Le théorème 2 se trouve ainsi démontré.

Théorème 3. *E étant un ensemble borné situé dans un espace euclidien à $m \geqslant 3$ dimensions tel que $m_e(E) > 0$, il existe pour tout nombre réel μ tel que $0 < \mu < m_e(E)$ un ensemble H (de même espace) qui est équivalent par décomposition finie à E et tel que $m_e(H) = \mu$.*

Démonstration. Vu le théorème 1, il suffira de démontrer que l'ensemble E est équivalent par décomposition finie à un ensemble de mesure extérieure $\leqslant \mu$. Soit Q un cube m-dimensionnel contenant E. D'après Banach et Tarski[3]), comme $m \geqslant 3$, Q équivaut par décomposition finie à un cube m-dimensionnel quelconque, donc, en particulier, à un cube K de mesure (m-dimensionnelle) μ. Comme $E \subset Q$, E équivaut par décomposition finie à une partie H de K et on a évidemment $m_e(H) \leqslant m(K) = \mu$.

Le théorème 3 est ainsi démontré.

Naturellement, dans un espace euclidien à un nombre fini quelconque de dimensions un ensemble équivalent par décomposition finie à un ensemble de mesure nulle est de mesure nulle.

[3]) Voir S. Banach et A. Tarski, Fund. Math. **6** (1924), p. 263, Théorème 24. Le théorème y est énoncé pour $m=3$, mais sa généralisation pour le cas $m \geqslant 3$ n'offre pas de difficulté.

On a Topological Problem Connected with the Cantor-Bernstein Theorem.

By

Casimir Kuratowski (Warszawa).

The purpose of this paper is to define two closed and bounded sets A and B (on the euclidean plane) which are not homeomorphic, although each of them is homeomorphic to a relatively open subset of the other[1]).

1. ***Definition.*** Let us consider the doubly infinite sequence $p_m = 2^{(-2^{-m})}$:

$$\ldots, p_{-3} = \frac{1}{2^8},\quad p_{-2} = \frac{1}{2^4},\quad p_{-1} = \frac{1}{2^2},\quad p_0 = \frac{1}{2},\quad p_1 = \frac{1}{\sqrt{2}},\quad p_2 = \frac{1}{\sqrt[4]{2}},\quad p_3 = \frac{1}{\sqrt[8]{2}}, \ldots$$

Let M_n (where $n = 1, 2, \ldots$) denote the dendrite composed of the segment

$$S_n = \mathop{E}_{xy}\left(x = \frac{1}{n}\right)(0 \leqslant y \leqslant 1)$$

and of the sequence of small dendrites $D_0^n, D_1^n, \ldots$ such that

(i) $\operatorname{Lim}_{k=\infty} D_k^n = \left(\frac{1}{n}, 1\right)$, (ii) $D_k^n \cdot D_m^n = 0$ (if $k \neq m$),

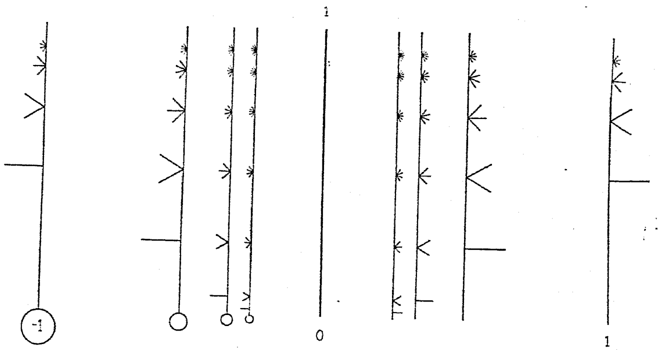
(iii) D_k^n is composed of $k+1$ segments having the point $\left(\frac{1}{n}, p_{k-n+1}\right)$ in common with S_n (this point being a point of M_n of order $k+3$),

(iv) $M_n \cdot M_m = 0$ if $n \neq m$.

[1]) A and B are 1-dimensional sets. The problem raised by Sikorski (Coll. Math. **1** (1947-48), p. 242) of defining two 0-dimensional sets A and B of that kind, is still unsolved.

Consider the dendrite Q_n symmetric to M_n relatively to the Y-axis and add to it a small circumference having but the point $\left(-\frac{1}{n}, 0\right)$ in common with Q_n. Denote the set thus obtained as N_n. Assume further that $N_n \cdot N_m = 0$ if $n \neq m$. Let P denote the segment 01 of the Y-axis and put

$$A = \sum_{n=1}^{\infty} M_n + P + \sum_{n=1}^{\infty} N_n, \qquad B = A - N_1 \quad \text{and} \quad C = B - M_1.$$



Obviously B *is an open subset of* A, *and* C *is an open subset of* B.

2. *The sets* A *and* C *are homeomorphic.*

The desired homeomorphism is defined as follows:

$$h\left(\frac{1}{n}, y\right) = \left(\frac{1}{n+1}, y^2\right) \quad \text{for} \quad n = 1, 2, \ldots,$$
$$h(0, y) = (0, y^2), \quad h(D_k^n) = D_k^{n+1},$$

the definition on the left side of A being symmetric.

It is easy to see that

$$h(S_n) = S_{n+1}, \quad h(P) = P \quad \text{and} \quad h\left(\frac{1}{n}, p_k\right) = \left(\frac{1}{n+1}, p_{k-1}\right).$$

It follows that $h(A) = C$.

3. *The sets* A *and* B *are not homeomorphic.*

Suppose they are. Let f be a homeomorphism of A onto B. It is easy to see that

(1) $f(0,0) = (0,0)$, (2) $f(0,1) = (0,1)$,

and that there exists a *permutation* $j_1, j_2, \ldots$ of the sequence of all positive integers such that $f(M_n) = M_{j_n}$.

Write $m_n = j_n - n$. Clearly

(3) $$f\left(\frac{1}{n}, p_k\right) = \left(\frac{1}{j_n}, p_{k-m_n}\right) \quad \text{for} \quad k > -n,$$

the points $\left(\frac{1}{n}, p_k\right)$ and $\left(\frac{1}{j_n}, p_{k-m_n}\right)$ having the same order.

We prove now that the sequence $\{m_n\}$ is convergent. Let $\{m_{r_n}\}$ be any subsequence of $\{m_n\}$. The continuity of f yields in view of (3) that

(4) $$f\left(0, \frac{1}{2}\right) = \lim_{n=\infty} f\left(\frac{1}{r_n}, \frac{1}{2}\right) = \lim_{n=\infty} \left(\frac{1}{j_{r_n}}, p_{-m_{r_n}}\right) = (0, \lim_{n=\infty} p_{-m_{r_n}}),$$

provided the last limit exists.

Now, this limit exists in the following three cases:

(5) $\lim_{n=\infty} m_{r_n} = \infty$, (6) $\lim_{n=\infty} m_{r_n} = -\infty$,

(7) $\lim_{n=\infty} m_{r_n} = m$ (where $-\infty < m < \infty$).

In cases (5) and (6) we have by (4)

$$f(0, \tfrac{1}{2}) = (0, 0) \quad \text{or} \quad f(0, \tfrac{1}{2}) = (0, 1)$$

respectively. But this is impossible by (1) and (2), f being a one-one correspondence.

Thus the cases (5) and (6) can be eliminated. Hence the sequence $\{m_n\}$ is bounded. Moreover it cannot contain two convergent subsequences with different limits, since (7) and (4) give

$$f(0, \tfrac{1}{2}) = (0, p_{-m}).$$

Thus the sequence $\{m_n\}$ is convergent. Put

$$m = \lim_{n=\infty} m_n.$$

The sequence $\{j_n\}$ being a permutation of the sequence of all positive integers, it follows at once that $m=0$. In other words: we have, for n sufficiently great, $m_n=0$, hence $j_n=n$. Therefore

$$f(0, \tfrac{1}{2})=(0, \tfrac{1}{2}). \tag{8}$$

Applying the same argument to the left side of the set A, it can be shown that there exists a sequence of integers $\{i_n\}$ such that $f(N_n)=N_{i_n}$ and that the sequence $\{i_n-n\}$ is convergent. Let q be its limit. We have then (cf. (4)): $f(0, \tfrac{1}{2})=(0, p_{-q})$ and therefore $q=0$ (by (8)). It readily follows that $\{i_n\}$ is again a permutation of *all* positive integers. But this is impossible since $i_n \geqslant 2$ in view of the definition of B.

Set Theoretical Approach to the Disconnection Theory of the Euclidean Space.

By

Karol Borsuk (Warszawa).

1. Introduction. In 1931 I gave [1]) an elementary proof of the qualitative part of the known theorem of L. E. J. Brouwer [2]) asserting that if a compactum A disconnects the $(n+1)$-dimensional Euclidean space E_{n+1} then so does every subset of E_{n+1} homeomorphic to A. That elementary proof consists in the characterization of the compacta $A \subset E_{n+1}$ which do not disconnect E_{n+1} by the connectivity of the functional space of continuous transformations of A in the n-dimensional Euclidean sphere S_n.

In 1935 S. Eilenberg [3]) showed how the continuous transformations of $A \subset E_2$ allow to prove also the invariance of the number of the regions in which A decomposes the Euclidean plane E_2. In his reasonings S. Eilenberg uses the fact that the continuous transformations of A in S_1 can be multiplied and thus constitute an Abelian group. A similar multiplication for arbitrary continuous transformations of A in S_n is for $n>1$ impossible. However it is possible to define an operation of multiplication *(homotopic multiplication)* for some pairs of homotopy classes (called henceforth ***multiplicatible classes***) and obtain in such a manner a group having as elements the homotopy classes of continuous transformations

---

[1]) K. Borsuk, *Über Schnitte der n-dimensionalen Euklidischen Räume*, Math. Annalen **106** (1932), p. 239-248.

[2]) L. E. J. Brouwer, *Beweis des Jordanschen Satzes für den n-dimensionalen Raum*, Math. Annalen **71** (1912), p. 314.

[3]) S. Eilenberg, *O zastosowaniach topologicznych odwzorowań na okrąg koła*, Wiadomości Matematyczne **41** (1935), p. 1-32. S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), p. 61-112.