The sequence $\{j_n\}$ being a permutation of the sequence of all positive integers, it follows at once that $m=0$. In other words: we have, for n sufficiently great, $m_n=0$, hence $j_n=n$. Therefore

$$f(0,\tfrac{1}{2})=(0,\tfrac{1}{2}). \tag{8}$$

Applying the same argument to the left side of the set A, it can be shown that there exists a sequence of integers $\{i_n\}$ such that $f(N_n)=N_{i_n}$ and that the sequence $\{i_n-n\}$ is convergent. Let q be its limit. We have then (cf. (4)): $f(0,\frac{1}{2})=(0,p_{-q})$ and therefore $q=0$ (by (8)). It readily follows that $\{i_n\}$ is again a permutation of *all* positive integers. But this is impossible since $i_n \geqslant 2$ in view of the definition of B.

Set Theoretical Approach to the Disconnection Theory of the Euclidean Space.

By

Karol Borsuk (Warszawa).

1. Introduction. In 1931 I gave[1]) an elementary proof of the qualitative part of the known theorem of L. E. J. Brouwer[2]) asserting that if a compactum A disconnects the $(n+1)$-dimensional Euclidean space E_{n+1} then so does every subset of E_{n+1} homeomorphic to A. That elementary proof consists in the characterization of the compacta $A \subset E_{n+1}$ which do not disconnect E_{n+1} by the connectivity of the functional space of continuous transformations of A in the n-dimensional Euclidean sphere S_n.

In 1935 S. Eilenberg[3]) showed how the continuous transformations of $A \subset E_2$ allow to prove also the invariance of the number of the regions in which A decomposes the Euclidean plane E_2. In his reasonings S. Eilenberg uses the fact that the continuous transformations of A in S_1 can be multiplied and thus constitute an Abelian group. A similar multiplication for arbitrary continuous transformations of A in S_n is for $n>1$ impossible. However it is possible to define an operation of multiplication *(homotopic multiplication)* for some pairs of homotopy classes (called henceforth ***multiplicatible classes***) and obtain in such a manner a group having as elements the homotopy classes of continuous transformations

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[1]) K. Borsuk, *Über Schnitte der n-dimensionalen Euklidischen Räume*, Math. Annalen **106** (1932), p. 239-248.

[2]) L. E. J. Brouwer, *Beweis des Jordanschen Satzes für den n-dimensionalen Raum*, Math. Annalen **71** (1912), p. 314.

[3]) S. Eilenberg, *O zastosowaniach topologicznych odwzorowań na okrąg koła*, Wiadomości Matematyczne **41** (1935), p. 1-32. S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), p. 61-112.

of A in S_n. This group, defined by myself in 1936 [4]), was in 1949 investigated from the point of view of algebraic topology by E. Spanier [5]), who termed it the n^{th} *cohomotopy group*. But the operation of the homotopic multiplication is quite elementary and does not involve any notion of combinatorial topology. It suggests that it can provide a useful tool in the elementary study of the topology of the Euclidean space E_{n+1} similarly as the multiplication of continuous transformations in the circumference S_1 does in the elementary study of the topology of the Euclidean plane E_2.

In this paper it is shown that the structure of the n^{th} cohomotopy group of a compact subset A of E_{n+1} depends only on the number k of the components of the set $E_{n+1}-A$ and conversely, that the n^{th} cohomotopy group of A determines the number k. Thus an elementary proof of the topological invariance of the number k is given. In order to emphasize the elementary character of reasonings only some quite elementary theorems without proof will be used; they are specified in the Preliminaries.

2. Preliminaries. Only metric spaces will be considered. E_n will denote the Euclidean *n-dimensional space* with the points $(x_1, x_2, \ldots, x_n)$, and S_n the *n-dimensional sphere* defined in the space E_{n+1} by the equation

$$x_1^2+x_2^2+\ldots+x_{n+1}^2=1.$$

If $m \leqslant n$ then the space E_m will be considered as a subset of the space E_n; we shall namely identify every point $(x_1, x_2, \ldots, x_m) \in E_m$ with the point $(x_1, x_2, \ldots, x_m, 0, \ldots, 0) \in E_n$. It follows that $S_m \subset S_n$ for $m \leqslant n$. In particular the sphere $S_{n-1} \subset S_n$ will be called the *equator* of S_n. The points $b_1^n=(0,0,\ldots,0,1)$ and $b_{-1}^n=(0,0,\ldots,0,-1)$ of S_n will be called *poles* of S_n.

By an *n-dimensional element* we understand any set homeomorphic with the n-dimensional (closed) simplex. In quite elementary manner [6]) we can distinguish between the *interior* and the *boundary* points of an n-dimensional element. The set of all boundary points is homeomorphic to the $(n-1)$-dimensional sphere S_{n-1}.

[4]) K. Borsuk, *Sur les groupes des classes de transformations continues*, C. R. de l'Ac. des Sc. **202** (1936), p. 1400-1403.

[5]) E. Spanier, *Borsuk's cohomotopy groups*, Annals of Math. **50** (1949), p. 203-245.

[6]) See for instance W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1941, p. 96.

Obviously the equator S_{n-1} cuts S_n in two regions: R_1^n containing b_1^n, and R_{-1}^n containing b_{-1}^n. The closures of this regions are n-dimensional elements having S_{n-1} as their common boundary. We shall denote them by Q_1^n and Q_{-1}^n.

By the projection of S_n of the centre b_1^n on the space E_n we obtain a homeomorphic correspondence between the points of $S_n-(b_1^n)$ and the points of E_n. Thus we can consider S_n as the Euclidean space E_n completed by an "infinitely distant" point corresponding to b_1^n. Let us observe that by this correspondence the geometrical spheres lying in E_n correspond to the geometrical spheres lying in $S_n-(b_1^n)$.

It is easily seen that:

(1) *For every finite set of points lying in a region $G \subset E_n$ there exists an n-dimensional element $Q \subset G$ containing all these points in its interior* [7]).

By the *Cartesian product* of two spaces X and Y the space $X \times Y$ is meant consisting of all ordered pairs (x,y) with $x \in X$, $y \in Y$, and with the distance defined by the formula

$$\varrho((x,y),(x',y'))=\sqrt{\varrho(x,x')^2+\varrho(y,y')^2}.$$

If X_0 is a subset of X, and f a transformation with the range X, then by f/X_0 will be denoted the *partial transformation* of f defined in X_0, i. e. the transformation f_0 defined in X_0 by the formula $f_0(x)=f(x)$. We shall say that f constitutes an *extension* of f_0 on X.

By a *mapping* we always understand a continuous transformation.

By a *retraction* we understand a mapping r of X onto $X_0 \subset X$ satisfying the condition

$$r(x)=x \quad \text{for every} \quad x \in X_0.$$

If there exists a retraction of X onto X_0, then X_0 is called a *retract* of X. If there exists a continuous mapping $r(x,t)$ (called a *deformation retraction* of X onto X_0) of the space $X \times \langle 0,1 \rangle$ (where $\langle 0,1 \rangle$ denotes the interval $0 \leqslant t \leqslant 1$) into X such that $r(x,0) \equiv x$ and $r(x,1)$ is a retraction of X onto X_0, then X_0 is said to be a *deformation retract* of X.

[7]) See for instance C. Kuratowski, *Topologie II*, Warszawa-Wrocław 1950, p. 343.

Examples:

(2) *S_n is a deformation retract of $E_{n+1}-(0)$.*

(3) $[(b_1^n)\times S_n]+[S_n\times(b_1^n)]$ *is a deformation retract of*

$$[S_n\times S_n]-(b_{-1}^n, b_{-1}^n)\ [8].$$

A compactum A is an *absolute retract* (abbreviation AR) whenever a topological image of A in any space X is necessarily a retract of X. A compactum A is an *absolute neighbourhood retract* (abbreviation ANR) whenever a topological image A^* of A in any space X is necessarily a retract of some neighbourhood of A^* in X. It is known that:

(4) *Every n-dimensional element is an AR* [9].

(5) *Every n-dimensional sphere S_n is an ANR* [10].

The following property of absolute retracts will be used in this paper:

(6) *If X_0 is a closed subset of a space X, and Y is an AR, then every mapping of X_0 in Y has a continuous extension on X with the values belonging to Y* [11].

A subset X_0 of X is said to be *contractible* in X if there exists a mapping $f(x,t)$ of $X_0\times\langle 0,1\rangle$ into X such that $f(x,0)\equiv x$ and $f(x,1)\equiv\text{const}$.

Given a compactum X all mappings f with the range X and the values belonging to another space Y constitute a metric space Y^X with the distance formula

$$\varrho(f,g)=\sup_{x\in X}\varrho(f(x),g(x)) \quad \text{for every} \quad f,g\in Y^X.$$

(7) *If the space Y is complete, then the space Y^X is also complete* [12].

Let X be a compactum and Y an arbitrary space. The component of the space Y^X containing a given mapping $f\in Y^X$ will be denoted by (f) and called the *homotopy class* of f. Clearly $(f_1)=(f_2)$

[8]) K. Borsuk, *Sur l'addition homologique des types de transformations continues en surfaces sphériques*, Annals of Math. **38** (1937), p. 734.

[9]) K. Borsuk, *Sur les rétracts*, Fund. Math. **17** (1931), p. 160.

[10]) K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), p. 227.

[11]) K. Borsuk, *Sur les rétracts*, Fund. Math. **17** (1931), p. 161.

[12]) See for instance C. Kuratowski, *Topologie I*, Warszawa-Wrocław 1948, p. 315.

means that f_1 and f_2 belong to the same component of Y^X, i. e. are *homotopic* in Y^X. The mapping $f\in Y^X$ is said to be *essential* if it is not homotopic to a constant. For instance:

(8) *The identical mapping of S_n is essential* [13]) *(in $S_n^{S_n}$).*

If to every $0\leqslant t\leqslant 1$ corresponds a mapping $f_t\in Y^X$ depending continuously on the parameter t, then we say that the mappings f_t constitute a *family* $\{f_t\}$ joining in Y^X the mapping f_0 with the mapping f_1.

(9) *If Y is an ANR then the space Y^X is locally connected* [14].

Since every component of a complete locally connected space is arcwise connected [15]), we infer by (7) and (9):

(10) *If Y is an ANR then two mappings $f,g\in Y^X$ are homotopic in Y^X if and only if there exists a family of mappings joining f with g in Y^X.*

The following statement is for us of importance [16]):

(11) *Let X_0 be closed subset of a compactum X, and Y an ANR. If $\{f_t\}$ is a family $\subset Y^{X_0}$, and $g_0\in Y^X$ an extension of f_0, then there exists a family $\{g_t\}\subset Y^X$ such that g_t is an extension of f_t for every $0\leqslant t\leqslant 1$.*

3. Some elementary lemmas.

Lemma 1. *Let X be a compactum. If $f_0,f_1\in S_n^X$ and $\varrho(f_0(x),f_1(x))<2$ for every $x\in X$, then $(f_0)=(f_1)$.*

Proof. The inequality $\varrho(f_0(x),f_1(x))<2$ implies that the segment $L_x=\overline{f_0(x)f_1(x)}$ does not pass through the centre 0 of S_n. Let us denote by $a_t(x)$, for every $0\leqslant t\leqslant 1$, the point of L_x such that

$$\varrho(f_0(x),a_t(x))=t\cdot\varrho(f_0(x),f_1(x)).$$

Putting

$$f_t(x)=\frac{a_t(x)}{\varrho(0,a_t(x))}$$

we obtain a family $\{f_t\}$ joining f_0 with f_1 in S_n^X.

[13]) See for instance W. Hurewicz and H. Wallman, l. c. p. 37.

[14]) K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), p. 224.

[15]) See for instance C. Kuratowski, *Topologie II*, Warszawa-Wrocław 1950, p. 184.

[16]) K. Borsuk, *Sur les prolongements des transformations continues*, Fund. Math. **28** (1936), p. 103. See also W. Hurewicz and H. Wallmann, l. c. p. 86.

Lemma 2. *If a compact set $A\subset E_n$ and a mapping $f\in E_n^A$ are given, then for every $\varepsilon>0$ there exists a natural number k and a mapping $g\in E_n^A$ such that $\varrho(f,g)<\varepsilon$, and for every $y\in E_n$ the inverse-image $g^{-1}(y)$ contains at most k points.*

Proof. Consider two n-dimensional simplexes $\Delta_0,\Delta_1\subset E_n$ such that $A\subset\Delta_0$ and $f(A)\subset\Delta_1$. By (6) there exists a mapping $\bar{f}\in\Delta_1^{\Delta_0}$, being an extension of f. Let Γ be a triangulation of Δ_0 such that for every simplex Δ of Γ the diameter of $\bar{f}(\Delta)$ is $<\frac{1}{5}\varepsilon$. Let $a_1,a_2,\dots,a_m$ be all vertices of the triangulation Γ. Obviously we can find m points $a_1',a_2',\dots,a_m'$ in such a manner that

$$\varrho(a_i',\bar{f}(a_i))<\tfrac{1}{5}\varepsilon \quad \text{for } i=1,2,\dots,m$$

and that any $m\leqslant n+1$ of these points are linearly independent.

For every point $x\in\Delta_0$ there exists a simplex $\Delta=\Delta(a_{i_0},a_{i_1},\dots,a_{i_l})$ of Γ containing x. Then

$$x=a_0\cdot a_{i_0}+a_1\cdot a_{i_1}+\dots+a_l\cdot a_{i_l} \quad\text{with}\quad a_\nu\geqslant 0 \quad\text{and}\quad a_0+a_1+\dots+a_l=1.$$

Putting

$$g(x)=a_0\cdot a_{i_0}'+a_1\cdot a_{i_1}'+\dots+a_l\cdot a_{i_l}'$$

we obtain a simplicial mapping g of A in E_n such that

$$\varrho(\bar{f}(x),g(x))\leqslant\varrho(\bar{f}(x),\bar{f}(a_{i_0}))+\varrho(\bar{f}(a_{i_0}),a_{i_0}')+\varrho(a_{i_0}',g(x))<\varepsilon,$$

since

$$\varrho(\bar{f}(x),\bar{f}(a_{i_0}))<\tfrac{1}{5}\varepsilon,\quad \varrho(\bar{f}(a_{i_0}),a_{i_0}')<\tfrac{1}{5}\varepsilon,\quad \varrho(a_{i_0}',g(x))\leqslant \operatorname{Max}\varrho(a_{i_\mu}',a_{i_\nu}'),$$

$$\varrho(a_{i_\mu}',a_{i_\nu}')\leqslant\varrho(a_{i_\mu}',\bar{f}(a_{i_\mu}))+\varrho(\bar{f}(a_{i_\mu}),\bar{f}(a_{i_\nu}))+\varrho(\bar{f}(a_{i_\nu}),a_{i_\nu}')<\tfrac{3}{5}\varepsilon.$$

Moreover let us observe that g maps each of the simplexes of Γ topologically. Consequently if k denotes the number of all simplexes in the triangulation Γ, the set $g^{-1}(x)$ contains at most k points.

Lemma 3. *Given a compactum $A\subset E_{n+1}$ and a mapping $f\in S_n^A$. There exists a finite subset N of $E_{n+1}-A$ such that f has a continuous extension g_0 on $E_{n+1}-N$ with the values belonging to S_n.*

Proof. Let Δ_0 be an $(n+1)$-dimensional simplex in E_{n+1} containing A in its interior. Let B denote the boundary of Δ_0. We extend the mapping f on the set $A+B$ putting

$$f(x)=b_1^n \quad\text{for every } x\in B.$$

Let Y be an AR such that $S_n\subset Y\subset E_{n+1}$. By (6) there exists a mapping $\bar{f}\in Y^{\Delta_0}$ being an extension of f. Applying the lemma 2 we infer that there exists a mapping g of Δ_0 into E_{n+1} such that

$$\varrho(\bar{f}(x),g(x))\leqslant\tfrac{1}{2}\quad\text{for every } x\in A+B \tag{12}$$

and that all inverse-images $g^{-1}(y)$ of single points $y\in E_{n+1}$ are finite.

Let us put $N=g^{-1}(0)$. Since $\bar{f}(x)\in S_n$ for every $x\in A+B$, we infer from (12) that

$$N\subset\Delta_0-A-B.$$

Putting

$$g_1(x)=\frac{g(x)}{\varrho(g(x),0)}\quad\text{for every } x\in\Delta_0-N$$

we obtain a mapping g_1 of Δ_0-N into S_n. By (12) it is

$$\varrho(g_1(x),\bar{f}(x))<\tfrac{3}{2}\quad\text{for every } x\in A+B,$$

and consequently, by the lemma 1, the mappings f and $g_1/(A+B)$ are homotopic in S_n^{A+B}. Applying (5) and (11) we infer that f has a continuous extension f' on Δ_0-N with the values belonging to S_n. It is sufficient to put

$$g_0(x)=f'(x)\quad\text{for every } x\in\Delta_0-N$$

and

$$g_0(x)=b_1^n\quad\text{for every } x\in E_{n+1}-\Delta_0$$

in order to obtain the required extension of f on $E_{n+1}-N$.

Lemma 4. *Given two mappings $f_0,f_1\in S_n^A$, where A is a compactum. If $(f_0)=(f_1)$ and for some $a\in A$ it is $f_0(a)=f_1(a)$, then there exists a continuous family $\{f_t\}\subset S_n^A$, $0\leqslant t\leqslant 1$, such that $f_t(a)=f_0(a)$ for every $0\leqslant t\leqslant 1$.*

Proof. By (10) there exists a family $\{f_t'\}$ joining $f_0'=f_0$ with $f_1'=f_1$ in S_n^A.

If $n=1$, then the points of S_n can be considered as complex numbers z with $|z|=1$. Putting

$$f_t(x)=\frac{f_0(a)}{f_t'(a)}\cdot f_t'(x)\quad\text{for every } x\in A$$

we obtain the required family.

If $n>1$, then there exists a finite system

$$t_0=0<t_1<t_2<\dots<t_m<t_{m+1}=1$$

of real numbers such that the diameter of the subset M_ν of S_n consisting of all points of the form $f_t'(a)$, with $t_\nu\leqslant t\leqslant t_{\nu+1}$ is $\leqslant\frac{1}{4}$ for $\nu=0,1,\dots,m$. Let a^* denote the point of S_n antipodical to $f_0(a)$ (i. e. such that $\varrho(a^*,f_0(a))=2$). Obviously there exists a mapping $\varphi_\nu(t)$ of the interval $t_\nu\leqslant t\leqslant t_{\nu+1}$ onto a simple arc L_ν joining $f_{t_\nu}'(a)$ with $f_{t_{\nu+1}}'(a)$ in the set $S_n-(a^*)$ and having the diameter $\leqslant\frac{1}{2}$. Hence

(13) $$\varrho(f_t'(a),\varphi_\nu(t))\leqslant 1 \quad \text{for every } t_\nu\leqslant t\leqslant t_{\nu+1}.$$

Consider now three mappings φ, ψ and χ defined in the closed subset

$$M=[A\times(0)]+[(a)\times\langle 0,1\rangle]+[A\times(1)]$$

of the set $P=A\times\langle 0,1\rangle$ by the formulae:

$$\begin{aligned}
\varphi(x,0)&=f_0(x) && \text{for } x\in A,\\
\varphi(a,t)&=\varphi_\nu(t) && \text{for } t_\nu\leqslant t\leqslant t_{\nu+1},\ \nu=0,1,\dots,m,\\
\varphi(x,1)&=f_1(x) && \text{for } x\in A,\\
\psi(x,0)&=f_0(x) && \text{for } x\in A,\\
\psi(a,t)&=f_t'(a) && \text{for } 0\leqslant t\leqslant 1,\\
\psi(x,1)&=f_1(x) && \text{for } x\in A,\\
\chi(x,0)&=f_0(x) && \text{for } x\in A,\\
\chi(a,t)&=f_0(a) && \text{for } 0\leqslant t\leqslant 1,\\
\chi(x,1)&=f_1(x) && \text{for } x\in A.
\end{aligned}$$

Evidently $\varphi,\psi,\chi\in S_n^M$, and by (13) $\varrho(\varphi,\psi)\leqslant 1$. Moreover, $\varphi(x,0)=\chi(x,0)$ and $\varphi(x,1)=\chi(x,1)$ for every $x\in A$, and $\varphi(a,t)\in S_n-(a^*)$ and $\chi(a,t)=f_0(a)$ for every $0\leqslant t\leqslant 1$. Consequently $\varrho(\varphi,\chi)<2$. By the lemma 1 the mappings φ and ψ are homotopic in S_n^M; so are the mappings φ and χ. It follows that ψ and χ are homotopic in S_n^M. Putting

(14) $$\overline{\psi}(x,t)=f_t'(x) \quad \text{for every } (x,t)\in A\times\langle 0,1\rangle$$

we obtain a continuous extension of ψ onto P with the values belonging to S_n. By (11) there exists a continuous extension $\overline{\chi}(x,t)$ of χ on P with the values belonging to S_n. Putting

$$f_t(x)=\overline{\chi}(x,t) \quad \text{for every } x\in A \text{ and } 0\leqslant t\leqslant 1$$

we obtain a family $\{f_t\}\subset S_n^A$ joining f_0 with f_1 and satisfying the condition

$$f_t(a)=\chi(a,t)=f_0(a) \quad \text{for every } 0\leqslant t\leqslant 1.$$

Lemma 5. *If X_0 is a deformation retract of a compactum X and Y an ANR, then the homotopy classes of the mappings $f\in Y^X$ are in a one-one correspondence with the homotopy classes of the partial mappings f/X_0.*

Proof. It will be sufficient to show that for any two mappings $f,f'\in Y^X$ the homotopy of the partial mappings $f_0=f/X_0$ and $f_0'=f'/X_0$ in Y^{X_0} implies the homotopy of f and f' in Y^X.

If $r(x,t)$ denotes the retraction by deformation of X to X_0, then putting

$$g_t(x)=fr(x,t) \quad \text{for every } x\in X,\ 0\leqslant t\leqslant 1$$

we see at once that f is homotopic to $f_0r(x,1)$. Similarly we show that f' is homotopic to $f_0'r(x,1)$. But the homotopy of f_0 and f_0' in Y^{X_0} implies the homotopy of $f_0r(x,1)$ and $f_0'r(x,1)$ in Y^X. Consequently the mappings f and f' are homotopic in Y^X.

4. A theorem on addition.

Theorem. *Let A_1 and A_2 be compacta and let A_1A_2 contains exactly one point a. If we assign to every homotopy class $(f)\subset S_n^{A_1+A_2}$ the pair of the homotopy classes $(f/A_1)\subset S_n^{A_1}$ and $(f/A_2)\subset S_n^{A_2}$, then we obtain a one-one correspondence between the set of all homotopy classes $\subset S_n^{A_1+A_2}$ and the set of all pairs of homotopy classes (f_1), (f_2) with $(f_1)\subset S_n^{A_1}$ and $(f_2)\subset S_n^{A_2}$.*

Proof. Obviously the homotopy class of $f_i=f/A_i$, $i=1,2$, is uniquely determined by the homotopy class of f. On the other hand it is evident that to every pair $f_1\in S_n^{A_1}$ and $f_2\in S_n^{A_2}$ there exists a pair $f_1'\in(f_1)$ and $f_2'\in(f_2)$ such that $f_1'(a)=f_2'(a)$. Putting $f'(x)=f_i'(x)$ for every $x\in A_i$, $i=1,2$, we obtain a mapping $f'\in S_n^{A_1+A_2}$ such that to its homotopy class $(f')\subset S_n^{A_1+A_2}$ correspond the homotopy classes $(f_1)=(f_1')\in S_n^{A_1}$ and $(f_2)=(f_2')\in S_n^{A_2}$. It remains to prove that the homotopy class (f') is uniquely determined by the homotopy classes (f_1) and (f_2), i. e. that if $f_i''\in(f_i)$, $i=1,2$, and $f_1''(a)=f_2''(a)$, then the mapping f'' given by the formula

$$f''(x)=f_i''(x) \quad \text{for } x\in A_i,\ i=1,2$$

is homotopic to f'.

It is easily seen that there exists a mapping $f^* \epsilon (f'')$ such that $f^*(a)=f'(a)$. Hence the partial mappings $f^*/A_i=f_i^*$, $i=1,2$, are homotopic to f_i'', and consequently also to f_i' in $S_n^{A_i}$. Moreover $f_i^*(a)=f_i'(a)$ for $i=1,2$. Applying the lemma 4 we infer that there exists a family $\{g_{i,t}\}\subset S_n^{A_i}$, $0\leqslant t\leqslant 1$ and $i=1,2$, such that $g_{i,0}=f_i'$, $g_{i,1}=f_i^*$, and $g_{i,t}(a)=f_0(a)$ for $0\leqslant t\leqslant 1$ and $i=1,2$. Putting

$$g_t(x)=g_{i,t}(x) \quad \text{for} \quad x \epsilon A_i,\ 0\leqslant t\leqslant 1,\ i=1,2$$

we obtain a family $\{g_t\}$ joining in $S_n^{A_1+A_2}$ the mappings f' and f^*. Hence $(f')=(f^*)$ and consequently also $(f')=(f'')$.

Let us apply the last theorem to the following

Example. Let $a_1,a_2,\ldots,a_m$ be a system of m different points of a segment $L\subset E_{n+1}$, and $\alpha_1,\alpha_2,\ldots,\alpha_m$ a system of positive numbers so small that the $(n+1)$-dimensional elements

$$Q_i=\underset{x}{E}[x \epsilon E_{n+1},\ \varrho(x,a_i)\leqslant\alpha_i], \quad i=1,2,\ldots,m$$

are disjoint. The boundary $S_{n,i}$ of Q_i is a set homeomorphic to S_n. Obviously the set

$$T=L-\sum_{i=1}^{m}Q_i+\sum_{i=1}^{m}S_{n,i}$$

is a continuum. Let us show that if we assign to every homotopy class $(f)\subset S_n^T$ the system of m homotopy classes $(f/S_{n,i})\subset S_n^{S_{n,i}}$, $i=1,2,\ldots,m$, then we obtain a one-one correspondence between the set of all homotopy classes of the mappings $f \epsilon S_n^T$ and the set of all systems of m homotopy classes $(f_i)\subset S_n^{S_{n,i}}$, $i=1,2,\ldots,m$.

If $m=1$, then evidently $S_{n,1}$ is a deformation retract of T and the statement follows by the lemma 5. Assume it now for $m\leqslant k$, and suppose that $m=k+1$. Choosing a point

$$a \epsilon L-\sum_{i=1}^{m}Q_i$$

lying between two neighbouring segments $L\cdot Q_1$ and $L\cdot Q_2$, we can decompose T into two continua T_1 and T_2 having only the point a in common and being of the form similar to T, but containing each $\leqslant k$ spheres. By the inductive hypothesis the homotopy classes of f/T_1 and f/T_2 are in a one-one correspondence with the systems of partial mappings $f/S_{n,i}$. Applying the last theorem we infer that the homotopy class of f is in a one-one correspondence with the systems of the homotopy classes of the partial mappings $f/S_{n,i}$.

5. Mappings of a closed subset of S_{n+1} into S_n.

Let A be a proper closed subset of the $(n+1)$-dimensional sphere S_{n+1}. Let us order all components of $S_{n+1}-A$ in a finite or infinite sequence $\{\Gamma_i\}$ with the index i running through all integers satisfying the inequality $0\leqslant i<\alpha$, where α denotes a positive integer or ∞.

Let us choose in every component Γ_i a point $a_i \epsilon \Gamma_i-(b_1^{n+1})-(b_{-1}^{n+1})$ and a spherical $(n+1)$-dimensional element $K_i\subset\Gamma_i$ with centre a_i, boundary $S_{n,i}$, and interior W_i. Putting

$$P=S_{n+1}-\sum_{i<\alpha}W_i$$

we obtain a compactum containing A.

Theorem. *If $f \epsilon S_n^A$, then:*

1⁰ *There exist extensions $g \epsilon S_n^P$ of f.*

2⁰ *If $g \epsilon S_n^P$ is an extension of f, then except for a finite number of i, $f/S_{n,i}$ is unessential.*

3⁰ *If $g \epsilon S_n^P$ is an extension of f, then the homotopy class $(f)\subset S_n^A$ is uniquely determined by the homotopy classes $(g/S_{n,i})\subset S_n^{S_{n,i}}$, $0<i<\alpha$.*

4⁰ *Given $g_i \epsilon S_n^{S_{n,i}}$ for $0<i<\alpha$ such that almost all g_i are unessential, there exists a mapping $g \epsilon S_n^P$ such that $(g_i)=(g/S_{n,i})$ for $0<i<\alpha$.*

5⁰ *The homotopy class $(g)\subset S_n^P$, where g is an extension of $f \epsilon S_n^A$, is uniquely determined by the homotopy class $(f)\subset S_n^A$.*

Proof. 1⁰ By lemma 3 there exists a finite set $N\subset S_{n+1}-A$ and an extension $\bar{f} \epsilon S_n^{S_{n+1}-N}$ of f. Let Λ denote the finite set of indices i such that $\Gamma_i\cdot N\neq 0$.

By (1) for every $i \epsilon \Lambda$ there exists an $(n+1)$-dimensional element $Q_i\subset\Gamma_i$ containing in its interior the finite set $N\cdot\Gamma_i+(a_i)$. Obviously there exists a mapping r_i retracting $Q_i-(a_i)$ to the boundary B_i of Q_i. It will be sufficient to put

$$f^*(x)=f(x) \quad \text{for every} \quad x \epsilon S_{n+1}-\sum_{i\epsilon\Lambda}Q_i,$$

$$f^*(x)=\bar{f}r_i(x) \quad \text{for every} \quad x \epsilon Q_i-(a_i),\ i \epsilon \Lambda$$

in order to obtain a continuous extension f^* of f onto $S_{n+1}-\sum_{i\epsilon\Lambda}(a_i)\supset P$. The partial mapping $g=f^*/P$ is the required extension of f on P.

2° Applying the above reasoning to a mapping $g \epsilon S_n^P$, instead to f, we infer that

(15) *For every $g \epsilon S_n^P$ there exists a finite set Λ' of indices such that g has a continuous extension $\bar{g}$ onto the set $S_{n+1} - \sum_{i \epsilon \Lambda'} (a_i)$.*

Hence for almost all indices i the mapping $\bar{g}$ is defined on K_i, and consequently the partial mapping $\bar{g}/S_{n,i} = g/S_{n,i}$ is unessential on $S_{n,i}$.

3° It will be convenient to consider S_{n+1} as an Euclidean space E_{n+1} for which the point $a_0 \epsilon \Gamma_0$ plays the role of the point in infinity. As we have already seen (in the Preliminaries) the sets $S_{n,i}$ are geometrical spheres also in E_{n+1}. The sphere $S_{n,0}$ bounds in E_{n+1} a region $V_0 = E_{n+1} - K_0$ containing $A + \sum_{1 \leqslant i < \alpha} W_i$. In E_{n+1} we choose an orthogonal system of coordinates having as its initial point the point a_0^* antipodal on S_{n+1} to a_0. In every region W_i, $1 \leqslant i < \alpha$, let us choose a point a_i' in such a manner that for every two different indices i_1 and i_2 the first coordinate x'_{i_1} of a'_{i_1} is allways different from the first coordinate x'_{i_2} of a'_{i_2}.

Let $g \epsilon S_n^P$ be an extension of $f \epsilon S_n^A$. By (15) there exists a finite set Λ' of natural indices and a continuous extension $\bar{g}$ of g onto the set $\bar{V}_0 - \sum_{i \epsilon \Lambda'} (a_i)$. We can assume that Λ' consists of the indices $1, 2, \ldots, k$ and that for $1 \leqslant i < j \leqslant k$ it is

(16) $$x_i' < x_j'.$$

Consider a positive number ϱ so small that

(17) $$\varrho < \tfrac{1}{2}(x_j' - x_i') \quad \text{for} \quad \text{every } 1 \leqslant i < j \leqslant k$$

and that the sphere $S'_{n,i}$ (in E_{n+1}) with centre a_i' and radius ϱ is contained in W_i. It is easily seen that each of the spheres $S_{n,i}$ and $S'_{n,i}$ is a deformation retract of $K_i - W_i'$, where W_i' is the interior region bounded by $S'_{n,i}$. By the lemma 5 we conclude that the homotopy class of $\bar{g}/S'_{n,i}$ is determined by the homotopy class of $\bar{g}/S_{n,i} = g/S_{n,i}$, for $i = 1, 2, \ldots, k$. Hence to prove 3° it is sufficient to show that the homotopy class of f is determined by the homotopy classes of $\bar{g}/S'_{n,i}$, $i = 1, 2, \ldots, k$.

Let us denote the point of $S_{n,i}$ (for $i = 0, 1, \ldots, k$) with the minimal first coordinate $\underline{x}_i$ by c_i, and that with the maximal first coordinate $\bar{x}_i$ — by d_i. By (16) and (17)

$$\underline{x}_0 < \underline{x}_1 < \bar{x}_1 < \underline{x}_2 < \bar{x}_2 < \ldots < \underline{x}_k < \bar{x}_k < \bar{x}_0.$$

Consider the segments

$$L_0 = \overline{c_0 c_1}; \quad L_1 = \overline{d_1 c_2}; \quad L_2 = \overline{d_2 c_3}; \quad \ldots; \quad L_{k-1} = \overline{d_{k-1} c_k}; \quad L_k = \overline{d_k d_0}.$$

Let us observe that the set

$$T_0 = \sum_{i=0}^{k} L_i + \sum_{i=1}^{k} S_{n,i}$$

is a deformation retract of the set

$$M_0 = \bar{V}_0 - \sum_{i=1}^{k} W_i'.$$

To prove this let us denote by $H(x)$, for every $x \epsilon M_0$, the n-dimensional hyperplane passing through x and perpendicular to x_1-axis. Obviously $H(x)$ has with T_0 either one point (lying on one of the segments L_i) or an $(n-1)$-dimensional sphere (lying on one of the spheres $S_{n,i}$) in common. In both cases there exists exactly one point of $H(x) \cdot T_0$ nearest to x. Denoting it by $\varphi(x)$ we obtain evidently a retraction of M_0 to T_0. It is sufficient to put

$$r(x, t) = t \cdot \varphi(x) + (1 - t) \cdot x \quad \text{for} \quad 0 \leqslant t \leqslant 1$$

in order to obtain a required rectracting deformation of M_0 to T_0.

By the lemma 5 we infer that the homotopy class of $\bar{g}/M_0$ is determined by the homotopy class of $\bar{g}/T_0$. Since $P \subset M_0$, the homotopy class of $g/P = \bar{g}/P$ is also determined by the homotopy class of $\bar{g}/T_0$. But the set T_0 is obviously homeomorphic to the set T investigated as example in section **4**. As it is shown there, the homotopy class of $\bar{g}/T_0$ is determined by the homotopy classes of the mappings $\bar{g}/S'_{n,i}$, $i = 1, 2, \ldots, k$. Thus the proof of 3° is completed.

Remark 1. In the case $A = P$ we infer by 3° that the homotopy classes of $g/S_{n,i}$, $0 < i < \alpha$, determine the homotopy class of $g/S_{n,0}$. If we observe that the role of the sphere $S_{n,0}$ is the same as the role of each of the spheres $S_{n,i}$, $0 < i < \alpha$, then we infer

(18) *For every $g \epsilon S_n^P$ the homotopy class of $g/S_{n,i}$ is determined by the homotopy classes of $g/S_{n,j}$ for $0 \leqslant j < \alpha$, $j \neq i$.*

4° Let g_i be unessential for $i > k$. Keeping the same notations as used in the proof of 3°, consider a continuous extension g_i^* of the mapping g_i on the set $k_i - W_i'$. Let g_i' denote the mapping $g_i^*/S'_{n,i}$. By the example considered in section **4** we infer that there exists a mapping $f \epsilon S_n^{T_0}$ such that $(f/S'_{n,i}) = (g_i^*/S'_{n,i})$ for $i = 1, 2, \ldots, k$.

Putting
$$g^*(x)=fq(x) \quad \text{for every} \quad x \in M_0$$
we obtain an extension $g^* \in S_n^{M_0}$ of f. Let us put $g=g^*/P$. For $0<i\leqslant k$ the mappings $g^*/(K_i-W_i')$ and g_i^* constitute two extensions of g_i'. If we observe that $S_{n,i}$ and $S'_{n,i}$ are retracts by deformation of the set $K_i-W'_i$, we infer (by the lemma 5) that for $i=1,2,\dots,k$ the mappings $g^*/S_{n,i}=g/S_{n,i}$ and $g_i^*/S_{n,i}=g_i$ are of the same homotopy class. For $i>k$ the mapping g^*/K_i constitutes an extension of $g/S_{n,i}$ onto K_i. Hence $g/S_{n,i}$ is unessential, i. e. $(g/S_{n,i})=(g_i)$.

5⁰ Applying 3⁰ to the mapping g which is an extension of f onto the set P we infer that the homotopy class (g) is determined by the homotopy classes $(g/S_{n,i})$, $0<i<\alpha$. Consequently, it suffices to show that every homotopy class $(g/S_{n,i})$ is determined by the homotopy class (f).

Let f_1 and f_2 be two homotopic mappings of A into S_n, and g_1 and g_2 their respective continuous extensions on P. It is to be shown that
$$(g_1/S_{n,i})=(g_2/S_{n,i}) \quad \text{for} \quad 0\leqslant i<\alpha.$$

By (11) there exists an extension $g'_1 \in S_n^P$ of f_1 homotopic to g_2. Consequently it is sufficient to show that
$$(g'_1/S_{n,i})=(g_1/S_{n,i}) \quad \text{for} \quad 0\leqslant i<\alpha.$$

Choosing an index i put
$$g''_i(x)=g_1(x) \quad \text{for every} \quad x \in P-\Gamma_i,$$
$$g''_i(x)=g'_1(x) \quad \text{for every} \quad x \in P\cdot\Gamma_i.$$

Clearly g''_i is also a mapping of P into S_n. But by (18) the homotopy class $(g''_i/S_{n,i})=(g'_1/S_{n,i})$ is determined by the homotopy classes $(g''_i/S_{n,j})=(g_1/S_{n,j})$, for $j\neq i$, and consequently it is identical with $(g_1/S_{n,i})$.

***Remark* 2.** Putting
$$A_i=A+\Gamma_i \quad \text{for} \quad 0<i<\alpha$$
we see at once that if $g/S_{n,i}$ is unessential; then f has an extension $f_i \in S_n^{A_i}$. Conversely, if there exists an extension $f_i \in S_n^{A_i}$, then putting
$$g^*(x)=g(x) \quad \text{for} \quad x \in P-\Gamma_i,$$
$$g^*(x)=f_i(x) \quad \text{for} \quad x \in P\cdot\Gamma_i$$
we obtain an extension $g^* \in S_n^P$ of f such that $g^*/S_{n,i}=f_i/S_{n,i}$ is unessential. By 5⁰ the mapping $g/S_{n,i}$ is also unessential. Hence f has a continuous extension on A_i if and only if $g/S_{n,i}$ is unessential.

Let us call the component Γ_i of the set $S_{n+1}-A$ ***unessential*** for a mapping $f \in S_n^A$ if there exists an extension $f_i \in S_n^{A_i}$ of f and *essential* if such extension does not exist. By 2⁰ there exists only a finite number of essential components for every $f \in S_n^A$. Choosing an arbitrary point in each of them we see by a reasoning used in the proof of 1⁰ that f has a continuous extension on the set obtained from S_{n+1} by the removal of this points. Hence we can formulate the following

Corollary. *For every $f \in S_n^A$ there exists only a finite number of essential components. If N is a subset of S_{n+1} containing at least one point of each of essential components, then there exists a continuous extension of f on the set $S_{n+1}-N$.*

6. Power transformations. Let ϱ, ϑ denote the polar coordinates in the plane E_2 related to the Cartesian coordinates x_1, x_2 by the formulae
$$x_1=\varrho\cdot\cos\vartheta; \quad x_2=\varrho\cdot\sin\vartheta.$$

The first of the polar coordinates ϱ is $\geqslant 0$, the second ϑ is in general defined merely modulo 2π and in the case $\varrho=0$ it is quite arbitrary.

Using the polar coordinates, we shall denote the point $(x_1,x_2,x_3,\dots,x_{n+1}) \in E_{n+1}$ also by $([\varrho,\vartheta];x_3,\dots,x_{n+1})$. Evidently the sphere S_n consists of all points $([\varrho,\vartheta];x_3,\dots,x_{n+1})$ with $\varrho^2+x_3^2+\dots+x_{n+1}^2=1$.

Let m be an integer. By a *power transformation* of S_n with the *exponent* m we understand the mapping $\pi_n^m \in S_n^{S_n}$ defined by the formula
$$\pi_n^m([\varrho,\vartheta];x_3,\dots,x_{n+1})=([\varrho,m\vartheta];x_3,\dots,x_{n+1}).$$

***Remark* 3.** By the homological theory of degree it is evident that the power transformations with different exponents are not homotopic. An elementary proof (without using the homology theory) is easy only in the case $n=1$ [17]).

Theorem. *Every mapping $f \in S_n^{S_n}$ is homotopic to a power transformation* [18]).

[17]) S. Eilenberg, *Sur les espaces multicohérents I*, Fund. Math. **27** (1936), p. 156.

[18]) This statement is commonly known in the general theory of degree. But the proof given here is entirely elementary. See also K. Borsuk, *Drei Sätze über die n-dimensionale Euklidische Sphäre*, Fund. Math. **20** (1933), p. 184.

Proof. If $n=1$, then we can suppose that S_1 is identical with the set of all complex numbers $z=e^{2\pi iu}$ with $0\leqslant u<1$. Let $f\in S_1^{S_1}$. We can assume that $f(1)=1$. Obviously there exists a branch of logarithm such that the real function

$$\varphi(t)=\frac{1}{2\pi i}\log f(e^{2\pi iu}),\quad 0\leqslant u<1$$

is continuous and tends to an integer m when $u\to 1$. Let us put $\varphi(1)=m$ and

$$\psi_t(u)=m\cdot t\cdot u+(1-t)\cdot\varphi(u)\quad\text{for every}\quad 0\leqslant t\leqslant 1,\quad 0\leqslant u\leqslant 1.$$

We get

$$\psi_0(u)=\varphi(u),\quad \psi_1(u)=mu\quad\text{for}\quad 0\leqslant u\leqslant 1,$$

and

$$\psi_t(1)=mt+(1-t)\cdot m=m\quad\text{for}\quad 0\leqslant t\leqslant 1.$$

It follows that putting

$$f_t(e^{2\pi iu})=e^{2\pi i\psi_t(u)}\quad\text{for}\quad 0\leqslant u\leqslant 1,\quad 0\leqslant t\leqslant 1$$

we obtain a family $\{f_t\}\subset S_1^{S_1}$ joining the mapping $f_0=f$ with the mapping f_1, being a power transformation.

Suppose now that $n>1$ and that the statement is true for spheres of dimension $<n$.

Applying (11) to the subset $(b_1^n)+(b_{-1}^n)$ of S_n we infer that for every $f\in S_n^{S_n}$ there exists a homotopic mapping $g\in S_n^{S_n}$ such that

$$g(b_1)=b_1\quad\text{and}\quad g(b_{-1})=b_{-1}.$$

Putting $A=g^{-1}(S_{n-1})$ let us consider the mapping $\varphi=g/A\in S_{n-1}^A$. By the lemma 3 there exists a finite set $N\subset S_n-A$ such that φ has a continuous extension φ' on S_n-N with the values belonging to S_{n-1}. Let us put

$$N_\nu=(b_\nu^n)+N\cdot g^{-1}(R_\nu^n)\quad\text{for}\quad \nu=\pm 1.$$

By (1) there exists an n-dimensional element $Q_1\subset S_n-N_{-1}$ containing in his interior the set N_1 and an n-dimensional element $Q_{-1}\subset S_n-Q_1$ containing in his interior N_{-1}. Evidently there exists a mapping $r_\nu(x)$ retracting the set $Q_\nu-(b_\nu^n)$ to the boundary B_ν of Q_ν. Putting

$$\psi(x)=\varphi'(x)\quad\text{for}\quad x\in S_n-Q_1-Q_{-1},$$
$$\psi(x)=\varphi' r_\nu(x)\quad\text{for}\quad x\in Q_\nu-(b_\nu^n),\ \nu=\pm 1$$

we obtain a mapping of $S_n-(b_1^n)-(b_{-1}^n)$ into S_{n-1}.

Let V_ν denote the region on S_n composed of all points $x\in S_n$ satisfying the inequality $\varrho(x,b_\nu^n)<1$. Putting

$$G_\nu=(Q_\nu-B_\nu)\cdot g^{-1}(R_\nu)\cdot V_\nu,$$
$$A_\nu=S_n-G_1-G_{-1}+(b_\nu^n)$$

let us consider the mapping f_ν defined in the set A_ν as follows:

$$f_\nu(x)=\psi(x)\quad\text{for}\quad x\in S_n-G_1-G_{-1},$$
$$f_\nu(b_\nu^n)=b_\nu^n.$$

By (6) the mapping f_ν has a continuous extension f_ν^* on the set $A_\nu+G_\nu$ with values belonging to $Q_\nu^n=\bar{R}_\nu^n$. Putting

$$f^*(x)=f_\nu^*(x)\quad\text{for}\quad x\in A_\nu+G_\nu,\quad \nu=\pm 1$$

we obtain a mapping $f^*\in S_n^{S_n}$. Let us show that

(19) $$\varrho(f^*(x),g(x))<2\quad\text{for every}\quad x\in S_n.$$

If $x\in A$, then $g(x)=f^*(x)=\psi(x)$. If $x\in A_\nu-A$, then $g(x)\in S_n-S_{n-1}$ and $f^*(x)=\psi(x)\in S_{n-1}$. Hence $\varrho(f^*(x),g(x))<2$. If $x\in S_n-A_1-A_{-1}$, then it is $x\in G_\nu$ for an $\nu=\pm 1$. Hence $g(x)\in R_\nu^n$, $f^*(x)\in\bar{R}_\nu^n$, and consequently $\varrho(f^*(x),g(x))<2$. Thus (19) is proved. We infer by the lemma 1 that f^* is homotopic to g; hence also to f.

By the construction of f^* the compactum

$$S_n-V_1-V_{-1}$$

containing in its interior the sphere S_{n-1}, is mapped by f^* in S_{n-1}. By applying the hypothesis of induction to the $(n-1)$-dimensional sphere S_{n-1} we infer that there exists a family $\{g_t\}\subset S_{n-1}^{S_{n-1}}$ joining the mapping $g_0=f^*/S_{n-1}$ with a power transformation $g_1=\pi_{n-1}^m$. We extend g_t to the boundary B_0 of the set $S_n-V_1-V_{-1}$ putting

$$g_t(x)=f^*(x)\quad\text{for every}\quad x\in B_0\quad\text{and}\quad 0\leqslant t\leqslant 1.$$

Applying (11) to the family $\{g_t\}$ defined in the closed subset $S_{n-1}+B_0$ of $S_n-V_1-V_{-1}$ we conclude that there exists a family $\{g_t^*\}\subset S_{n-1}^{S_n-V_1-V_{-1}}$ joining $g_0^*=f^*/(S_n-V_1-V_{-1})$ with $g_1^*/(S_n-V_1-V_{-1})$. Moreover, if we put

$$g_t^*(x)=f^*(x)\quad\text{for every}\quad x\in V_1+V_{-1}\ \text{and}\ 0\leqslant t\leqslant 1$$

we obtain a family $\{g_t^*\}\subset S_n^{S_n}$ joining f^* with the mapping g_1^*. By the construction $g_1^*/S_{n-1}=g_1=\pi_{n-1}^m=\pi_n^m/S_{n-1}$. Moreover, for every $x\in R_\nu^n$, $\nu=\pm 1$

$$g_1^*(x)\in\bar{R}_\nu^n\quad\text{and}\quad \pi_n^m(x)\in R_\nu^n.$$

Hence $\varrho(g_1^*(x), \pi_n^m(x)) < 2$ for every $x \in S_n$. By the lemma 2 the mapping π_n^m is homotopic to g_1^* and consequently also to f^* and to f; thus the proof is complete.

7. Homotopic multiplication of two homotopy classes. Let φ be a mapping of a compactum X in an arbitrary space Y. An open subset G of X will be called a *fundamental set* for φ if φ is constant on $X-G$. If in the homotopy class $(\varphi)\subset Y^X$ there exists a mapping φ' such that G is fundamental for φ', then G will be called a *fundamental set for the homotopy class* (φ).

Evidently, if G is a fundamental set for the homotopy class (φ) then the mapping $\varphi/(X-G)$ is unessential. In the case where Y is an ANR the inverse is also true, since if $\varphi/(X-G)$ is unessential, then by (11) φ is homotopic in Y^X to a mapping φ' constant on $X-G$. It follows that if $X-G$ is contractible in X, then G is a fundamental set for every homotopy class $(\varphi)\subset Y^X$. In particular

(20) *If Y is an ANR, then every open set $G\neq 0$ of S_n is fundamental for every homotopy class $(\varphi)\subset Y^{S_n}$.*

We shall say that two homotopy classes $(\varphi_1), (\varphi_2)\subset Y^X$ are *multiplicable* if there exist two disjoint open sets G_1 and G_2 such that G_1 is fundamental for (φ_1), and G_2 is fundamental for (φ_2).

Examples.

1. If Y is an ANR, then by (20) every two homotopy classes $(\varphi_1), (\varphi_2)\subset Y^{S_n}$ are multiplicable.

2. If we set for every two mappings $\varphi_1, \varphi_2 \in S_n^X$

$$\psi(x) = (\varphi_1(x), \varphi_2(x)) \quad \text{for} \quad x \in X,$$

then we obtain a mapping ψ of X into the $2n$-dimensional manifold $T = S_n \times S_n$. If we suppose that the dimension of the compactum X is $<2n$, then it is seen at once that ψ is homotopic in T^X to a mapping ψ_0 such that $\psi_0(X)\subset T-(b_{-1}^n, b_{-1}^n)$. By (3) there exists a mapping $r(x,t)$ retracting by deformation the set $T-(b_{-1}^n, b_{-1}^n)$ to the set

$$Z = [(b_1^n)\times S_n] + [S_n \times (b_1^n)].$$

Putting

$$\psi_t(x) = r(\psi_0(x), t) \quad \text{for every} \quad x \in X \text{ and } 0 \leqslant t \leqslant 1$$

we obtain a continuous family $\{\psi_t\}\subset T^X$ joining ψ_0 with $\psi_1 \in Z^X$.

Thus we have shown that every mapping $\psi \in T^X$ is homotopic (in T^X) to a mapping $\psi_1 \in Z^X$. This means that for every two mappings $\varphi_1, \varphi_2 \in S_n^X$ there exist two homotopic mappings $\overline{\varphi}_1, \overline{\varphi}_2 \in S_n^X$ such that for every $x \in X$ either $\overline{\varphi}_1(x) = b_1^n$ or $\overline{\varphi}_2(x) = b_1^n$. It follows that the open subsets of X

$$G_1 = \overline{\varphi}_1^{-1}(S_n - (b_1^n)), \quad G_2 = \overline{\varphi}_2^{-1}(S_n - (b_1^n))$$

are disjoint, and G_1 is fundamental for $\overline{\varphi}_1$ and G_2 for $\overline{\varphi}_2$.

Thus we have shown that

(21) *If X is a compactum of dimension $<2n$, then every two homotopy classes $(\varphi_1), (\varphi_2)\subset S_n^X$ are multiplicable.*

Let X be a compactum and Y a continuum being an ANR. If the homotopy classes $(\varphi_1), (\varphi_2)\subset Y^X$ are multiplicable, then there exist two mappings $\varphi_1^0 \in (\varphi_1)$ and $\varphi_2^0 \in (\varphi_2)$, and two disjoint sets G_1 and G_2 open in X such that φ_1^0 is constant on $X-G_1$ and φ_2^0 constant on $X-G_2$. Let us observe that we can assume that these constant values are equal to an arbitrarily given point $a \in Y$. In fact, if a_i denotes the value of φ_i^0 in $X-G_i$, then there exists a continuous function $a_i(t)$ of the parameter $0\leqslant t\leqslant 1$ with values belonging to Y such that $a_i(0)=a_i$ and $a_i(1)=a$. Putting $f_t(x)=a_i(t)$ for every $0\leqslant t\leqslant 1$ and $x \in X-G_i$ we obtain a family $\{f_t\}\subset Y^{X-G_i}$ joining $\varphi_i^0/(X-G_i)$ with the mapping $f_1(X-G_i)=(a)$. We conclude by (11) that φ_i^0 is homotopic in Y^X with a mapping transforming $X-G_i$ into a.

Hence we can assume that

$$\varphi_i^0(x) = a \qquad \text{for every} \quad x \in X-G_i, \quad i=1,2.$$

Putting

$$\text{(22)} \qquad \begin{aligned} \psi(x) &= \varphi_i^0(x) && \text{for } x \in G_i, \quad i=1,2, \\ \psi(x) &= a && \text{for } x \in X-G_1-G_2 \end{aligned}$$

we obtain a mapping $\psi \in Y^X$. The homotopy class $(\psi)\subset Y^X$ will be called a *homotopic product* of the homotopy classes (φ_1) and (φ_2).

It is obvious by this definition that

(23) *If X_0 is a closed subset of a compactum X, and $(\psi)\subset Y^X$ is a homotopic product of $(\varphi_1), (\varphi_2)\subset Y^X$, then $(\psi/X_0)\subset Y^{X_0}$ is a homotopic product of $(\varphi_1/X_0), (\varphi_2/X_0)\subset Y^{X_0}$.*

We do not assert that a homotopic product exists for every two homotopy classes $(\varphi_1), (\varphi_2)\subset Y^X$, or that it is uniquely determined.

Lemma 6. *Let X be a compactum, and Y an ANR. If $\varphi_1,\varphi_2 \in Y^X$ and there exist two disjoint open subsets G_1, G_2 of X such that*

$$\varphi_1(x)=\varphi_2(x) \quad \textit{for every} \quad x \in X-G_1-G_2,$$

and a closed subset Y_0 of Y contractible in Y and such that

$$\varphi_1(X-G_1)+\varphi_2(X-G_2)\subset Y_0,$$

then putting

$$\varphi(x)=\varphi_1(x) \quad \textit{for} \quad x \in X-G_2,$$
$$\varphi(x)=\varphi_2(x) \quad \textit{for} \quad x \in X-G_1$$

we obtain a mapping $\varphi \in Y^X$; its homotopy class (φ) is a homotopic product of the homotopic classes (φ_1) and (φ_2).

Proof. Let $f(y,t)$ be a mapping contracting Y_0 in X to a point a, i. e. a mapping of $Y_0 \times \langle 0,1\rangle$ into Y such that $f(y,0)\equiv y$ and $f(y,1)\equiv a$. By (11) the mapping $f(y,t)$ has a continuous extension $\bar{f}(y,t)$ on the set $Y\times\langle 0,1\rangle$ with the values lying in Y satisfying the condition

$$\bar{f}(y,0)=y \quad \text{for every} \quad y \in Y.$$

Putting

$$\bar{\varphi}_1(x)=\bar{f}[\varphi_1(x),1];\ \ \bar{\varphi}_2(x)=\bar{f}[\varphi_2(x),1] \quad \text{for every} \quad x\in X$$

we have $\bar{\varphi}_1 \in (\varphi_1)$ and $\bar{\varphi}_2 \in (\varphi_2)$. Since $\bar{\varphi}_1(x)=a$ for every $x \in X-G_1$, and $\bar{\varphi}_2(x)=a$ for every $x\in X-G_2$, it follows that the homotopic classes (φ_1) and (φ_2) are multiplicable. If we put

$$\bar{\varphi}(x)=\bar{\varphi}_1(x) \quad \text{for every} \quad x\in X-G_2,$$
$$\bar{\varphi}(x)=\bar{\varphi}_2(x) \quad \text{for every} \quad x\in X-G_1,$$

then we obtain a mapping $\bar{\varphi} \in Y^X$ such that $(\bar{\varphi})$ is a homotopic product of $(\bar{\varphi}_1)=(\varphi_1)$ and $(\bar{\varphi}_2)=(\varphi_2)$. It remains to show that $(\bar{\varphi})=(\varphi)$. Putting

$$\varphi'_t(x)=\bar{f}[\varphi(x),t] \quad \text{for every} \quad x\in X \text{ and } 0\leqslant t\leqslant 1$$

we obtain a family $\{\varphi'_t\}\subset Y^X$ joining $\varphi'_0=\varphi$ with $\varphi'_1=\bar{\varphi}$.

8. Mappings in S_n. If X is a compactum of dimension $<2n$, then by (21) every two homotopy classes $(\varphi_1),(\varphi_2)\subset S_n^X$ are multiplicable. If $\dim X<2n-1$, then holds the following more exact

Lemma 7[19]. *If $\dim X<2n-1$, then for every two homotopy classes $(\varphi_1),(\varphi_2)\subset S_n^X$ there exists exactly one homotopic product.*

Proof. By (21) it is sufficient to show that any two homotopic products (ψ) and (ψ') of (φ_1) and (φ_2) are identical. Let ψ be given by the formula (22) where G_1, G_2 are two disjoint open subsets of X and φ_i^0 is homotopic with φ_i and $\varphi_i^0(x)=a$ for every $x\in X-G_i$, $i=1,2$. Similarly let G'_1,G'_2 be two disjoint open subsets of X, and φ'_i a mapping homotopic to φ_i and such that $\varphi'_i(x)=a'$ for every $x\in X-G'_i$, $i=1,2$. Let ψ' be given by the formula

$$(24) \qquad \begin{aligned} \psi'(x)&=\varphi'_i(x) &&\text{for} \quad x\in G'_i,\ i=1,2 \\ \psi'(x)&=a' &&\text{for} \quad x\in X-G'_1-G'_2. \end{aligned}$$

Since the mappings φ_i^0 and φ'_i are homotopic, there exists by (10) a family $\{\varphi_{t,i}\}\subset S_n^X$ joining $\varphi_{0,i}=\varphi_i^0$ with $\varphi_{1,i}=\varphi'_i$. Let us denote by T the $2n$-dimensional manifold $S_n\times S_n$. Putting

$$\chi(x,t)=(\varphi_{t,1}(x),\varphi_{t,2}(x)) \quad \text{for every} \quad x\in X \text{ and } 0\leqslant t\leqslant 1$$

we obtain a mapping $\chi\in T^{X\times\langle 0,1\rangle}$. By (22) and (24) the values of $\chi(x,0)$ and $\chi(x,1)$ lie in the set

$$Z=[(b_1^n)\times S_n]+[S_n\times(b_1^n)].$$

Putting

$$\varkappa(b_1^n,y)=\varkappa(y,b_1^n)=y \quad \text{for every} \quad y\in S_n$$

we obtain a mapping $\varkappa\in S_n^Z$ such that

$$(25) \qquad \varkappa\chi(x,0)=\psi(x) \quad \text{and} \quad \varkappa\chi(x,1)=\psi'(x) \quad \text{for every} \quad x\in X.$$

As $\dim(X\times\langle 0,1\rangle)<2n$, it follows that for every $\varepsilon>0$ there exists a mapping $\chi'\in T^{X\times\langle 0,1\rangle}$ such that $\varrho(\chi,\chi')<\varepsilon$ and

$$\chi'(X\times\langle 0,1\rangle)\subset T-(b_{-1}^n,b_{-1}^n).$$

By (3) there exists a retraction r of the set $T-(b_{-1}^n,b_{-1}^n)$ to Z.

[19]) More exactly, the homotopy classes $\subset S_n^X$ constitute an Abelian group with homotopic multiplication group operation. See K. Borsuk, *Sur les groupes des classes de transformations continues*, C. R. de l'Ac. des Sc. **202** (1936), p. 1402 and E. Spanier, l. c. p. 211. For our purposes the more elementary partial statement given here is sufficient.

Putting

$$\chi_t'(x)=r[\chi'(x,t)] \quad \text{for every } x\in X \text{ and } 0\leqslant t\leqslant 1$$

we obtain a family $\{\chi_t'\}$ joining in Z^X the mapping $\chi_0'(x)=r[\chi'(x,0)]$ with the mapping $\chi_1'(x)=r[\chi'(x,1)]$.

But for ε sufficiently small the distance between the mappings $r[\chi'(x,0)]$ and $r[\chi(x,0)]=\chi(x,0)$, and the distance between the mappings $r[\chi'(x,1)]$ and $r[\chi(x,1)]=\chi(x,1)$ is arbitrarily small. By (9) we conclude that the mappings $\chi(x,0)$ and $\chi(x,1)$ are homotopic in Z^X. It follows by (25) that also the mappings ψ and ψ' are homotopic in S_n^X.

Lemma 8. *For every two integers m_1 and m_2 the homotopy class of the power transformation $\pi_n^{m_1+m_2}$ is a homotopic product of the homotopy classes of the power transformations $\pi_n^{m_1}$ and $\pi_n^{m_2}$.*

Proof. Let us define in the set $\langle 0,1\rangle\times\langle 0,2\pi\rangle$ three real continuous functions in the following manner:

If $0\leqslant t\leqslant 1$ and $0\leqslant\vartheta\leqslant\pi$, then:

$$\alpha_{1,t}(\vartheta)=m_1\cdot\vartheta+m_1\cdot\vartheta\cdot t,$$
$$\alpha_{2,t}(\vartheta)=m_2\cdot\vartheta-m_2\cdot\vartheta\cdot t,$$
$$\alpha_{3,t}(\vartheta)=(m_1+m_2)\cdot\vartheta\cdot(1-t)+2m_1\cdot\vartheta\cdot t.$$

If $0\leqslant t\leqslant 1$ and $\pi\leqslant\vartheta\leqslant 2\pi$, then:

$$\alpha_{1,t}(\vartheta)=m_1\cdot\vartheta+m_1\cdot(2\pi-\vartheta)\cdot t,$$
$$\alpha_{2,t}(\vartheta)=m_2\cdot\vartheta-m_2\cdot(2\pi-\vartheta)\cdot t,$$
$$\alpha_{3,t}(\vartheta)=(m_1+m_2)\cdot\vartheta\cdot(1-t)+2(m_1\cdot\pi+m_2\cdot\vartheta-m_2\cdot\pi)\cdot t.$$

Putting for every $x=([\varrho,\vartheta],x_3,\dots,x_{n+1})\in S_n$, where $\varrho\geqslant 0$ and $0\leqslant\vartheta\leqslant 2\pi$, and for $0\leqslant t\leqslant 1$

$$\varphi_{\nu,t}(x)=([\varrho,\alpha_{\nu,t}(\vartheta)];x_3,\dots,x_{n+1}) \quad \text{for } \nu=1,2,3$$

we obtain three families $\{\varphi_{1,t}\}$, $\{\varphi_{2,t}\}$, $\{\varphi_{3,t}\}\subset S_n^{S_n}$. It is easy to see that the first familly joins the power transformation $\varphi_{1,0}=\pi_n^{m_1}$ with the mapping $\varphi_{1,1}$ satisfying the condition

$$\varphi_{1,1}([\varrho,\vartheta];x_3,\dots,x_{n+1})=([\varrho,0];x_3,\dots,x_{n+1}) \quad \text{if } \pi\leqslant\vartheta\leqslant 2\pi.$$

The second family joins the power transformation $\varphi_{2,0}=\pi_n^{m_2}$ with the mapping $\varphi_{2,1}$ satisfying the condition

$$\varphi_{2,1}([\varrho,\vartheta];x_3,\dots,x_{n+1})=([\varrho,0];x_3,\dots,x_{n+1}) \quad \text{if } 0\leqslant\vartheta\leqslant\pi.$$

The third family joins the power transformation $\varphi_{3,0}=\pi_n^{m_1+m_2}$ with the mapping $\varphi_{3,1}$, such that

$$\varphi_{3,1}([\varrho,\vartheta];x_3,\dots,x_{n+1})=\varphi_{1,1}([\varrho,\vartheta];x_3,\dots,x_{n+1}) \quad \text{if } 0\leqslant\vartheta\leqslant\pi,$$
$$\varphi_{3,1}([\varrho,\vartheta];x_3,\dots,x_{n+1})=\varphi_{2,1}([\varrho,\vartheta];x_3,\dots,x_{n+1}) \quad \text{if } \pi\leqslant\vartheta\leqslant 2\pi.$$

By the lemma 6 (where the role of the contractible set Y_0 plays the subset of S_n composed of all points of the form $([\varrho,0];x_3,\dots,x_{n+1})$ we conclude that $(\varphi_{3,1})$ is a homotopic product of $(\varphi_{1,1})$ and $(\varphi_{2,1})$. Hence $(\pi_n^{m_1+m_2})=(\varphi_{3,1})$ is a homotopic product of $(\pi_n^{m_1})=(\varphi_{1,1})$, and $(\pi_n^{m_2})=(\varphi_{2,1})$.

Theorem. *The homotopy classes $(\varphi)\subset S_n^{S_n}$ constitute a cyclic group if by the product we mean the homotopic product. This group contains at least two elements.*

Proof. If $n=1$, the statement follows from the theorem of section **6** lemma 8 and the remark 3. It follows also from this last remark that the group of homotopy classes $\subset S_1^{S_1}$ is infinite.

If $n>1$ then we infer by the lemma 7 that the operation of homotopic product is univalent. Applying the theorem of section **6** we infer that if we assign to every integer m the homotopy class $(\pi_n^m)\subset S_n^{S_n}$, then we obtain a transformation of the group of integers onto the set of all homotopy classes $\subset S_n^{S_n}$. By the lemma 8 this transformation is additive relatively to the homotopic product. We conclude that the homotopy classes $\subset S_n^{S_n}$ form a cyclic group having as the unit element the class (π_n^0), i. e: the class of all unessential mappings. By (8) the class (π_n^1) is different from π_n^0.

The cyclic group of homotopy classes $(\varphi)\subset S_n^{S_n}$ defined in such manner will be denoted by $(S_n^{S_n})$.

9. Structure of the group (S_n^A) in the case $A\subset E_{n+1}$.

Now we shall apply the mappings in S_n and in particular the notion of homotopic product to the theory of the disconnection of the Euclidean space E_{n+1}. We shall confine ourselves to the case of $n>1$, i. e. to the case of the Euclidean space of dimension $\geqslant 3$. The case of E_1 is trivial, the case of E_2, though also accessible for the method used here (slightly modified), can be treated by still more elementary means, as shown by S. Eilenberg[20]).

[20]) S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), p. 61-112.

We begin with the following elementary algebraic remark.

Let $\{J_i\}$ $1 \leqslant i < \alpha$ be a finite or infinite sequence of abelian groups J_i (α denotes a natural number or ∞).

We denote by $\underset{1\leqslant i<\alpha}{\mathbf{P}} J_i$ the so called[21]) *weak product* of J_i, i. e. the Abelian group constituted by all sequences $\{e_i\}$ with $e_i \in J_i$, where $e_i = 0$ for almost all indices i and where the multiplication is defined by the formula

$$\{e_i\} \cdot \{e_i'\} = \{e_i \cdot e_i'\}.$$

Let us observe that if all J_i, are free cyclic groups, then the rank of the group $\underset{1\leqslant i<\alpha}{\mathbf{P}} J_i$ is $\alpha - 1$. If all J_i are cyclic groups of the same finite order $k \neq 1$, then the rank modulo k of the group $\underset{1\leqslant i<\alpha}{\mathbf{P}} J_i$ is $\alpha - 1$.

Thus we have the following

Lemma 9. *If in two weak products $\underset{1\leqslant i<\alpha}{\mathbf{P}} J_i$ and $\underset{1\leqslant i<\alpha'}{\mathbf{P}} J_i'$ all groups J_i and J_i' are cyclic, and either all are free or all of the same finite order $k \neq 0$, then $\underset{1\leqslant i<\alpha}{\mathbf{P}} J_i$ is isomorphic to $\underset{1\leqslant i<\alpha'}{\mathbf{P}} J_i'$ if and only if $\alpha = \alpha'$.*

Now we proceed to the

Theorem. *Let A be a subcompactum of E_{n+1} and let α denote the number (natural or ∞) of components of $E_{n+1} - A$. Then the set of all homotopy classes $(f) \subset S_n^A$ with the homotopic multiplication as operation is an abelian group isomorphic to the weak product $\underset{1\leqslant i<\alpha}{\mathbf{P}} J_i$ where all groups J_i are isomorphic to the group $(S_n^{S_n})$.*

Proof. By the theorem of section **5** there exists a finite or infinite sequence of n-dimensional spheres $\{S_{n,i}\}$, $1 \leqslant i < \alpha$, such that if we assign to every $(f) \subset S_n^A$ the sequence of homotopy classes $\{(f_i)\}$ where $f_i = f/S_{n,i} \in S_n^{S_{n,i}}$, then we obtain a one-one correspondence between the set (S_n^A) of all homotopy classes $(f) \subset S_n^A$ and the set $\{S_n^{S_{n,i}}\}$ of all sequences $\{(f_i)\}$, where $(f_i) \subset S_n^{S_{n,i}}$ and where almost all f_i are unessential. But by the theorem of section **8** the homotopy classes $(f_i) \subset S_n^{S_{n,i}}$ constitute a cyclic group $(S_n^{S_{n,i}})$ isomorphic to the group $(S_n^{S_n})$, and the class of all unessential mappings is its unit element. In this group the homotopic multiplication plays the role of the operation. It follows that the set $\{S_n^{S_{n,i}}\}$ is identical with the weak product $\underset{1\leqslant i<\alpha}{\mathbf{P}} (S_n^{S_{n,i}})$.

Now to prove our theorem it suffices to observe that with regard to (23) the just defined one-one correspondence between the homotopy classes $(f) \subset S_n^A$ and the sequences $\{(f/S_{n,i})\} \in \underset{1\leqslant i<\alpha}{\mathbf{P}} (S_n^{S_{n,i}})$ is such that to the homotopic product of two homotopy classes $(f') \subset S_n^A$ and $(f'') \in S_n^A$ corresponds necessarily the product of the corresponding elements of the group $\underset{1\leqslant i<\alpha}{\mathbf{P}} (S_n^{S_{n,i}})$.

Remark. The group of all homotopy classes $(f) \subset S_n^A$ with group operation defined as homotopic multiplication will be denoted by (S_n^A). Obviously, its definition is purely topological. Hence

(26) *If A and A' are homeomorphic, then the groups (S_n^A) and $(S_n^{A'})$ are isomorphic.*

By the last theorem the structure of the group (S_n^A) is determined by the number α of the components of $E_{n+1} - A$. On the other hand by the lemma 9 the number α is determined by the structure of the group (S_n^A). Applying (26) we obtain the following

Corollary[2]). *If A and A' are two homeomorphic subcompacta of E_{n+1}, then the number of components of $E_{n+1} - A$ is the same as the number of components of $R_{n+1} - A'$.*

Remark. It is commonly known on the base of the theory of degree, that $S_n^{S_n}$ contains an infinite number of components. It follows that $(S_n^{S_n})$ is a free cyclic group. Consequently, if A is a subcompactum of E_{n+1} and α denotes the number of components of $E_{n+1} - A$, then the group (S_n^A) is isomorphic to the weak product of $\alpha - 1$ cyclic free groups. In particular, if α is a positive integer then (S_n^A) is a free Abelian group with $\alpha - 1$ generators.

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[21]) See for instance S. Lefschetz, *Algebraic Topology*, Princeton 1942, p. 47.