.3 that

8.2. *The system* $\langle \mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau, \{\tilde{\boldsymbol{S}}_\tau/\boldsymbol{J}\}, \{h_\tau\}\rangle$, *where* $h_\tau(A)=\pi_\tau(\mathfrak{s}_\tau(A))$, *is the greatest element of the partly ordered sets* $\mathfrak{L}_0$, $\mathfrak{L}$, *and* $\mathfrak{L}^*$.

9. Minimal products. Elementary properties. The Boolean algebra $\boldsymbol{S}^\sigma/\mathfrak{J}$ is called the *minimal σ-product* of the Boolean algebras $\boldsymbol{A}_\tau$ $(\tau\in T)$, and denoted by $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$. The Boolean algebra $\boldsymbol{S}^\sigma/\mathfrak{J}^*$ is called the *minimal σ^*-product* of the Boolean algebras $\boldsymbol{A}_\tau$ $(\tau\in T)$ and denoted by $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$.

We assume also the following notation for the finite cartesian products:

$$\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau=\boldsymbol{S}/\mathfrak{J},\quad \mathbf{P}^{a*}_{\tau\in T}\boldsymbol{A}_\tau=\boldsymbol{S}/\mathfrak{J}^*.$$

It follows from the definition and theorem 3.2 that

9.1. $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$ *is a σ-complete Boolean algebra.* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ *is a dense subalgebra of* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$.

$\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ *is a σ-complete Boolean algebra.* $\mathbf{P}^{a*}_{\tau\in T}\boldsymbol{A}_\tau$ *is a subalgebra of* $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ *and of* $\boldsymbol{S}^*/\mathfrak{J}^*$. $\boldsymbol{S}^*/\mathfrak{J}^*$ *is a dense subalgebra of* $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$.

By 1.5, 4.1, 4.2 and 5.3 we have

9.2. $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^{a*}_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^\alpha_{\tau\in T}\boldsymbol{S}_\tau=\boldsymbol{S}$.

Therefore we shall not study the products $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ and $\mathbf{P}^{a*}_{\tau\in T}\boldsymbol{A}_\tau$. We note only that

9.3. $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}$ *is a regular subalgebra of* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$.

Consequently $\tilde{\boldsymbol{S}}_\tau/\boldsymbol{J}$ *is a regular subalgebra of* $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$; $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}^*$ *is a regular subalgebra of* $\mathbf{P}^{a*}_{\tau\in T}\boldsymbol{A}_\tau$.

[48] $h^1_\tau(h^2_\tau)^{-1}$ maps $\boldsymbol{B}^2_\tau$ on $\boldsymbol{B}^1_\tau$.

It is sufficient to prove that $(\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau)\prod_{u\in U}(\pi_{\tau_0}(\mathfrak{s}_{\tau_0}(A_u)))=0$ whenever $(\boldsymbol{A}_{\tau_0})\prod_{u\in U}A_u=0$. In fact, if $(\boldsymbol{A}_{\tau_0})\prod_{u\in U}A_u=0$, then $\prod_{u\in U}\mathfrak{s}_{\tau_0}(A_u)$ is nowhere dense in $\mathcal{S}_{\tau_0}$. By 4.3 the set $\prod_{u\in U}\pi_0(\mathfrak{s}_{\tau_0}(A_u))$ is nowhere dense in $\mathcal{S}$, which proves the required equality.

The following theorems can be proved in the same way as the analogous theorems in § 5.

9.4. *If* $\boldsymbol{A}_\tau\approx\boldsymbol{B}_\tau$ *for every* $\tau\in T$, *then* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^b_{\tau\in T}\boldsymbol{B}_\tau$ *and* $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{B}_\tau$.

9.5. *For every* $\tau\in T$, $\boldsymbol{A}_\tau\approx\boldsymbol{S}_\tau/\mathfrak{J}_\tau\approx\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}\approx\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}^*$ *and* $\boldsymbol{S}^\sigma_\tau/\mathfrak{J}_\tau\approx\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}\approx\approx\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}^*$. *If* $\boldsymbol{A}_\tau$ *is σ-complete, then* $\boldsymbol{A}_\tau\approx\boldsymbol{S}^\sigma_\tau/\mathfrak{J}_\tau\approx\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}\approx\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}^*$.

More exactly:

The mapping $A\to\pi_\tau\mathfrak{s}_\tau(A)/\mathfrak{J}$ *(for* $A\in\boldsymbol{A}_\tau$*) is an isomorphism of* $\boldsymbol{A}_\tau$ *on* $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}$. *The mapping* $S/\mathfrak{J}_\tau\to\pi_\tau(S)/\mathfrak{J}$ *(for* $S\in\boldsymbol{S}^\sigma_\tau$*) is an isomorphism of* $\boldsymbol{S}^\sigma_\tau/\mathfrak{J}_\tau$ *(the minimal σ-extension of* $\boldsymbol{A}_\tau$*) on* $\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}$ *which transforms* $\boldsymbol{S}_\tau/\mathfrak{J}_\tau$ *on* $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}$.

The mapping $A\to\pi_\tau\mathfrak{s}_\tau(A)/\mathfrak{J}^*$ *(for* $A\in\boldsymbol{A}_\tau$*) is an isomorphism of* $\boldsymbol{A}_\tau$ *on* $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}^*$. *The mapping* $S/\mathfrak{J}_\tau\to\pi_\tau(S)/\mathfrak{J}^*$ *(for* $S\in\boldsymbol{S}^\sigma_\tau$*) is an isomorphism of* $\boldsymbol{S}^\sigma_\tau/\mathfrak{J}_\tau(A)$ *(the minimal σ-extension of* $\boldsymbol{A}_\tau$*) on* $\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}^*$ *which transforms* $\boldsymbol{S}_\tau/\mathfrak{J}_\tau(A)$ *on* $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}^*$.

9.6. $\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}$ *is a regular σ-subalgebra of* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$. $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}$ *is a dense subalgebra of* $\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}$ *and a regular subalgebra of* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$.

$\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}^*$ *is a regular subalgebra of* $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$. $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}^*$ *is a dense subalgebra of* $\tilde{\boldsymbol{S}}^\sigma_\tau/\mathfrak{J}^*$ *and a regular subalgebra of* $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$.

9.7. *The class* $\sum_{\tau\in T}\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}$ *(i. e. the class of all elements of Boolean algebras* $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}$*) is a σ-generator of* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$.

The class $\sum_{\tau\in T}\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}^*$ *is a σ-generator of* $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$.

9.8. *The subalgebras* $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}$ *are independent in* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$. *The subalgebras* $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}^*$ *are σ-independent in* $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$.

9.9. *If every algebra* $\boldsymbol{A}_\tau$ *has more than two elements, then* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau=\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ *if and only if* T *is finite.*

For if T is infinite, the subalgebras $\tilde{\boldsymbol{S}}_\tau/\mathfrak{J}$ are not σ-independent in $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$. In fact, let $A_n\in\boldsymbol{A}_{\tau_n}$ $(\tau_i\neq j$ for $i\neq j)$, $0\neq A_n\neq 0'$. Then $\prod_{n=1}^{\infty}\pi_{\tau_n}(\mathfrak{s}_{\tau_n}(A_n))$ belongs to $\mathfrak{J}-\mathfrak{J}^*$.

9.10. *If* $\overline{\overline{T}}=1$, *that is, if the family* $\{\boldsymbol{A}_\tau\}_{\tau\in T}$ *contains only one Boolean algebra* $\boldsymbol{A}$, *then* $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau=\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau=\boldsymbol{S}^\sigma(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$.

Theorem 8.1 b′-c′) implies

9.11. *The system* $\langle \mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau, \{\widetilde{\mathfrak{S}}_\tau/\mathfrak{J}\}, \{g_\tau\}\rangle$, *where* $g_\tau(A)=\pi_\tau(\mathfrak{s}_\tau(A))/\mathfrak{J}$ *for* $A\in\boldsymbol{A}_\tau$, *is a minimal element of* $\mathfrak{L}$. *The system* $\langle \mathbf{P}^{\mathfrak{b}*}_{\tau\in T}\boldsymbol{A}_\tau, \{\widetilde{\mathfrak{S}}_\tau/\mathfrak{J}^*\}, \{h_\tau\}\rangle$, *where* $h_\tau(A)=\pi_\tau(\mathfrak{s}_\tau(A))/\mathfrak{J}^*$ *for* $A\in\boldsymbol{A}_\tau$, *is a minimal element of* $\mathfrak{L}^*$.

10. Characteristic properties of minimal products. It follows from theorems 3.2 and 3.7 that

10.1. $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$ *is a minimal σ-extension of* $\boldsymbol{S}$.

10.2. $\mathbf{P}^{\mathfrak{b}*}_{\tau\in T}\boldsymbol{A}_\tau$ *is a minimal σ-extension of* $\boldsymbol{S}^*$.

These theorems imply:

10.3. *In order that a σ-complete Boolean algebra* $\boldsymbol{B}$ *be isomorphic to* $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$ *it is necessary and sufficient that there be a family* $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ *of subalgebras of* $\boldsymbol{B}$ *such that*

a) $\boldsymbol{B}_\tau\approx\boldsymbol{A}_\tau$ *for every* $\tau\in T$;

b) *the subalgebras* $\boldsymbol{B}_\tau$ *are independent in* $\boldsymbol{B}$;

c) *the smallest subalgebra* $\boldsymbol{B}_0$ *containing all the subalgebras* $\boldsymbol{B}_\tau$ $(\tau\in T)$ *is dense in* $\boldsymbol{B}$;

d) *the set* $\sum_{\tau\in T}\boldsymbol{B}_\tau$ *is a σ-generator of* $\boldsymbol{B}$.

The necessity follows from 9.5, 9.8, 9.1 and 9.7.

Suppose the conditions a-d) are satisfied. By a), b) and theorem 6.2′ we infer that $\boldsymbol{B}_0\approx\mathbf{P}^{\mathfrak{a}}_{\tau\in T}\boldsymbol{A}_\tau\approx\boldsymbol{S}$. By c), d), and theorem 3.7, the Boolean algebra $\boldsymbol{B}$ is a minimal σ-extension of $\boldsymbol{B}_0$. Hence, by 10.1, $\boldsymbol{B}\approx\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$.

10.4. *In order that a σ-complete Boolean algebra* $\boldsymbol{B}$ *be isomorphic to* $\mathbf{P}^{\mathfrak{b}*}_{\tau\in T}\boldsymbol{B}_\tau$, *it is necessary and sufficient that there be a family* $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ *of subalgebras of* $\boldsymbol{B}$ *such that*

a) $\boldsymbol{B}_\tau\approx\boldsymbol{A}_\tau$ *for every* $\tau\in T$;

b) *the subalgebras* $\boldsymbol{B}_\tau$ *are σ-independent in* $\boldsymbol{B}$;

c) *the smallest subalgebra* $\boldsymbol{B}^*\subset\boldsymbol{B}$ *which contains all elements* $(\boldsymbol{B})\prod_{n=1}^{\infty}B_n$ *where* $B_n\in\boldsymbol{B}_{\tau_n}$, $\tau_i\neq\tau_j$ *for* $i\neq j$, *is dense in* $\boldsymbol{B}$;

d) *the set* $\sum_{\tau\in T}\boldsymbol{B}_\tau$ *is a σ-generator of* $\boldsymbol{B}$.

The necessity follows from 9.4, 9.8, 9.6, and 9.7.

Suppose the conditions a-d) are satisfied. Let h_τ be an isomorphism of $\boldsymbol{B}_\tau$ on $\boldsymbol{S}_\tau$, and let $\boldsymbol{B}_0^*$ be the class of all elements $(\boldsymbol{B})\prod_n B_n$ where $B_n\in\boldsymbol{B}_{\tau_n}$, $\tau_i\neq\tau_j$ for $i\neq j$. The formula

$$h_0((\boldsymbol{B})\prod_{n=1}^{\infty}B_n)=\prod_{n=1}^{\infty}h_{\tau_n}(B_n)$$

defines a one-one mapping h_0 of $\boldsymbol{B}_0^*$ on $\boldsymbol{S}_0^*$. The hypothesis b) implies that h_0 satisfies Kuratowski-Posament's condition [49]). Therefore the mapping h_0 can be extended to an isomorphism of $\boldsymbol{B}^*$ on $\boldsymbol{S}^*$. By c), d), and theorem 3.7, the Boolean algebra $\boldsymbol{B}$ is a minimal σ-extension of $\boldsymbol{B}^*$. Hence, by 10.2, $\boldsymbol{B}\approx\mathbf{P}^{\mathfrak{b}*}_{\tau\in T}\boldsymbol{A}_\tau$.

11. Commutativity and associativity of minimal products. It follows immediately from the definition of minimal products that

11.1. *If* $t(\tau)$ *is a one-one mapping of* T *on* T, *then* $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_{t(\tau)}$ *and* $\mathbf{P}^{\mathfrak{b}*}_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^{\mathfrak{b}*}_{\tau\in T}\boldsymbol{A}_{t(\tau)}$.

The minimal σ-product and the minimal σ*-product are thus completely commutative. They are also completely associative. In fact,

11.2. *If the set* T *is the sum of mutually disjoint sets* $T_u\neq 0$ $(u\in U)$, *then* $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^{\mathfrak{b}}_{u\in U}(\mathbf{P}^{\mathfrak{b}}_{\tau\in T_u}\boldsymbol{A}_\tau)$ *and* $\mathbf{P}^{\mathfrak{b}*}_{\tau\in T}\boldsymbol{A}_\tau\approx\mathbf{P}^{\mathfrak{b}*}_{u\in U}(\mathbf{P}^{\mathfrak{b}*}_{\tau\in T_u}\boldsymbol{A}_\tau)$.

We shall only prove that $\boldsymbol{B}=\mathbf{P}^{\mathfrak{b}}_{u\in U}(\mathbf{P}^{\mathfrak{b}}_{\tau\in T_u}\boldsymbol{A}_\tau)$ is isomorphic to $\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$. The proof of the remaininig part of 11.2 is analogous.

The σ-complete Boolean algebra $\boldsymbol{B}$ contains, by 10.3, a family $\{\boldsymbol{B}'_u\}_{u\in U}$ of subalgebras such that

a′) $\boldsymbol{B}'_u\approx\mathbf{P}^{\mathfrak{b}}_{\tau\in T_u}\boldsymbol{A}_\tau$ for $u\in U$;

b′) the subalgebras $\boldsymbol{B}'_u$ are independent in $\boldsymbol{B}$;

c′) the smallest subalgebra $\boldsymbol{B}'_0$ containing all the subalgebras $\boldsymbol{B}'_u$ $(u\in U)$ is dense in $\boldsymbol{B}$;

d′) the set $\sum_{u\in U}\boldsymbol{B}'_u$ is a σ-generator of $\boldsymbol{B}$.

By a), $\boldsymbol{B}'_u$ is σ-complete. By a) and 10.3 there is a family $\{\boldsymbol{B}_\tau\}_{\tau\in T_u}$ of subalgebras of $\boldsymbol{B}'_u$ such that

a″) $\boldsymbol{B}_\tau\approx\boldsymbol{A}_\tau$ for every $\tau\in T_u$;

b″) the subalgebras $\boldsymbol{B}_\tau$ $(\tau\in T_u)$ are independent in $\boldsymbol{B}'_u$ (thus in $\boldsymbol{B}$ also);

c″) the smallest subalgebra $\boldsymbol{B}^0_u$ containing all the subalgebras $\boldsymbol{B}_\tau$ $(\tau\in T_u)$ is dense in $\boldsymbol{B}'_u$;

d″) the set $\sum_{\tau\in T_u}\boldsymbol{B}_\tau$ is a σ-generator of $\boldsymbol{B}'_u$.

The conditions a′-d′) and a″-d″) imply that the family $\{\boldsymbol{B}_\tau\}_{\tau\in T}$ of subalgebras of $\boldsymbol{B}$ satisfies the condition a-d) of theorem 10.3. Consequently $\boldsymbol{B}\approx\mathbf{P}^{\mathfrak{b}}_{\tau\in T}\boldsymbol{A}_\tau$, q. e. d.

[49]) Kuratowski and Posament [1], p. 282.

Theorem 11.2 (where $U=T$ and $T_u=(u)$) and theorem 9.10 imply:

11.3. *If, for every $\tau\in T$, $\boldsymbol{B}_\tau$ is a minimal σ-extension of $\boldsymbol{A}_\tau$, then $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{B}_\tau\approx\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$ and $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{B}_\tau\approx\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$.*

12. The connexion between set-theoretical and Boolean products of fields of sets. Theorems 0.1 and 9.2 imply

12.1 *For every family $\{\boldsymbol{X}_\tau\}_{\tau\in T}$ of fields of sets:*

$$\mathbf{P}^{\alpha}_{\tau\in T}\boldsymbol{X}_\tau\approx\mathbf{P}^{a}_{\tau\in T}\boldsymbol{X}_\tau\approx\mathbf{P}^{a}_{\tau\in T}\boldsymbol{X}_\tau\approx\mathbf{P}^{a*}_{\tau\in T}\boldsymbol{X}_\tau.$$

If $\{\boldsymbol{X}_\tau\}_{\tau\in T}$ is a famly of σ-fields of sets, then, in general, the products $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{X}_\tau$ and $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{X}_\tau$ differ from $\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau$. However:

12.2. *For every family $\{\boldsymbol{X}_\tau\}_{\tau\in T}$ of σ-fields (of subsets of sets $\mathfrak{X}_\tau$ respectively), $\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau\approx\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{X}_\tau$.*

Let $\boldsymbol{X}_0^*$ be the class of all sets $\mathbf{P}_{\tau\in T}X_\tau$ where $X_\tau\in\boldsymbol{X}_\tau$ and the inequality $X_\tau\neq\mathfrak{X}_\tau$ holds only for an at most enumerable number of elements $\tau\in T$.

By 10.4 it is sufficient to prove that the least field $\boldsymbol{X}^*$ containing the class $\boldsymbol{X}_0^*$ is dense in $\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau$.

Let $\boldsymbol{X}$ be the class of all sets $X\in\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau$ such that

(*) if $x\in X$, there is a set $Y\in\boldsymbol{X}_0^*$ such that $x\in Y\subset X$.

We have:

(i) $\boldsymbol{X}^*\subset\boldsymbol{X}$;

(ii) if $X_n\in\boldsymbol{X}$, then $\sum_{n=1}^{\infty}X_n\in\boldsymbol{X}$;

(iii) if $X_n\in\boldsymbol{X}$, then $\prod_{n=1}^{\infty}X_n\in\boldsymbol{X}$.

(i) and (ii) is obvious. If $X_n\in\boldsymbol{X}$ and $x\in\prod_{n=1}^{\infty}X_n$, then, for every positive integer n, there is a set $Y_n\in\boldsymbol{X}_0^*$ such that $x\in Y_n\subset X_n$. Clearly $x\in\prod_{n=1}^{\infty}Y_n\subset\prod_{n=1}^{\infty}X_n$ and $\prod_{n=1}^{\infty}Y_n\in\boldsymbol{X}_0^*$.

(i), (ii), and (iii) imply that $\boldsymbol{X}=\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau$. Thus the class $\boldsymbol{X}_0^*$ is dense in $\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau$. Consequently, the field $\boldsymbol{X}^*$ is also dense in $\mathbf{P}^{\beta}_{\tau\in T}\boldsymbol{X}_\tau$, which proves 12.2.

12.3. *In order that $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ be isomorphic to a σ-field of sets it is necessary and sufficient that, for every $\tau\in T$, the minimal σ-extension of $\boldsymbol{A}_\tau$ be isomorphic to a σ-field of sets.*

Let $\boldsymbol{B}_\tau$ be a minimal σ-extensions of $\boldsymbol{A}_\tau$. By 11.3 $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ is isomorphic to a σ-field of sets if and only if $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{B}_\tau$ is so.

Suppose $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{B}_\tau$ is isomorphic to a σ-field $\boldsymbol{X}$ of sets. By 9.5 and 9.6 $\boldsymbol{B}_\tau$ is isomorphic to a σ-subalgebra of $\boldsymbol{X}$, that is, $\boldsymbol{B}_\tau$ is also isomorphic to a σ-field. The necessity is proved.

The sufficiency follows from 12.2.

12.4. *If, for every $\tau\in T$, $\boldsymbol{A}_\tau$ is isomorphic to a σ-field of sets, then $\langle\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau, \{\widetilde{\mathfrak{S}}_\tau/\mathfrak{J}^*\}, \{g_\tau\}\rangle$, where $g_\tau(A)=\pi_\tau\mathfrak{s}_\tau(A)/\mathfrak{J}^*$ for $A\in\boldsymbol{A}_\tau$, is the least element of $\mathfrak{L}^*$* [50].

Let $\langle\boldsymbol{B}, \{\boldsymbol{B}_\tau\}, \{h_\tau\}\rangle\in\mathfrak{L}^*$. Then $g_\tau h_\tau^{-1}$ is an isomorphism of $\boldsymbol{B}_\tau$ on $\widetilde{\mathfrak{S}}_\tau/\mathfrak{J}^*$. By 5.6 and 9.6, $g_\tau h^{-1}$ is a σ-homomorphism of $\boldsymbol{B}_\tau$ in $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ which is isomorphic to a σ-field of sets on account of 12.3. The σ-subalgebras $\boldsymbol{B}_\tau$ being σ-independent, the σ-homomorphisms $g_\tau h_\tau^{-1}$ can be extended to a σ-homomorphism of $\boldsymbol{B}$ in $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ on account of Theorem VII in my paper [6].

We note else that if $\langle\boldsymbol{B}, \{\boldsymbol{B}_\tau\}, \{h_\tau\}\rangle\in\mathfrak{L}^*$ and $\boldsymbol{B}$ is isomorphic to a σ-field of sets, then $\boldsymbol{B}\approx\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$. This follows easily from theorem II in my paper [5].

13 [51]**. The case $\overline{\overline{T}}=1$.** Suppose now that $\overline{\overline{T}}=1$, that is the family $\{\boldsymbol{A}_\tau\}_{\tau\in T}$ contains only one Boolean algebra $\boldsymbol{A}$.

By 5.9 and 6.3 we obtain

13.1. *Every σ-homomorphism of $\boldsymbol{S}(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A})$ in a σ-complete Boolean algebra $\boldsymbol{C}$ can be extended to a σ-homomorphism of $\boldsymbol{S}^\sigma(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A})$ in $\boldsymbol{C}$.*

The partly ordered set $\mathfrak{L}_0$ is then the class of all systems $\langle\boldsymbol{B}, \boldsymbol{B}_0, h\rangle$ such that

(i) $\boldsymbol{B}$ is a σ-complete Boolean algebra;

(ii) $\boldsymbol{B}_0$ is a σ-regular subalgebra of $\boldsymbol{B}$, and $\boldsymbol{B}_0\approx\boldsymbol{A}$;

(iii) h is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}_0$;

(iv) $\boldsymbol{B}_0$ is a σ-generator of $\boldsymbol{B}$.

Otherwise speaking, $\mathfrak{L}_0$ is the class of all σ-extensions $\boldsymbol{B}$ of the algebra $\boldsymbol{A}$. Obviously $\mathfrak{L}_0=\mathfrak{L}=\mathfrak{L}^*$.

By 5.9 and 8.2,

13.2. *$\langle\boldsymbol{S}^\sigma(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A}), \boldsymbol{S}(\boldsymbol{A})/\boldsymbol{J}(\boldsymbol{A}), g\rangle$ where $g(A)=\mathfrak{s}(A)/\boldsymbol{J}(A)$ for $A\in\boldsymbol{A}$, is the greatest element of $\mathfrak{L}$. $\langle\boldsymbol{S}^\sigma(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A}), \boldsymbol{S}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A}), h\rangle$, where $h(A)=\mathfrak{s}(A)/\mathfrak{J}(\boldsymbol{A})$ for $A\in\boldsymbol{A}$, is a minimal element of $\mathfrak{L}$.*

[50]) I do not known whether theorem 12.4 is true for arbitrary σ-complete Boolean algebras.

[51]) Obviously the theorems formulated in this section can be also proved immediately.

This theorem explains the terminology: „the maximal σ-extension“ and „the minimal σ-extension“.

13.3. *If* $\boldsymbol{S}^\sigma(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A})$ *is isomorphic to a σ-field of sets, then* $\langle \boldsymbol{S}^\sigma(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A}), \boldsymbol{S}(\boldsymbol{A})/\mathfrak{J}(\boldsymbol{A}), g\rangle$, where $g(A)=\mathfrak{s}(A)/\mathfrak{J}$, *is the least element of* $\mathfrak{L}$.

This theorem which is a particular case of 12.4, follows immediately from Theorem VI in my paper [6].

The following theorem is a particular case of 6.4:

13.4. *In order that a σ-complete Boolean algebra $\boldsymbol{B}$ be a maximal σ-extension of $\boldsymbol{A}$, it is necessary and sufficient that*

a) *$\boldsymbol{B}$ contain a σ-regular subalgebra $\boldsymbol{B}_0$ which is a σ-generator of $\boldsymbol{B}$ and an isomorph of $\boldsymbol{A}$;*

b) *every σ-homomorphism of $\boldsymbol{B}_0$ in any σ-complete Boolean algebra $\boldsymbol{C}$ can be extended to a σ-homomorphism of $\boldsymbol{B}$ in $\boldsymbol{C}$.*

The following theorem is a particular case of 8.1:

13.5. *Every σ-extension $\boldsymbol{B}$ of $\boldsymbol{A}$ is isomorphic to the quotient algebra $\boldsymbol{S}^\sigma(\boldsymbol{A})/\boldsymbol{I}$ where $\boldsymbol{I}$ is a σ-ideal of subsets of $\mathcal{S}(\boldsymbol{A})$ such that*

a) $\boldsymbol{J}(\boldsymbol{A})\subset\boldsymbol{I}$;

b) *no open non-empty subset of $\mathcal{S}(\boldsymbol{A})$ belongs to $\boldsymbol{I}$.*

By 12.2, the class $\mathfrak{L}$ contains at most one system $\langle\boldsymbol{B}, \boldsymbol{B}_0, h\rangle$ such that $\boldsymbol{B}$ is isomorphic to a σ-field of sets. If it exists, $\boldsymbol{B}$ is a minimal σ-extension of $\boldsymbol{A}$.

If $\boldsymbol{A}$ is σ-complete, then $\mathfrak{L}$ contains only one element on account of 13.5 and 2.5.

14. Extending of measures. The following theorem [52]) follows immediately from 5.4 and 0.4.

14.1. *For every $\tau\in T$ let μ_τ be a normalized measure on $\boldsymbol{A}_\tau$. Then there exists a measure μ on $\mathbf{P}^a_{\tau\in T}\boldsymbol{A}_\tau$ such that*

$$\mu(\mathbf{P}_{\tau\in T}\mathfrak{s}(A_\tau)/\boldsymbol{J})=\prod_{\tau\in T}\mu_\tau(A_\tau)$$

where $A_\tau\in\boldsymbol{A}_\tau$ *and the inequality* $A_\tau\neq 0'$ *(that is,* $\mathfrak{s}(A_\tau)\neq\mathcal{S}(\boldsymbol{A}_\tau)$*) holds only for a finite number of elements* $\tau\in T$.

An analogous theorem holds also for σ-measures and the maximal σ-product:

[52]) See Kappos [1], p. 61-64.

14.2. *For every $\tau\in T$ let μ_τ be a normalized σ-measure on $\boldsymbol{A}_\tau$. Then there is a σ-measure μ on $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$ such that*

$$\mu(\mathbf{P}_{\tau\in T}\mathfrak{s}(A_\tau)/\boldsymbol{J})=\prod_{\tau\in T}\mu_\tau(A)$$

where $A_\tau\in\boldsymbol{A}_\tau$ *and the inequality* $A_\tau\neq 0'$ *holds only for an at most enumerable number of elements* $\tau\in T$.

The formula

$$\nu^0_\tau(\mathfrak{s}(A))=\mu_\tau(A) \quad \text{for} \quad A\in\boldsymbol{A}_\tau$$

defines a measure ν^0_τ on $\boldsymbol{S}_\tau$. The space $\boldsymbol{S}_\tau$ being bicompact, the measure ν^0_τ can be extended [53]) to a σ-measure ν_τ on $\boldsymbol{S}^\sigma_\tau$. Since μ_τ is a σ-measure on $\boldsymbol{A}_\tau$, we infer that

(i) $$\nu(S)=0 \quad \text{for} \quad S\in\boldsymbol{I}_\tau.$$

On account of 0.4 there exists a σ-measure ν on $\mathbf{P}^\beta_{\tau\in T}\boldsymbol{S}_\tau=\boldsymbol{S}^\sigma$ such that

(ii) $$\nu(\mathbf{P}_{\tau\in T}S_\tau)=\prod_{\tau\in T}\nu_\tau(S_\tau) \quad \text{for every set} \quad \mathbf{P}_{\tau\in T}S_\tau\in\mathbf{P}^\alpha_{\tau\in T}\boldsymbol{S}_\tau.$$

By (i) and (ii) $\nu(S)=0$ for every $S\in\boldsymbol{J}$. Thus the formula

$$\mu(S/\boldsymbol{J})=\nu(S) \quad \text{for} \quad S\in\boldsymbol{S}^\sigma$$

defines the required σ-measure on $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$.

Theorem 14.2 fails if we replace the σ-product $\mathbf{P}^b_{\tau\in T}\boldsymbol{A}_\tau$ by $\mathbf{P}^{b}_{\tau\in T}\boldsymbol{A}_\tau$ or $\mathbf{P}^{b*}_{\tau\in T}\boldsymbol{A}_\tau$ [54]).

References.

Andersen E. S. and Jessen B. [1] *Some limit theorems on integrals in an abstract set*, Det. Kgl. Danske Videnskabernes Selskab, Matematisk-Fysiske Meddelelser **22**, n° 14 (1946).

Birkhoff G. [1] *Lattice theory*, revised edition, New York 1948.

Kappos D. A. [1] *Die Cartesischen Produkte und die Multiplikation von Massfunktionen in Boolschen Algebren*, Mathematische Annalen **120** (1947), pp. 43-74.

Kuratowski C. [1] *Topologie I* (new edition), Warszawa-Wrocław 1948.

— et Posament T. [1] *Sur l'isomorphie algébro-logique et les ensembles relativement boreliens*. Fund. Math. **22** (1934), pp. 281-286.

Loomis L. H. [1] *On the representation of σ-complete Boolean algebras*, Bull. Am. Math. Soc. **53** (1947), pp. 757-760.

[53]) See Marczewski [2], p. 24 (ii) and p. 16 (0).

[54]) See Sikorski [7].

Łomnicki Z. et Ulam S. [1] *Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités. I. Variables indépendantes*, Fund. Math. **23** (1934), pp. 237-278.

MacNeille H. [1] *Partially ordered sets*, Trans. Am. Math. Soc. **42** (1937), pp. 416-460.

Marczewski E. [1] *Indépendance d'ensembles et prolongement de mesures (Résultats et problèmes)*, Colloquium Mathematicum **1** (1948), pp. 122-132.

— [2] *Ensembles indépendants et leurs applications à la théorie de la mesure*, Fund. Math. **35** (1948), pp. 13-28.

Sikorski R. [1] *On the representation of Boolean algebras as fields of sets*, Fund. Math. **35** (1948), pp. 247-258.

— [2] *On the inducing of homomorphisms by mappings*, Fund. Math. **36** (1949), pp. 7-22.

— [3] *A theorem on the structure of homomorphisms*, Fund. Math. **36** (1949), pp. 245-247.

— [4] *A theorem on extension of homomorphisms*, Annales Soc. Pol. Math. **21** (1948), pp. 332-335.

— [5] *Independent fields and cartesian products*, Studia Math. **11** (1950), pp. 171-184.

— [6] *On an analogy between measures and homomorphisms*, Annales Soc. Pol. Math. **23** (1950), pp. 1-20.

— [7] *On measures in cartesian products of Boolean algebras*, to appear in Coll. Math. **2** (1950).

Stone M. H. [1] *The theory of representations for Boolean algebras*, Trans. Am. Math. Soc. **40** (1936), pp. 36-111.

— [2] *Applications of the theory of Boolean rings to general topology*, Trans. Am. Math. Soc. **41** (1937), pp. 375-481.

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Sur les suites doubles de fonctions.

Par

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Théorème 1. *Soit $f_{m,n}(x)$ $(m=1,2,...;\ n=1,2,...)$ une suite double infinie de fonctions mesurables d'une variable réelle, assujettie à la condition suivante:*

$k_1, k_2, ...$ et $l_1, l_2, ...$ étant deux suites infinies quelconques de nombres naturels, telles que

$$\lim_{n=\infty} k_n = \lim_{n=\infty} l_n = +\infty,$$

on a

$$\lim_{n=\infty} f_{k_n,l_n}(x) = f(x)$$

pour tout x, abstraction faite d'un ensemble de mesure nulle (dépendant des suites $k_1, k_2, ...$ et $l_1, l_2, ...$).

On a alors

$$\lim_{m,n} f_{m,n}(x) = f(x)$$

pour tout x abstraction faite d'un ensemble de mesure nulle[1]).

Démonstration. Il suffit évidemment de démontrer le théorème pour les fonctions $f_{m,n}(x)$ $(m=1,2,...;\ n=1,2,...)$ définies dans l'intervalle $I=[0\leqslant x\leqslant 1]$, en posant $f(x)=0$ pour $x\in I$.

Soit $f_{m,n}(x)$ $(m=1,2,...;\ n=1,2,...)$ une suite double de fonctions satisfaisant aux hypothèses du théorème 1 dans l'intervalle I, où $f(x)=0$ pour $x\in I$ et supposons que l'ensemble E de tous les nombres x de I pour lesquels l'égalité $\lim\limits_{m,n} f_{m,n}(x)=0$ est en défaut ne soit pas de mesure nulle. Les fonctions $f_{m,n}(x)$ étant mesurables, l'ensemble E est donc mesurable et de mesure positive.

Pour tout nombre $x\in E$, il existe deux suites infinies de nombres naturels $k_1, k_2, ...$ et $l_1, l_2, ...$, telles que $\lim\limits_{n=\infty} k_n = \lim\limits_{n=\infty} l_n = +\infty$ sans que $\lim f_{k_n,l_n}(x)=0$. Il en résulte, comme on le voit sans peine,

[1]) Ce théorème résout un problème de M. Sikorski.