et, comme $\overline{\overline{f(G)-G}}<2^{\aleph_0}$, on trouve, à plus forte raison: $\overline{\overline{H_1-G}}<2^{\aleph_0}$. Or on a $H_1 \subset H = I - G$. d'où $H_1 = H_1 - G$: on trouve ainsi $\overline{\overline{H_1}}<2^{\aleph_0}$, ce qui est impossible.

Nous avons ainsi démontré qu'il n'existe aucun ensemble E de puissance $2^{\aleph_0}$, tel que $\overline{E}<\overline{G}$ et $\overline{E}<\overline{H}$. Les ensembles G et H étant linéaires, il en résulte tout de suite, vu le théorème 7 que, $\overline{G}<\lambda$ et $\overline{H}<\lambda$. Le théorème 10 se trouve ainsi démontré.

Il résulte tout de suite du théorème 10 le

Corollaire. *Si $2^{\aleph_0}=\aleph_1$, il existe deux types ordinaux indénombrables $\varphi_1<\lambda$ et $\varphi_2<\lambda$, tels qu'il n'existe aucun type ordinal indénombrable ψ tel que $\psi<\varphi_1$ et $\psi<\varphi_2$.*

Voici une démonstration directe de ce colloraire.

D'après N. Lusin, si $2^{\aleph_0}=\aleph_1$, il existe un ensemble linéaire indénombrable L qui admet un ensemble au plus dénombrable de points communs avec chaque ensemble (linéaire) parfait non dense [6]). Or, j'ai démontré que si $2^{\aleph_0}=\aleph_1$, il existe un ensemble linéaire indénombrable S qui admet un ensemble au plus dénombrable de points communs avec chaque ensemble linéaire de mesure nulle [7]).

On a évidemment $\overline{L}<\lambda$ et $\overline{S}<\lambda$. Je dis qu'il n'existe aucun ensemble indénombrable E tel que $\overline{E}<\overline{L}$ et $\overline{E}<\overline{S}$.

En effet, admettons que E soit un tel ensemble. Comme $\overline{E}<\overline{L}$ et $\overline{E}<\overline{S}$, l'ensemble E est à la fois semblable à un sous-ensemble L_1 de L et à un sous-ensemble S_1 de S, et il existe une fonction croissante f définie dans L_1 qui transforme L_1 en S_1. Or, l'ensemble L_1, en tant que sous-ensemble de L, jouit de la propriété P suivante: tout ensemble linéaire parfait non dense admet un ensemble au plus dénombrable de points de l'ensemble L_1. Or, comme j'ai démontré [8]) chaque fonction de Baire d'une variable réelle transforme tout ensemble jouissant de la propriété P en un ensemble de mesure nulle. Une fonction croissante dans l'ensemble L_1 pouvant être étendue à une fonction de Baire d'une variable réelle, l'ensemble $S_1=f(L_1)$ est donc de mesure nulle. Or, vu la propriété de l'ensemble S et vu que $S_1 \subset S$, l'ensemble S_1 est au plus dénombrable, de même que l'ensemble E (qui est semblable à S_1), contrairement à l'hypothèse.

Notre corollaire se trouve ainsi démontré.

[6]) Comptes Rendus Acad. des Sc. Paris **158** (1914), p. 1259.

[7]) Fund. Math. **5** (1924), p. 184.

[8]) Voir mon livre *Hypothèse du continu*, Monografie Matematyczne t. IV (Warszawa-Lwów 1934), p. 39.

Note on arithmetic models for consistent formulae of the predicate calculus.

By

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Introduction.

The Skolem model.

1. The researches of Loewenheim, Skolem, Gödel [1], and Bernays [1]) have established the following result:

Suppose the formula

(1) $$(x_{11})\dots(x_{1n_1})\,(Ey_{11})\dots(Ey_{1m_1})\,(x_{21})\dots(Ey_{nm_n})A(x_{11}\dots y_{nm_n}),$$

where $A(a_{11}\dots b_{nm_n})$ is a free variable formula not containing function symbols or formula variables without arguments, has the normal Skolem form (HB, II, 179-182)

(2) $$(x_1)\dots(x_r)\,(Ey_1)\dots(Ey_s)B(x_1\dots x_r y_1\dots y_s),$$

where $B(a_1\dots b_s)$ is also a free variable formula, and $B_{0j}(a_1\dots b_s)$, $1\leqslant j\leqslant N$, are the formula variables occurring in (2). Both (1) and (2) are understood to be formulae of the predicate calculus (HB, II, 375-380) without free variables.

If (2), and therefore (1), is consistent with respect to the predicate calculus, that is their negations cannot be proved in the predicate calculus, formulae $B^*_{0j}(a_1\dots b_s)$ of the formalism Z_μ (HB, II, 293) of arithmetic can be defined so that the implication

(3) $$(n)\,[q(n)=0]\to(x_1)\dots(x_r)\,(Ey_1)\dots(Ey_s)B^*(x_1\dots x_r y_1\dots y_s)$$

[1]) Hilbert-Bernays, *Grundlagen der Mathematik*, II, 178-189, and particularly 234-253. This work is referred to in the text by „HB". We usually refer to proofs in HB rather than in the original papers since the proofs are more detailed, and also the book is more accessible.

can be proved in Z_μ: B^* is got from B by substituting B^*_{0j} for B_{0j}, and $q(n)$ is a primitive recursive function so that $q(n)=0$ is verifiable if and only if (2) is consistent with respect to the predicate calculus (HB, II, 243, 244-252).

Actually instead of (3) the stronger form

$$(n)\,[q(n)=0]\to B^*[x_1\dots x_r,\ s\eta^r(x_1\dots x_r)+1\dots s\eta^r(x_1\dots x_r)+s] \tag{4}$$

can be proved in Z_μ, where $\eta^r(x_1\dots x_r)$ is the primitive recursive numbering of r-tuples of integers $\langle x_1\dots x_r\rangle$ given in HB, II, 235.

(4) is called an *arithmetic model* of the formula (2).

Applications of the model.

2. The model is used to establish

(i) the *completeness* of the predicate calculus (with respect to predicates of ω-consistent extensions, [2], para. 18 (b), of Z_μ) in Gödel [1],

(ii) the *relativity* theorems of Skolem [1], which, since Gödel [3], are more appropriately known as *undecidability* theorems,

(ii) has been used for a critique of the concept of infinite *cardinals*.

3. 2(i) raises the *problem* whether the predicate calculus is complete with respect to *decidable* predicates, (HB, II, 191).

The proof of 2(ii) was only indicated in Skolem's paper [1], and no precise conditions were laid down on the formula (1) under which undecidability results can be established. Also the relation between 2(ii) and Gödel's work [3] does not seem to have been discussed in the literature in satisfactory detail.

Development of the model.

4. It is shown in Theorem I below that 2(ii) holds for formulae (1) in which recursive number theory can be „developed" in a sense which is made precise there. The basic method of proof is *Cantor's diagonal argument* just as in the classical proof [3] of Gödel's first undecidability theorem.

Also, under suitable „derivability" conditions on systems (1), the formula $(n)[q(n)=0]$ cannot be proved from (1); this is our Theorem II, which is an analogue to Gödel's second undecidability theorem since $(n)[q(n)=0]$ is Bernays arithmetization of „consistency of (1) with respect to the predicate calculus".

In section II the relation between Gödel's undecidability results and the present ones is discussed. It appears there that our Theorem I is a *pure* undecidability result; that is while Gödel decides the *ω-consistency* of the formula which he shows to be undecided by the axioms of the formalism considered, the proof of Theorem I leaves open the question whether the relevant undecided formula is ω-consistent or not.

The present note concludes with some remarks on the diagonal argument, and the connection between non enumerability and undecidability.

Limitations of the model.

5. It is an easy consequence of Theorem I that the model (4) is *ω-inconsistent*: that is, there is a formula (=predicate, set) $I(n)$ of the system (1) which „represents" the set of integers, and a formula $R(n)$ which represents a set to which all *recursive* integers $0,1,\dots$ belong, yet there is an $\mathfrak{n}$ so that

$$I^*(\mathfrak{n}) \quad \text{and} \quad \neg R^*(\mathfrak{n}).$$

Another consequence of Theorem I is that the predicates B^*_{0j} of a system satisfying the conditions of Theorem I are *not decidable* in Z_μ, and therefore not decidable predicates in any of the general senses of Church, Kleene, Post or Turing [4]. In the literature there is, as far as I know, no system (1) (set theory) which has been *proved* to satisfy the conditions of Theorem I (the crucial condition is ω-consistency). Such a system would show that the predicate calculus *is not complete* with respect to decidable predicates. This would prove Bernays' conjecture (HB, II, 191).

I. Undecidability results.

6. In discussing the Skolem model (4) we use the notation of HB, II, except that German letters are generally replaced by Roman ones.

Thus the j^{th} r-tuple of integers is denoted by $(n_{1j}\dots n_{rj})$, the formula $B(n_{1j}\dots n_{rj}, sj+1\dots sj+s)$ by B_j, and the conjunction $B_0\&\dots\&B_k$ by F_k. The *distinct* prime formulae of B_0 are labelled $P_1\dots P_{r_0}$, those of B_{k+1} which do not appear in F_k by $P_{r_k+1},\dots,P_{r_{k+1}}$, where two prime formulae are called distinct either if the formula variables are different symbols or if the arguments are different numbers.

If the distinct prime formulae of F_k are replaced by independent formula variables $V_1 \dots V_{r_k}$, F_k becomes a truth function of the propositional calculus which we denote by $T^k(V_1 \dots V_{r_k})$, or, for short, by T^k. A distribution of truth values on $V_1 \dots V_{r_k}$ is called *true making at stage* k if they make T^k true.

It is shown in HB, II, 187 that if in *all* true making distributions on T^k, V_r is true, V_r is also true in all true making distributions on T^m, $m \geqslant k$, provided only (2) is consistent. Thus if P_r is the formula $B_{0j}(\mathfrak{n}_{1t} \dots \mathfrak{n}_{rt}, st+1 \dots st+s)$ also $B^*_{0j}(\mathfrak{n}_{1t} \dots \mathfrak{n}_{rt}, st+1 \dots st+s)$ of (3) is true.

We say $B^*_{0j}(\mathfrak{n}_{1t} \dots \mathfrak{n}_{rt}, st+1 \dots st+s)$ is *decided at stage* k. Note that if $B^*_{0j}(\mathfrak{n}_{1t} \dots \mathfrak{n}_{rt}, st+1 \dots st+s)$ is decided at some finite stage, the formula $B^*_{0j}(\mathfrak{n}_{1t} \dots \mathfrak{n}_{rt}, st+1 \dots st+s)$ can be *proved* in Z_μ from the formula $(n)[q(n)=0]$, HB, II, 240, formula (2).

Remark. The definition of B^*_{0j} from true making distributions on T^k is a particular case of the *Unendlichkeitslemma* of König [5]. Conversely, the elimination of the selection axiom in the definition of B^*_{0j} by the least number symbol can be adapted to the proof of König's lemma. The most familiar application of this lemma is the bisection definition of limits in analysis.

7. We now give *numbers* $\mathfrak{t}^*$ to terms $\mathfrak{t}$ of the system which *represent* these terms in the model. More precisely, we prove the

Lemma. Let formula (1) be written in *free variable form* (*aufgelöste Form* of HB, II, 1-9),

$$A[a_{11} \dots a_{1n_1} \dots a_{nn_n} \dots \Phi_{ij}(\dots a_{rs} \dots) \dots], \quad 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m_n, \ 1 \leqslant r \leqslant i, \ 1 \leqslant s \leqslant n_r \tag{5}$$

We define primitive recursive functions $\Phi^*_{ij}(\dots a_{rs} \dots)$ so that if $B_{0j}(\mathfrak{t}_{11} \dots \mathfrak{t}_{nm_n})$ has been proved from (5) by the predicate calculus[2]), where the terms $\mathfrak{t}$ are made up of the symbols Φ_{ij} and free variables, then $B^*_{0j}(\mathfrak{t}^*_{11} \dots \mathfrak{t}^*_{nm_n})$ is decided at some finite stage k, and conversely; here $\mathfrak{t}^*$ is got from $\mathfrak{t}$ by substituting Φ^* for Φ in $\mathfrak{t}$, and the stage k is got from the proof of $B_{0j}(\mathfrak{t}_{11} \dots \mathfrak{t}_{nm_n})$. Also B^* is the predicate of Z_μ which replaces the formula variable B in the arithmetic model of the normal Skolem form (2).

[2]) cf. HB, I, 105: provable in the predicate calculus — ableitbar im Prädikatenkalkül, provable by the predicate calculus — ableitbar durch den Prädikatenkalkül.

Note. It can be shown that this result is also true if instead of formulae $B_{0j}(\mathfrak{t}_{11} \dots \mathfrak{t}_{nm_n})$ we consider quantified formulae whose formula variables are B_{0j}; but the extension is not needed below.

It is convenient to prove the result for normal Skolem forms first:

(i) Suppose a formula $B_{0j}(\mathfrak{s}_1 \dots \mathfrak{s}_{r+s})$ has been proved from the free variable form $B[a_1 \dots a_r \psi_1(a_1 \dots a_r) \dots \psi_s(a_1 \dots a_r)]$ of (2), and $\mathfrak{s}$ are made up of ψ.

Then $B^*_{0j}(\mathfrak{s}^*_1 \dots \mathfrak{s}^*_{r+s})$ is decided at a stage k where $\mathfrak{s}^*$ is got from $\mathfrak{s}$ by substituting the functions $\psi^*_i \ (= s\eta^r(a_1 \dots a_r)+i)$ for ψ_i in $\mathfrak{s}$.

For, if $B_{0j}(\mathfrak{s}_1 \dots \mathfrak{s}_{r+s})$ has been proved from the free variable form of (2), by the *Deduktionstheorem* (HB, I, 150),

$$(Ex_1) \dots (Ex_r) \{ \neg B[x_1 \dots x_r \psi_1(x_1 \dots x_r) \dots \psi_s(x_1 \dots x_r)] \} \vee B_{0j}(\mathfrak{s}_1 \dots \mathfrak{s}_{r+s})$$

can be proved in the predicate calculus. By the first ε-theorem (HB, II, 18), we find terms $\mathfrak{u}^i_1 \dots \mathfrak{u}^i_r$, $1 \leqslant i \leqslant N$, made up of the symbols ψ_i so that the disjunction

$$\neg B[\mathfrak{u}^1_1 \dots \mathfrak{u}^1_r \psi_1(\mathfrak{u}^1_1 \dots \mathfrak{u}^1_r) \dots \psi_s(\mathfrak{u}^1_1 \dots \mathfrak{u}^1_r)] \vee \dots \vee$$
$$\neg B[\mathfrak{u}^N_1 \dots \mathfrak{u}^N_r \psi_1(\mathfrak{u}^N_1 \dots \mathfrak{u}^N_r) \dots \psi_s(\mathfrak{u}^N_1 \dots \mathfrak{u}^N_r)] \vee B_{0j}(\mathfrak{s}_1 \dots \mathfrak{s}_{r+s})$$

can be proved in the *propositional calculus*. Replace in the terms $\mathfrak{u}$ and $\mathfrak{s}$ the symbols ψ_i by the functions ψ^*_i, and let the highest $\mathfrak{u}^*$ be less than M. Then the above disjunction can only be identically true in the propositional calculus if $\mathfrak{s}^* < M$, and therefore $B^*_{0j}(\mathfrak{s}^*_1 \dots \mathfrak{s}^*_{r+s})$ is certainly decided at a stage $< \eta^r(M \dots M)$.

Note that this part of the proof does not use the fact that the ψ^* are disparate.

(ii) The proof of the converse is a little less direct.

Suppose that at some stage, k say, $B^*_{0j}(\mathfrak{s}^*_1 \dots \mathfrak{s}^*_{r+s})$ is decided (to be true), that is the formula

$$\neg B(0 \dots 0\ 1 \dots s) \vee \dots \vee \neg B(n_{1k} \dots n_{rk}, sk+1 \dots sk+s) \vee B_{0j}(\mathfrak{s}^*_1 \dots \mathfrak{s}^*_{r+s}) \tag{6}$$

is proved by the propositional calculus.

Then we can prove $B_{0j}(\mathfrak{s}_1 \dots \mathfrak{s}_{r+s})$ from the free variable form of (2).

Recall that every integer can be expressed *uniquely* by terms made up of the symbol 0 and the functions ψ^*; for a method of doing this see HB, II, 193. Let the integers other than 0 occurring in (6) be replaced by such expressions, and let **0** be replaced by the free variable a; denote the term replacing n_{ij} by $\bar{n}_{ij}$. Then the disjunction

$$(7)\qquad \neg B[a\ldots a\psi_1^*(a\ldots a)\ldots\psi_s^*(a\ldots a)]\vee\ldots\vee \neg B[\bar{n}_{1k}\ldots\bar{n}_{rk}\psi_1^*(\bar{n}_{1k}\ldots\bar{n}_{rk})\ldots\psi_s^*(\bar{n}_{1k}\ldots\bar{n}_{rk})]\vee B_{0j}(\bar{\mathfrak{s}}_1^*\ldots\bar{\mathfrak{s}}_{r+s}^*)$$

is also an identity of the propositional calculus, where ψ^* are now arbitrary function symbols, and $\bar{\mathfrak{s}}^*$ is got from $\mathfrak{s}^*$ simply by replacing 0 by the variable a.

Now, from the free variable form

$$B[a_1\ldots a_r\psi_1^*(a_1\ldots a_r)\ldots\psi_s^*(a_1\ldots a_r)]$$

of (2) we prove the conjunction (8):

$$B[a\ldots a\psi_1^*(a\ldots a)\ldots\psi_s^*(a\ldots a)]\,\&\ldots\&B[\bar{n}_{1k}\ldots\bar{n}_{rk}\psi_1^*(\bar{n}_{1k}\ldots\bar{n}_{rk})\ldots\psi_s^*(\bar{n}_{1k}\ldots\bar{n}_{rk})]$$

by substituting $\bar{n}_{il}$ for a_i in turn, $1\leqslant l\leqslant k$. From (7) and (8) we get $B_{0j}(\mathfrak{s}_1\ldots\mathfrak{s}_{r+s})$.

Note that this proof uses the disparateness of the system ψ_i^*.

(iii) While the usual set theories could be thrown into normal Skolem form, and the development of integers, ordinals etc. carried out in the Skolem form, this is usually not done. It is therefore convenient to give the construction of φ^* for proofs of $B_{0j}(\mathfrak{t}_{11}\ldots\mathfrak{t}_{nm_n})$ from formulae (5) as stated in the lemma. The construction is obtained from the standard reduction of a general prenex formula of the predicate calculus to its normal Skolem form (HB, II, 181; HB, I, 159). (In other words, we show explicitly how from a model of the normal Skolem form (2) can be obtained a model of (1)).

(a) We define functions $\varphi_{ij}^0(\ldots a_{rs}\ldots)$, made up of the function symbols $\psi_i(a_1\ldots a_r)$, so that

$$B[a_1\ldots a_r\psi_1(a_1\ldots a_r)\ldots\psi_s(a_1\ldots a_r)]$$

and $A[a_{11}\ldots a_{n_n}\varphi_{11}^0(a_{11}\ldots a_{1n_1})\ldots\varphi_{nm_n}^0(a_{11}\ldots a_{nm_n})]$ are together consistent or inconsistent with respect to the predicate calculus; actually, the latter can be proved from the former by the elementary calculus with free variables.

(The usual reduction is stated in the form that (1) and (2), and *not* their free variable forms are equivalent in the sense above).

We denote the complex of all-variables $(x_{i1})\ldots(x_{in_i})$ by $(\mathfrak{x}_i)$ (cf. Gödel [1]), $(Ey_{i1})\ldots(Ey_{im_i})$ by $(E\mathfrak{y}_i)$ so that (1) becomes

$$(9)\qquad (\mathfrak{x}_1)(E\mathfrak{y}_1)\ldots(\mathfrak{x}_n)(E\mathfrak{y}_n)A(\mathfrak{x}_1\ldots\mathfrak{x}_n\mathfrak{y}_1\ldots\mathfrak{y}_n).$$

Let P_0 be the formula $A(\mathfrak{x}_1\ldots\mathfrak{y}_n)$, and P_{r+1} be the formula following the quantifiers of

$$(10)\qquad \begin{gathered}(\mathfrak{x}_1)\ldots(\mathfrak{x}_{r+1})\ldots(\mathfrak{y}_{1+j}^{i-j})\ldots(E\mathfrak{y}_s^{r+1-s})\ldots(\mathfrak{x}_{r+2})(E\mathfrak{y}_{r+2})\ldots(\mathfrak{x}_n)(E\mathfrak{y}_n)\\ \{[F_r(\mathfrak{x}_1\ldots\mathfrak{x}_r;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots;\ldots\mathfrak{y}_s^{r+1-s}\ldots)]\&\\ \&[F_r(\mathfrak{x}_1\ldots\mathfrak{x}_r;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots;\ldots\mathfrak{y}_v^{r-v}\ldots)\to P_r]\},\end{gathered}$$

where $0\leqslant j\leqslant i$, $\mathfrak{y}_{i+1}^0$ is the variable $\mathfrak{y}_{i+1}$, $1\leqslant i\leqslant r-1$, $1\leqslant s\leqslant r+1$, $1\leqslant t\leqslant r-2$, $1\leqslant u\leqslant t$, $1\leqslant v\leqslant r$, and F_r is a formula variable with the number of arguments indicated above; P_r does not contain the symbols $\mathfrak{y}_s^{r+1-s}$, $1\leqslant s\leqslant r$, and $F_r(\mathfrak{x}_1\ldots\mathfrak{x}_r;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots;\ldots\mathfrak{y}_v^{r-v}\ldots)$ does not contain $\mathfrak{x}_{r+1}$. We call (10) the *Skolem form* of (9) *after* r *steps* (of the reduction).

The normal Skolem form of (1) is (10) with $r=n-1$.

Now, suppose we replace the existence symbols $\mathfrak{y}_s^{r+1-s}$ in (10) by the function symbols $\psi_s^{r+1-s}(\mathfrak{x}_1\ldots\mathfrak{x}_{r+1};\ldots\mathfrak{y}_{1+j}^{i-j}\ldots)$, and $\mathfrak{y}_{r+w}$, $w\geqslant 2$, by $\psi_{r+w}(\mathfrak{x}_1\ldots\mathfrak{x}_{r+1};\ldots\mathfrak{y}_{1+j}^{i-j}\ldots;\mathfrak{x}_{r+2}\ldots\mathfrak{x}_{r+w})$. Let the resulting formula be denoted by (10′). From (10′) we get by the propositional calculus

$$(11)\qquad F_r[\mathfrak{x}_1\ldots\mathfrak{x}_r;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots;\ldots\psi_s^{r+1-s}(\mathfrak{x}_1\ldots\mathfrak{x}_r a;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots a)\ldots],$$

where an arbitrary symbol a is substituted for the terms $\mathfrak{x}_{r+1}$ and $\mathfrak{y}_1^{r-1}\ldots\mathfrak{y}_r$. Denote $\psi_s^{r+1-s}(\mathfrak{x}_1\ldots\mathfrak{x}_r a;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots a)$ by $\bar{\psi}_s^{r+1-s}(\mathfrak{x}_1\ldots\mathfrak{x}_r;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots)$.

Now substitute the terms

$$\bar{\psi}_1^r(\mathfrak{x}_1\ldots\mathfrak{x}_r;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots),\ldots\bar{\psi}_r^1(\mathfrak{x}_1\ldots\mathfrak{x}_r;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots)$$

for the all variables $\mathfrak{y}_1^{r-1}\ldots\mathfrak{y}_r$ in (10′). We prove thereby from (10′) the implication

$$(12)\qquad F_r(\mathfrak{x}_1\ldots\mathfrak{x}_r;\ldots\mathfrak{y}_{1+u}^{t-u}\ldots;\bar{\psi}_1^r\ldots\bar{\psi}_r^1)\to\tilde{P}_r,$$

where $\tilde{P}_r$ is got from P_r by substituting ψ_{r+w} for $\mathfrak{y}_{r+w}$, and $\bar{\psi}_v^{r-v}$ for the all variables $\mathfrak{y}_v^{r-v}$. From (11) and (12) we get $\tilde{P}_r$, a free variable form for the normal Skolem form at stage $(r-1)$ of the reduction.

Consider then the free variable form $\widetilde{P}_{n-1}$ of (2) with function symbols $\psi_i(a_1 \dots a^r)$. From this we prove a free variable form $\widetilde{P}_{n-2}$, where the function symbols of the new form are made up of symbols ψ_i and a as described above. Going back, we eventually get a free variable form of (9)

$$A[a_1 \dots a_n \varphi_1(a_1) \dots \varphi_n(a_1 \dots a_n)],$$

where the φ are made up of ψ. The free variable formula is proved from $\widetilde{P}_{n-1}$ by the elementary free variable calculus.

Note also that if in the formula (9) we replace the formula variable $F_r(\mathfrak{x}_1 \dots \mathfrak{x}_r; \dots \mathfrak{y}_{1+u}^{t-u} \dots; \dots \mathfrak{y}_v^{r-v} \dots)$ by $\mathfrak{A}_r$, i. e.

$$(13) \qquad (\mathfrak{x}_{r+1})\,(E\mathfrak{y}_{r+1}) \dots (\mathfrak{x}_n)\,(E\mathfrak{y}_n) P_{r-1}$$

(when necessary, avoiding clashing of variables by suitable change of name), (10) turns into the normal Skolem form of (9) after $(r-1)$ steps of the reduction. In particular, if $F_1, F_2, \dots$ are replaced in turn by the formulae (13), the Skolem normal form turns back into the original (9).

Example. Consider the intersection axiom

$$(x_{11})\,(x_{12})\,(Ey_1)\,(x_2)\,(x_2 \in y_1 \sim \cdot\, x_2 \in x_{11} \,\&\, x_2 \in x_{12}),$$

whose normal Skolem form is got after one step to be

$$(14) \qquad \begin{gathered}(x_{11})\,(x_{12})\,(y_1)\,(x_2)\,(Ey_1^1)\{F(x_{11} x_{12} y_1^1)\,\& \\ \&[F(x_{11} x_{12} y_1) \rightarrow (x_2 \in y_1 \sim \cdot\, x_2 \in x_{11} \,\&\, x_2 \in x_{12})]\}.\end{gathered}$$

Eliminate the symbol y_1^1 by $\psi_1^1(x_{11} x_{12} y_1 x_2)$. If now

$$(15) \quad F[x_{11} x_{12}\, \psi_1^1(x_{11} x_{12} y_1 x_2)] \,\&\, [F(x_{11} x_{12} y_1) \rightarrow (x_2 \in y_1 \rightarrow x_2 \in x_{11} \,\&\, x_2 \in x_{12})]$$

is true for all x_{11}, x_{12}, y_1, x_2, in particular for $y_1 = x_2 = a$, then $F[x_{11} x_{12} \psi_1^1(x_{11} x_{12} a a)]$ is true for all x_{11}, x_{12}, where a is a free variable. Also (15) is true for all x_{11}, x_{12}, x_2 and $y_1 = \psi_1^1(x_{11} x_{12} a a)$, and hence

$$(16) \qquad x_2 \in \psi_1^1(x_{11} x_{12} a a) \sim \cdot\, x_2 \in x_{11} \,\&\, x_2 \in x_{12}.$$

Thus from the free variable form (15) of the normal Skolem form (14) of the intersection axiom, we get a free variable form (16) of the intersection axiom itself, $\varphi_1(x_{11} x_{12}) = \psi_1^1(x_{11} x_{12} a a)$.

Plainly, from (15) we have proved (16) by the predicate calculus.

(b) Now let us define φ_{ij}^* as the functions which replace the $\mathfrak{y}$ of (9) in the reduction described in (a). For the function symbols ψ_i in the free variable form $\widetilde{P}_{n-1}$ we take the functions $s\eta^r(a_1 \dots a_r) + i$.

If $B_{0j}(\mathfrak{t}_{11} \dots \mathfrak{t}_{nm_n})$ has been proved from (5), $B_{0j}(\bar{\mathfrak{t}}_{11} \dots \bar{\mathfrak{t}}_{nm_n})$ can be proved by (a) from the free variable form of the normal Skolem form, where $\bar{\mathfrak{t}}$ is got from $\mathfrak{t}$ by replacing the function symbol φ_{ij} in $\mathfrak{t}$ by the function symbol φ_{ij}^* made up of the symbols ψ_i.

By part (i) of the lemma, proved for normal Skolem forms, $B_{0j}^*(\bar{\mathfrak{t}}_{11}^* \dots \bar{\mathfrak{t}}_{nm_n}^*)$ is decided at a finite stage.

Next suppose $B_{0j}^*(\bar{\mathfrak{t}}_{11}^* \dots \bar{\mathfrak{t}}_{nm_n}^*)$ is decided at stage k. Then, by part (ii) of the lemma, $B_{0j}(\bar{\mathfrak{t}}_{11} \dots \bar{\mathfrak{t}}_{nm_n})$ can be proved from the Skolem form by the predicate calculus. Since F_r is a formula variable, we may replace it throughout the proof by the formula $\mathfrak{A}_r$ defined in (13), and still retain a proof by the predicate calculus. But by this substitution $B_{0j}(\bar{\mathfrak{t}}_{11} \dots \bar{\mathfrak{t}}_{nm_n})$ is not affected since it does not contain F_r, and the normal form is turned into (5); thus, if $B_{0j}^*(\bar{\mathfrak{t}}_{11}^* \dots \bar{\mathfrak{t}}_{nm_n}^*)$ is decided at stage k, $B_{0j}(\mathfrak{t}_{11} \dots \mathfrak{t}_{nm_n})$ can be proved from (5).

In the language of set theory the result of the lemma is this: not only are the predicates B_{0j}^* arithmetic models of the predicates B_{0j}, but the numbers $\mathfrak{t}^*$ are numbers (in the model) of the sets (defined by the terms) $\mathfrak{t}$.

Note an obvious consequence of the lemma: if $B_{0j}(\mathfrak{t}_{11} \dots \mathfrak{t}_{nm_n})$ is undecided by the axiom system, $B_{0j}^*(\mathfrak{t}_{11}^* \dots \mathfrak{t}_{nm_n}^*)$ is not decided at any finite stage: it will be explained in section II how the ω-consistency of $B_{0j}^*(\mathfrak{t}_{11}^* \dots \mathfrak{t}_{nm_n}^*)$ is then decided by the minimum condition used in defining B_{0j}^*, together with suitable free variable formulae of arithmetic.

8. We prove the undecidability theorem for a set theory containing the formula variable characterizing set membership: $a \in b$. For typographical convenience we write $\mathfrak{e}(a,b) = 0$ instead of $a \in^* b$.

Theorem I. Suppose the formula (1), a set theory, is written in free variable form (5) where the function symbols φ_{ij} replace the existence variables of (1), and suppose it satisfies the following conditions:

(a) *Representation of Z_μ in the set theory*:

The recursively defined integer $\mathfrak{n}$ is represented by a term $i_\mathfrak{n}$ made up of the symbols φ_{ij} — but without the letter $\mathfrak{n}$! — and $i_\mathfrak{n}$ has the number $i^*(\mathfrak{n})$ in the sense of the lemma, where $i^*(\mathfrak{n})$ is a primitive recursive function of $\mathfrak{n}$.

A predicate $\mathfrak{P}(\mathfrak{n})$ of Z_μ is represented by a term p with number p^*.

By representation we mean:

(i) if $\mathfrak{P}(\mathfrak{n})$ can be proved in Z_μ for the *recursively* defined integer $\mathfrak{n}$, then $i_\mathfrak{n} \epsilon p$ can be proved from (5);

(ii) if $\neg\mathfrak{P}(\mathfrak{n})$ can be proved in Z_μ for the recursively defined integer $\mathfrak{n}$, $\neg i_\mathfrak{n} \epsilon p$ can be proved from (5).

(b) *ω-consistency of the set theory.*

If $\neg\mathfrak{P}(\mathfrak{n})$ can be proved in an ω-consistent extension of Z_μ, then $i_\mathfrak{n} \epsilon p$ cannot be proved from (5).

Then we find a formula U made up of the symbols φ_{ij} and the formula variables of (1) so that U is undecided by (5).

Proof. Observe first that by condition (b) the formula (1) is consistent, and therefore the formula $q(n)=0$ of para. 1 is verifiable; i. e. the formal system consisting of Z_μ and the axiom $q(n)=0$ is an ω-consistent extension of Z_μ.

Consider the predicate of Z_μ

(17) $$\mathfrak{e}[i^*(n), n]=1$$

and suppose it is represented in the set theory by the term u.

Then

(18) $$i_{u^*} \epsilon u$$

is undecided by (5).

For, suppose (18) were proved from (5). By the lemma, $\mathfrak{e}[i^*(u^*), u^*]=0$ would be decided (to be true) at some finite stage. Therefore $\mathfrak{e}[i^*(u^*), u^*]=0$ would be proved in the ω-consistent extension of Z_μ by the formula $q(n)=0$, as observed at the end of para. 6. This conflicts with the condition that u represents the predicate (17) in the sense of (a) (i) and (b).

Next, suppose (18) were disproved by (5). By the lemma, $\mathfrak{e}[i^*(u^*), u^*]=1$ would be decided (to be true) at a finite stage, u^* would have the property (17), and the supposition conflicts with (a) (ii).

Note on the Theorem.

Theorem I is still hypothetical. The construction of a set theory which satisfies the conditions of the theorem, is still outstanding. Conditions (a) are satisfied by many of the usual set theories, e. g. Gödel [6], 26-29. It is probable that Gödel's model of a set theory based on ordinals up to the first ε-number provides a system which satisfies also condition (b) of the theorem.

Corollary to the Theorem.

Under the conditions of the theorem the formula (1) has no computable model. For, if $\mathfrak{e}(a,b)$ were a computable function it could be represented by a term of Z_μ, and $\mathfrak{e}[i^*(u^*), u^*]=0$ would be decided in Z_μ.

The result is true not only of the particular model (4), but of any arithmetic model of (2) which is defined in Z_μ, and where the formula variables B_{0j} of (2) are replaced by computable predicates B_{0j}^* so that

$$(x_1)\dots(x_r)\,(Ey_1)\dots(Ey_s)B^*(x_1\dots x_r y_1\dots y_s)$$

can be proved in some ω-consistent extension of Z_μ. For, if the predicates B_{0j}^* are decidable, and the formula above is proved from a verifiable free variable formula (the ω-consistent extension), by para. 13 of (2), we find *computable* functions $\psi_i(x_1\dots x_r)$, $1 \leqslant i \leqslant s$, so that

$$B^*[x_1\dots x_r \psi_1(x_1\dots x_r)\dots\psi_s(x_1\dots x_r)]$$

is verifiable. Part (a) of the lemma, the only part used in the proof of Theorem I, is still valid for introducing a numbering of terms.

9. *Remark on Theorem I.*

For reference below, we must examine what the undecided formula $\mathfrak{e}[i^*(u^*), u^*]=0$ means.

To fix ideas suppose that the formula $i^*(u^*) \epsilon u^*$ occurs in some F_k.

Consider now the (finite) number of *true making* distributions in F_k whose numbers, defined by the rule at the bottom of p. 186 of HB, II, are

$$\mathfrak{m}_1^1 < \dots < \mathfrak{m}_{s_1}^1 < \mathfrak{m}_1^2 < \dots < \mathfrak{m}_{s_2}^2 < \mathfrak{m}_1^3 < \dots < \mathfrak{m}_1^p < \dots < \mathfrak{m}_{s_p}^p$$

where in $\mathfrak{m}^{2r+1}$, $1 \leqslant 2r+1 \leqslant p$, $i^*(u^*) \epsilon u^*$ is put *false*, and in $\mathfrak{m}^{2r}$, $1 < 2r \leqslant p$, $i^*(u^*) \epsilon u^*$ is put *true*; also $s_1=0$ means that in the truth distribution with lowest number $i^*(u^*) \epsilon u^*$ is put true.

Recall that on p. 240 of HB, II, the recursive formula $H(k,l,m,n)$ is defined which holds if and only if n is the number of a truth distribution on T^l, and m the number of that truth distribution applied to T^k. Also, since n is restricted, $(En)\,H(k,l,m,n)$ is a recursive formula; it holds if the truth distribution on T^k with number m can be continued up to l, and we denote the formula by $H_1(k,l,m)$.

Then $\mathfrak{e}[i^*(u^*), u^*]=0$ is equivalent to

(19) $(l)(El_1)(\neg P^1 \& Q_2 \cdot \vee \cdot \neg P^3 \& Q_4 \cdot \vee \ldots \vee \cdot P^{p'} \& Q_{p'+1})$,

where $p'=p-1$ if p is even, $p'=p-2$ if p is odd; P^1 is the disjunction

$$H_1(k, \mathfrak{m}_1^1, l_1) \vee \ldots H_1(k, \mathfrak{m}_{s_1}^1, l_1),$$

P^{2r+1} the disjunction

$$P^{2r-1} \vee H_1(k, \mathfrak{m}_1^{2r+1}, l_1) \vee \ldots \vee H_1(k, \mathfrak{m}_{s_{2r+1}}^{2r+1}, l_1),$$

and Q_{2r} is the disjunction

$$H_1(k, \mathfrak{m}_1^{2r}, l) \vee \ldots \vee H_1(k, \mathfrak{m}_{s_{2r}}^{2r}, l).$$

That is, either

(i) the distributions $\mathfrak{m}_1^1 \ldots \mathfrak{m}_{s_1}^1$ cannot be continued beyond the stage l_1, and one of the distributions $\mathfrak{m}_1^2 \ldots \mathfrak{m}_{s_2}^2$ can be continued indefinitely,

or

(ii) $\mathfrak{m}_1^1 \ldots \mathfrak{m}_{s_1}^1$, $\mathfrak{m}_1^3 \ldots \mathfrak{m}_{s_3}^3$ cannot be continued beyond the stage l_1 and one of the distributions $\mathfrak{m}_1^4, \ldots \mathfrak{m}_{s_4}^4$ can be continued indefinitely;

and so forth until all distributions $\mathfrak{m}^{2r}$ with even index are exhausted.

The formula (19) is the prenex form of the undecided formula of our system, considered in Theorem I.

Note that even if the formula (19) were *proved* in an ω-consistent extension of Z_μ it would not show *which* truth distribution can be continued indefinitely; we could only be certain of finding the number l_1 so that $\neg H_1(k_1 \mathfrak{m}_i^1 l_1)$, $1 \leqslant i \leqslant s_1$. A condition by which the formula (19) is decided, is given in Theorem II; its ω-consistency is discussed in section II.

10. The next theorem is a (rather weak) analogue of Gödel's second undecidability theorem, first proved in detail in HB, II.

We denote the term of the set theory considered, which represents the property „n is an integer" by ω, the property $q(n)=0$ by q, and the property $\mathfrak{e}(n,m)=0$ by e; further we suppose that ordered pairs can be defined by terms made up of φ_{ij}, and we denote by $\langle a^* b^* \rangle$ the number given by the lemma to $\langle ab \rangle$.

Theorem II. Suppose the formula (5), which we denote by $\mathfrak{A}$, satisfies the conditions of Theorem I, contains the set of all integers, and a set of pairs of integers; further we suppose that the formula

(20) $\mathfrak{A} \rightarrow :\cdot\, i_{u^*} \in u^* : \rightarrow : \mathfrak{A}^* \rightarrow \langle i^*(u^*) u^* \rangle \in e$

can be proved in the predicate calculus. Then

$$(n)\,[q(n)=0]$$

is undecided by $\mathfrak{A}$.

For, by HB, II, 243-252, the formula $(n)[q(n)=0] \rightarrow \mathfrak{A}^*$ can be proved in Z_μ, and hence from $\mathfrak{A}$. Hence we get

$$\mathfrak{A} \rightarrow :\cdot\, i_{u^*} \in u \rightarrow : (n)\,(n \in \omega \rightarrow n \in q) \rightarrow \langle i^*(u^*), u^* \rangle \in e.$$

By the propositional calculus

$$\mathfrak{A} \rightarrow : (n)\,(n \in \omega \rightarrow n \in q) \rightarrow \cdot \neg\, i_{u^*} \in u \vee \langle i^*(u^*) u^* \rangle \in e.$$

But, by the definition (17) of u, $\langle i^*(u^*) u^* \rangle \in e \rightarrow \neg\, i_{u^*} \in u$, so that $\mathfrak{A} \rightarrow : (n)\,(n \in \omega \rightarrow n \in q) \rightarrow \neg\, i_{u^*} \in u$.

If $(n)\,[q(n)=0]$ were decided by $\mathfrak{A}$, i. e. $(n)\,(n \in \omega \rightarrow n \in q)$ were proved from $\mathfrak{A}$, we should also have a proof of $\neg\, i_{u^*} \in u$ from $\mathfrak{A}$, contrary to Theorem I.

Note that if we add $(n)\,(n \in \omega \rightarrow n \in q)$ to $\mathfrak{A}$, and *if* (20) holds, the present theorem shows that $\mathfrak{e}[i^*(u^*), u^*)=0$, i. e. provable in an ω-consistent formalism.

Remark. (20) is the analogue to condition (3) on p. 286 of HB, II, where sufficient conditions for the second undecidability theorem of Gödel are enumerated. But while it is established in HB, II, 312-323, that the condition (3) holds in the formalism Z_μ, and therefore in any formalism containing Z_μ in the sense of Theorem I(a), the derivation of (20) for the usual formalisms of set theory is highly problematic.

As far as the undecidability of the formula $(n)\,[q(n)=0]$ in $\mathfrak{A}$ is concerned, Theorem II is uninteresting because under our conditions, Gödel's second undecidability theorem can be applied, which already establishes that $(n)[q(n)=0]$ is undecided by $\mathfrak{A}$.

II. Discussion.

Relation between Gödel [3] *and Theorems I and II.*

11. Gödel's proofs are much more direct and general than those of the present paper. In fact as far as the construction of undecided propositions in the relevant logical systems is concerned, the interest of our work is restricted to having shown how the undecidability results are actually got from the Skolem model where they were first suspected.

12. *ω-consistency*[3]). The main point of difference, which is, perhaps, worth mentioning, in the *form* of the undecided formula (18): the proof of Theorem I does not show whether (18) is *ω-consistent* or not. Concerning this business of ω-consistency recall that Gödel's formula is of the form $(x)A(x)$ where $A(a)$ is a decidable, in fact primitive recursive, predicate. Now, from this it follows that his formula, if undecided in the system, is *verifiable* (HB, I, 238) or ω-consistent, that is $A(\mathfrak{m})$ is true for any recursively defined integer $\mathfrak{m}$; for if $A(\mathfrak{m})$ were false, we would have a disproof of the general formula by an example. Conversely, if $(Ex)A(x)$ is undecidable in a suitable system then it is ω-inconsistent, because $A(\mathfrak{m})$ cannot hold for any recursive $\mathfrak{m}$.

All this is clearly true for any undecided formula $(x)A(x)$ or $(Ex)A(x)$, where $A(a)$ is decidable. But now suppose that the formula known to be undecidable is of the form $(x)(Ey)A(x,y)$; its ω-consistency (in the sense that we can find a computable function $f(n)$ so that $(x)A[x,f(x)]$ is ω-consistent) cannot be decided on inspection: for we *may* be able to find an $f(n)$ so that $A[n,f(n)]$ is verifiable, but of course undecided in the system considered if $f(n)$ can be represented in the system; or, again, it may be that $(Ey)A(0y)$ is undecidable. In the former case we should say that the formula is ω-consistent, in the latter ω-inconsistent. Trivial examples of both cases can be got from a Gödel undecidable formula, namely $(x)(Ey)[A(x)\,\&\,y=1]$ or $(x)(Ey)[x\geqslant 0\ \&\ A_1(y)]$, where $(x)A(x)$, $(Ey)A_1(y)$ areunde cidables of the system.

The previous paragraph applies to our formula (18). If we recall para. 9, it may be that

(i) we find a number $\mathfrak{l}_1$ for which $\neg H_1(k,\mathfrak{m}_i^1,\mathfrak{l}_1)$, $1\leqslant i\leqslant s$, holds, and the disjunction $H_1(k,\mathfrak{m}_1^2,\mathfrak{l})\vee\ldots\vee H_1(k,\mathfrak{m}_{s_2}^2,\mathfrak{l})$ is undecidable (verifiable), in which case $\mathfrak{e}[i^*(u^*),u^*]=0$ is ω-consistent,

or

(ii) $(El_1)[\neg H_1(k,\mathfrak{m}_1^1,l_1)\,\&\ldots\&\,\neg H_1(k_1,\mathfrak{m}_{s_1}^1,l_1)]$ is undecidable, when $\mathfrak{e}[i^*(u^*)u^*]=1$ is ω-consistent.

In case (i) $\mathfrak{e}[i^*(u^*)u^*]=0$ is *proved* in the extension of Z_μ by the verifiable formula $H_1(k,m_1^2,l)\vee\ldots\vee H_1(k,\mathfrak{m}_{s_2}^2,l)$; in case (ii) $\mathfrak{e}[i^*(u^*)u^*]=1$ is *proved* in the extension of Z_μ by the verifiable formula $H_1(k,\mathfrak{m}_1^1\, l)\vee\ldots\vee H_1(k,\mathfrak{m}_{s_1}^1,l)$.

[3]) Contrary to custom we speak also of ω-consistent formulae, not only ω-consistent systems.

Note that we only consider the first disjunct of the formula (19), but this definition of ω-consistency is easily extended to the whole of (19).

As pointed out at the end of para. 9 the ω-consistency of $\mathfrak{e}[i^*(u^*)u^*]=0$ is established by Theorem II provided

(i) the system $\mathfrak{A}$ is itself ω-consistent,

(ii) (20) is provable in the predicate calculus.

Since, however, it is doubtful whether (ii) holds for general systems we have the following unsolved

Problem. Can we set up formulae $\mathfrak{A}$, $\mathfrak{A}_0$, satisfying the conditions of Theorem I so that the formulae

$$\mathfrak{e}[i^*(u^*),u^*]=0 \qquad \mathfrak{e}_0[i_0^*(u_0^*),u_0^*]=1$$

can be proved in some ω-consistent extension of Z_μ, where $\mathfrak{e}(a,b)$, $i^*(a)$, u^* are the relevant terms of the model of $\mathfrak{A}$, $\mathfrak{e}_0(a,b)$, $i_0^*(a)$, u_0^* those of $\mathfrak{A}_0$.

13. ***Remark.*** Note particularly that we restrict here as in [2] the notion of ω-consistent extensions to extensions by ω-consistent formulae of the form $(x)A(x)$, where $A(a)$ is a decidable predicate. These extensions are sufficient for our purpose because they decide any formula of the form $\mathfrak{e}(a,b)$ by para. 12.

Since however several definitions of ω-consistency have been given, e. g. Mostowski [7] (4.1), we must explain briefly our restriction. It depends on the general principle of para. 6 in [2] according to which *a logical problem should be so formulated that its solution consists in the proof of a free variable decidable formula*; in particular, the necessary and sufficient condition for the ω-consistency of a (prenex) formula $\mathfrak{K}$ should be that a free variable formula K, of a given sequence K_n associated with $\mathfrak{K}$, is verifiable. It will also be demanded that if $\mathfrak{K}$ is ω-consistent, $\neg\mathfrak{K}$ should be ω-inconsistent.

The definitions of ω-consistency, or of truth functions (e. g. HB, II, 329-338), which are naturally given, either do not satisfy the above demands, or make *proved* formula of Z_μ ω-inconsistent; e. g. the definition of ω-consistency of $\mathfrak{K}$ by the condition that $\mathfrak{K}$ has an *Erfüllung* (para. 1 of [2]) makes the proved formula $\mathfrak{M}$ of Z_μ in appendix I of [2] ω-inconsistent. Conversely, the definition that $\mathfrak{K}$ is ω-consistent if there is a computable counter example to any Erfüllung of $-\mathfrak{K}$, makes both $\mathfrak{M}$ and $\neg\mathfrak{M}$ ω-consistent. The definition that $\mathfrak{K}$ is ω-consistent if $\neg\mathfrak{K}$ cannot be proved in extensions of Z_μ by ω-consistent formulae $(x)A(x)$ means only, by para. 38 of [2], that there isn o counter example of finite order to every Erfüllung of $\mathfrak{K}$. There might, it seems, be a computable counter example of a higher type, and thus the condition is rather weak. The truth definition of HB, II, 329-338, is, of course, not of free variable form.

In view of these doubts about a satisfactory definition of ω-consistency we feel it is an advantage to restrict ω-consistent extensions to extensions by free variable formulae. At any rate they are sufficient for our purpose.

We note in passing that the definition of ω-consistency of a system:

„If for every recursive integer $\mathfrak{n}\,\mathfrak{A}(\mathfrak{n})$ can be proved (in the system considered) then we cannot prove in the system the formula $(Ex) \neg \mathfrak{A}(x)$".

[7] 4.1, is in general *not* equivalent to

„If $(Ex) \neg \mathfrak{A}(x)$ can be proved, then we can find a number $\mathfrak{n}$ so that $\neg \mathfrak{A}(\mathfrak{n})$ can be proved".

The equivalence holds in Z_μ if $\mathfrak{A}(\mathfrak{n})$ is a decidable predicate, but not if $\mathfrak{A}(\mathfrak{n})$ is the predicate $(x)[A(x) \vee \neg A(\mathfrak{n})]$, where $(x)A(x)$ is undecidable in Z_μ: for, we can prove $(Ey)\,(x)\,[A(x) \vee \neg A(y)]$, but for no $\mathfrak{n}$ can we prove $(x)[A(x) \vee \neg A(\mathfrak{n})]$. (However, we cannot find an ω-inconsistency, i. e. a function $f_0(n)$ so that $f_0(n)$ is the number of a proof of $\mathfrak{A}(n)$ in Z_μ; for from a proof of $(Ex) \neg \mathfrak{A}(x)$ we find a recursive functional $q_c[f(c)]$, $q_c[f_0(c)] = \mathfrak{n}_0$ say, so that $f_0(\mathfrak{n}_0)$ is not the number of a proof of $\mathfrak{A}(\mathfrak{n}_0)$).

The diagonal argument.

14. Both the present paper and Gödel [3] use the diagonal (non-enumerability) argument to construct undecided propositions. Though this point is obvious, it seems worth mentioning: for one thing it connects undecidability proofs which are usually referred to paradoxes and self references, with a familiar technique of mathematics, and, roughly speaking, allows one to convert non-enumerability proofs into those of undecidability. But also it throws light on the diagonal argument, and its „permissibility" e. g. [8].

When one uses the diagonal definition one usually thinks of a sequence of *decidable*, say recursive, predicates $A_n(m)$ from which we get a new predicate $B(m)$. The formula by which $B(m)$ is defined is not one of the sequence $A_n(m)$. This argument is used, e. g. to construct to any logical system all of whose formulae are decidable a new decidable predicate, as in HB, I, 330.

But what Gödel [3] or we do is to apply the diagonal definition to a system of predicates which are *not systematically decidable*, but quantified; now we must expect that the formal definition of the diagonal predicate is one of the given sequence $\mathfrak{A}_n(m)$, say the p^{th}; then $\mathfrak{A}_p(p)$ is undecided in the system. This situation occurs in Theorem I, and also in Gödel's argument. Recall that in the latter a formula *Prov* $(a\,b)$ is set up which holds if and only if a is the number of a proof of the formula with number b; and $s(a,b)$ is a function whose value is the number of the expression got when the free variable in the expression with number b is replaced by the number a. Then Gödel orders all expressions of a formalism by his numbering, so that, say, $\mathfrak{A}_n(a)$ with the free variable a has the number n. He considers the sequence of formulae $(Ey)\,\mathit{Prov}\,[y,s(m,n)]$ which will be provable if $\mathfrak{A}_n(m)$ can be proved in the system. The diagonal definition is

$$(y) \neg \mathit{Prov}\,[y,s(n,n)]$$

and this formula has the number q; i. e. the diagonal definition is one of the sequence, and here the diagonal argument establishes undecidability.

The diagonal argument does not provide a new (formula for a) predicate as it does when applied to a sequence of decidable predicates, but — if one looks for something „new" — it provides a new axiom [4]).

The Skolem Paradox.

15. It was mentioned in para. 2 that the definition of an arithmetic model (4) was thought to constitute a difficulty for axiom systems of the predicate calculus with non-enumerable cardinals. While we do not pretend to give a coherent account of cardinals, it seems worth while to discuss by the light of nature what exactly is at the bottom of these difficulties.

Three points which are more or less naively made on the basis of the model, can be distinguished.

(i) *The paradox proper.*

It was thought that by applying Cantor's diagonal argument to the model, we should somehow get a contradiction. This is not so because if one applies the argument to the model in the manner intended, we get an undecided proposition, and not a contradiction, provided of course that the conditions of Theorem I are satisfied.

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[4]) A great deal has been written since Poincaré on diagonal definitions occurring in a system of definitions. A very neat way of putting the point is due to Prof. Wittgenstein:

Suppose we have a sequence of rules for writing down rows of 0 and 1, suppose the p^{th} rule, the diagonal definition, say: write 0 at the n^{th} place (of the p^{th} row) if and only if the n^{th} rule tells you to write 1 (at the n^{th} place of the n^{th} row); and write 1 if and only if the n^{th} rule tells you to write 0. Then, for the p^{th} place, the p^{th} rule says: write nothing!

Similarly, suppose the q^{th} rule says: write at the n^{th} place what the n^{th} rule tells you to write at the n^{th} place of the n^{th} row. Then, for the q^{th} place, the q^{th} rule says: write what you write!

(ii) *Indefinability of sets.*

If, then, we do not get a contradiction, what happens to the diagonal class? Is it „lost"?

Now, as has been pointed out in para. 14, such a question may well depend on a naive misunderstanding of the diagonal definition where it is supposed that it provides a „new" class. Yet it is worth while to make the question precise.

Skolem [1] stated that the enumeration of classes, and the diagonal class are *not definable* in the model. But that is not yet precise: what is to be meant by saying that a class C is not definable in a system S, C being defined in some system S'? It is to be expected that the answer to this question will also make clear *why* the diagonal class is indefinable in the model, *how* it is excluded.

To consider the notion of definability (of classes of integers) we ask: Suppose $\mathfrak{A}(n)$, $\mathfrak{A}'(n)$ are two predicates of systems S and S', both systems containing recursive integers; under what conditions are these predicates (or their classes) *different*?

(α) *Strong difference.*

If we can find an $\mathfrak{n}$ so that $\mathfrak{A}(\mathfrak{n})$, $\mathfrak{A}'(\mathfrak{n})$ are decided in S, S', one is proved, the other disproved, we say that the two predicates are different.

If, further, $\mathfrak{A}'(n)$ is a predicate of S' so that every predicate of S can be shown to be different from $\mathfrak{A}'(n)$ in the sense above, $\mathfrak{A}'(n)$ is said to be indefinable in S.

Let us apply this to the model. We say that $\mathfrak{A}'(n)$ differs from the class with number $\mathfrak{a}$ of the model if we can find an $\mathfrak{n}$ so that

$$\mathfrak{e}[i^*(\mathfrak{n}), \mathfrak{a}] = 0$$

is decided in the model, $\mathfrak{A}'(\mathfrak{n})$ in S', and one is disproved, the other proved.

From this point of view the diagonal class u of the set theory (1) defined by the predicate $\mathfrak{e}[i^*(n), n] = 1$, is not indefinable in the model, i. e. does not differ from the class with number u^*. Provided the set theory is consistent, if $\mathfrak{e}[i^*(\mathfrak{n}), \mathfrak{n}] = 1$ has been proved from (1), $\mathfrak{e}[i^*(\mathfrak{n}), u^*] = 0$ can be proved in Z_μ from $(n)\,[q(n) = 0]$; and, conversely, if $\mathfrak{e}[i^*(\mathfrak{n}), \mathfrak{n}] = 0$ has been proved from (1), $\mathfrak{e}[i^*(\mathfrak{n}), u^*] = 1$ can be proved in Z^μ from $(n)\,[q(n) = 0]$. In other words, *where one expects the two predicates of* n, $\mathfrak{e}[i^*(n), n] = 1$ $\mathfrak{e}[i^*(n), u^*] = 0$ *to be different, one is undecidable in the systems considered.* (Note however that we have not established that both are undecidable because, though when $\mathfrak{e}[i^*(n), n] = 1$ is undecided in the set theory, $\mathfrak{e}[i^*(n), u^*] = 0$ is not decided at a *finite* stage, the minimum condition *may* decide $\mathfrak{e}[i^*(n), u^*] = 0$ in $Z_\mu + (n)\,[q(n) = 0]$.

(β) *Threefold difference.*

If we can find an $\mathfrak{n}$ so that either (α) applies, or $\mathfrak{A}(\mathfrak{n})$ is decided in S, $\mathfrak{A}'(\mathfrak{n})$ undecided in S', or $\mathfrak{A}'(\mathfrak{n})$ decided in S', $\mathfrak{A}(\mathfrak{n})$ undecided in S, we call $\mathfrak{A}(n)$, $\mathfrak{A}'(n)$ different predicates. This amounts to considering predicates not as a division of integers into two classes T, F, but three T, F, U. With this definition it has neither been *shown* that the diagonal class is not indefinable in the model, see note to (α), nor that it is indefinable.

(γ) *ω-consistent difference.*

If $\mathfrak{A}(\mathfrak{n})$ is decided in some ω-consistent extension (see para. 13) of S, and $\mathfrak{A}'(\mathfrak{n})$ in some ω-consistent extension of S', one true, the other false, we call $\mathfrak{A}(n)$, $\mathfrak{A}'(n)$ different predicates.

With this definition the diagonal class is not definable in the model. For, by the argument of para. 12, a formula $\mathfrak{e}(a, b) = 0$ is decidable in the ω-consistent extensions of Z_μ considered, and clearly the class $\mathfrak{y}$ of the model is different from the class $\mathfrak{e}[i^*(n), n] = 1$ since $\mathfrak{e}[i^*(\mathfrak{y}), \mathfrak{y}]$ is computable for every $\mathfrak{y}$ in the extensions considered.

This fact may also be stated thus: *the model* (4) *is only a model of the set theory* (1), *but not of all ω-consistent extensions of* (1). The model is ω-inconsistent in the following sense: by para. 12 it can be decided in an ω-consistent extension of Z_μ whether the integer u^* has the property $\mathfrak{e}[i^*(n), n] = 1$; if it does, its representative $i^*(u^*)$ does not belong, in the sense of the model, to the representative u^* of the property $\mathfrak{e}[i^*(n), n] = 1$; „representatives" were defined in the lemma and in Theorem I. The impossibility of finding a model (4) for all ω-consistent extensions of a set theory containing Z_μ may be regarded as the counterpart in the formal theory of the naive notion of non-enumerability; and the ω-inconsistency of the model as the reason why the diagonal class is excluded.

Remark. The last definition (γ) is sufficient for our purposes. But it must not be regarded as a satisfactory general definition of indefinability.

(i) It is easy to see that there are predicates $\mathfrak{A}(n)$ of Z_μ so that $\mathfrak{A}(\mathfrak{n})$ is not decided in the ω-consistent extensions of Z_μ which we consider, and two predicates $\mathfrak{A}(n)$, $\mathfrak{A}'(n)$ may fail to be called different, because at the crucial values of $\mathfrak{n}$, one of them is undecided.

(2) The definition says nothing about the difference between two predicates defined in ω-inconsistent formal systems.

(iii) Lastly, return to the most naive (picturesque) interpretation of the diagonal argument: that there are *more* predicates than integers.

Since [1] and [3] this idea is rejected because the formulae by which the predicates are defined can be numbered. This argument is not convincing since, when „difference" between predicates has been explained, the same formula may define different predicates in systems with different axioms. The matter is easily settled for our $(\alpha)-(\gamma)$ by showing that any predicate defined in formalized systems is equal [for $(\alpha)-(\gamma)$!] to a predicate of ω-consistent extensions of Z_μ, $\mathfrak{Z}$ say, which is a system whose formulae can be enumerated by a primitive recursive function (but not its proofs).

(α) Since here only decidably different predicates of integers are counted different, and decidable predicates can be represented in Z_μ, any predicate considered is equal in sense (α) to a predicate of $\mathfrak{Z}$.

(β) By a formalized system we mean one whose formulae and proofs can be enumerated by primitive recursive functions: with the notation of para. 14, where $\mathfrak{A}(b)$ with the free variable b has the number $\mathfrak{a}$, $\neg\mathfrak{A}(n)$ the number $t(n)$, and $\mathfrak{A}'(n)$ is the formula

$$(Ex)\,Prov\,[x,s(n,\mathfrak{a})] \vee b=0 \cdot \& \cdot (x) \neg Prov\,[x,t(n)],$$

then $\mathfrak{A}'(n)$ defined in $\mathfrak{Z}$ is the same predicate in sense (β) as $\mathfrak{A}(n)$. For if $\mathfrak{A}(n)$ is proved in a consistent S, the second conjunct of $\mathfrak{A}'(n)$ is verifiable, hence provable in $\mathfrak{Z}$, and the first provable by an example for x; if $\neg\mathfrak{A}(n)$ is proved in S, the second conjunct of $\mathfrak{A}'(n)$ is disproved by an example, and hence $\neg\mathfrak{A}'(n)$ provable in $\mathfrak{Z}$; if $\mathfrak{A}(n)$ is undecided, the second conjunct is verifiable, the first equivalent in $\mathfrak{Z}$ to $b=0$, and therefore also undecided (with free variable b).

(γ) The result is trivial if S and S' are the system Z_μ. We do not discuss the general case.

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