$\overline{V}_{n,J}(x)$ est le cube de centre x_J et de coté $1/n$; on a, avec les notations ci-dessus, et en vertu du th. de Fubini

$$\int\limits_{V_{n,J}} f d\mu = \int\limits_{\overline{V}_{n,J}} f_J d\mu_J;$$

d'autre part, $\mu V_{n,J}(x) = \mu_J \overline{V}_{n,J}(x)$. Le th. de Vitali rappelé ci-dessus montre donc que, pour tout J, on a

$$\lim_{n\to\infty} g_{n,J}(x) = f_J(x) \quad \text{presque partout.}$$

Comme l'ensemble $\boldsymbol{F}$ est dénombrable, il existerait donc dans $P = I^{\mathfrak{N}}$ un ensemble de mesure nulle dans le complémentaire duquel on aurait

$$\lim_{n\to\infty} g_{n,J}(x) = f_J(x) \quad \text{pour } \textit{tout} \quad J \in \boldsymbol{F}.$$

Mais alors si $g_{n,J}(x)$ tendait presque partout vers $f(x)$ suivant $\mathfrak{N} \times \boldsymbol{F}$, le th. de la double limite ([1], p. 49) montrerait que $f_J(x)$ tend presque partout vers $f(x)$ suivant $\boldsymbol{F}$, ce que nous avons reconnu être inexact.

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Some Theorems on the Theory of Sets.

By

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W. Sierpiński and S. Piccard have considered the following problem:

Let E be a given non countable set and suppose that there exists a relation R between the elements of E, such that, for any $x \in E$, the power of the elements $y \in E$ for which xRy, is smaller than the power of E. The problem is whether E has a subset E_1 of the same power and having the property that no two elements $x, y \in E_1$ bear the relation R to each other?

In the present Note we shall consider relations between *elements* and *subsets* of a set:

Let E be a non void set. Denote by H the set of all subsets r of E. Let R be a relation between the elements $x \in E$ and $r \in H$, having the following property:

(A) *for any $r \in H$, there is one and only one $x \in r$ such that xRr holds.*

Problem I. Let E_1 be the subset of E consisting of all the elements $x \in E$ for which the power of the set of the elements $r \in H$ connected with x by the relation xRr, is $\leqslant n$ (n is at most equal to the power of E).

The question is: what is the power of E_1?

Theorem I. *Denoting by z the power of E_1, we have for z and n*

$$2^z - 1 \leqslant n \cdot z. \tag{B}$$

Proof. For any given $x \in E_1$, the power of the set of the $r \in H$ for which xRr, is by definition $\leqslant n$. Thus there are at most nz elements of H in the relation R with some elements of E_1. But the power of the set of all subsets of E_1 is

$$2^z - 1,$$

the void subset being not counted. It is therefore evident that the inequality

$$2^z - 1 > nz$$

would contradict to the condition (A), consequently:

$$2^z - 1 \leqslant nz.$$

It may be seen from (B) that

when	$n=1$	then	$z \leqslant 1$	when	$n=4$	then	$z \leqslant 4$
	$n=2$		$z \leqslant 2$		$n \geqslant 5$		$z < n$
	$n=3$		$z \leqslant 3$		$n=\aleph_0$		$z=$ finite.

Problem II. Let E be a countably infinite set. Denote by H the set of all finite subsets of E. Denote further by E^* the subset of E consisting of the elements $x \in E$ which are in the relation rRx with countably many $r \in H$.

The question is: what is the power of E^*?

Theorem II. *The power of E^* is $\aleph_0$*.

Proof. Denote by E_1 the subset of E consisting of those $x \in E$, for which there are only a finite number of $r \in H$ such that xRr. By Theorem I E_1 is finite. The power of E^* cannot be finite, because E is countably infinite and by condition (A) each element of E is in the relation with at least one element of H. The theorem is proved.

Let E be again an arbitrary set. Let H be the set of all subsets of E and n a cardinal number less than the power of E. Denote by E_1 the set of those $x \in E$ which are in the relation xRr with only such subsets $r \in H$ for which $\overline{\overline{r}} \leqslant n$.

Theorem III. *The power of E_1 is at most n.*

Proof: Suppose the contrary, i. e. that the power of E_1 is greater than n. Then by (A) there is an $x \in E_1$ such that xRE_1 holds. But, by the definition of E_1, this element x cannot belong to E_1, which is a contradiction.

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Remark on an Invariance Theorem.

By

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Borsuk gave a set theoretic proof of the following theorem [1]), which was previously proved by algebraic topology [2]).

Let $\mathfrak{X}$ denote the n-dimensional sphere; then:

(i) *For any closed set $F \subset \mathfrak{X}$ the number of components of the set $\mathfrak{X} - F$ is a topological invariant of the set F.*

Call (i*) the more general statement obtainable from (i) by omitting the assumption that F is closed [3]).

Theorem (i*) has been shown by Eilenberg using algebraic topology [4]).

In this note I shall deduce (i*) from (i) using set theoretic method.

In fact, I shall show that:

(ii) *Theorem* (i*) *holds in every locally connected continuum $\mathfrak{X}$ satisfying* (i).

The proof of (ii) will be based on the following theorem [5]):

(iii) *Let E be a subset of a locally connected space $\mathfrak{X}$. In order that the set $\mathfrak{X} - E$ be decomposable into n separated non void subsets, it is necessary and sufficient that E contain a closed set F such that for each closed set H satisfying the condition $F \subset H \subset E$ the set $\mathfrak{X} - H$ is decomposable into n separated non void subsets.*

Theorem (iii) may be established as follows.

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[1]) See *A Set Theoretical Approach to the Disconnection Theory of the Euclidean Space*, this volume, p. 217-241.

[2]) Cf. J. W. Alexander, Trans. Amer. Math. Soc. **23** (1922), p. 333, P. Alexandroff, Annals of Math. **30** (1928), p. 163.

[3]) If $\mathfrak{X} - F$ consists of an infinity of components, the „number of components" is to be understood as being equal to ∞ (not to be confused with the cardinal number of the set of components).

[4]) Bull. Amer. Math. Soc. **47** (1941), p. 73. See also P. Alexandroff, Dokłady Akad. Nauk SSSR **57** (1947), p. 110.

[5]) See my *Topologie II* (1950), p. 174.