Le théorème 2 se trouve ainsi démontré.

**Théorème 3.** Étant un ensemble borné situé dans un espace euclidien à \( m \geq 3 \) dimensions tel que \( m(E) > 0 \), il existe pour tout nombre réel \( \mu \) tel que \( 0 < \mu < m(E) \) un ensemble \( H \) (de même espace) qui est équivalent par décomposition finie à \( E \) et tel que \( m(H) = \mu \).

**Démonstration.** Vu le théorème 1, il suffira de démontrer que l'ensemble \( E \) est équivalent par décomposition finie à un ensemble de mesure extérieure \( \leq \mu \). Soit \( Q \) un cube \( m \)-dimensionnel contenant \( E \). D'après Banach et Tarski\(^3\), comme \( m \geq 3 \), \( Q \) équivaut par décomposition finie à un cube \( m \)-dimensionnel quelconque, donc, en particulier, à un cube \( K \) de mesure \( (m \)-dimensionnelle) \( \mu \). Comme \( E \subset Q \), \( E \) équivaut par décomposition finie à une partie \( H \) de \( K \) et on a évidemment \( m(H) \leq m(K) = \mu \).

Le théorème 3 est ainsi démontré.

Naturellement, dans un espace euclidien à \( m \) nombre fini quelconque de dimensions un ensemble équivalent par décomposition finie à un ensemble de mesure non nulle est de mesure non nulle.

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\(^3\) Voir S. Banach et A. Tarski, Fund. Math. 8 (1924), p. 263, Théorème 34. Le théorème \( \gamma \) est énoncé pour \( m = 3 \), mais sa généralisation pour le cas \( m \geq 3 \) n'offre pas de difficulté.

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On a Topological Problem Connected with the Cantor-Bernstein Theorem.

By

Casimir Kuratowski (Warszawa).

The purpose of this paper is to define two closed and bounded sets \( A \) and \( B \) (on the euclidean plane) which are not homeomorphic, although each of them is homeomorphic to a relatively open subset of the other\(^4\).

1. **Definition.** Let us consider the doubly infinite sequence

\[
p_m = 2^{m^2 - m};
\]

\[
\ldots - \frac{1}{2}, \quad \frac{1}{2}; \quad \frac{1}{3}, \quad - \frac{1}{3}; \quad \frac{1}{4}, \quad - \frac{1}{4}; \quad \frac{1}{5}, \quad - \frac{1}{5}; \quad \ldots
\]

Let \( M_n \) (where \( n = 1, 2, \ldots \)) denote the dendrite composed of the segment

\[
S_n = \bigcap_{x \in \mathbb{R}} \left( x = \frac{1}{n} \right) \quad (0 \leq y \leq 1)
\]

and of the sequence of small dendrites \( D_n^0, D_n^1, \ldots \) such that

(i) \( \lim_{k \to \infty} D_k^0 = \left( \frac{1}{n}, 1 \right) \),

(ii) \( D_k^0 \cdot D_k^m = 0 \) (if \( k \neq m \)),

(iii) \( D_k^0 \) is composed of \( k + 1 \) segments having the point \( \left( \frac{1}{n}, P_{k+1} \right) \) in common with \( S_n \) (this point being a point of \( M_n \) of order \( k + 3 \)),

(iv) \( M_n \cdot M_m = 0 \) if \( n \neq m \).

\(^4\) \( A \) and \( B \) are 1-dimensional sets. The problem raised by Sikorski (Coll. Math. 1 (1947-48), p. 243) of defining two 0-dimensional sets \( A \) and \( B \) of that kind, is still unsolved.
Consider the dendrite $Q_*$ symmetric to $M_*$ relatively to the Y-axis and add to it a small circumference having but the point $\left(-\frac{1}{n},0\right)$ in common with $Q_*$. Denote the set thus obtained as $X_*$. Assume further that $X_*$ is $n - m = 0$ if $n + m$. Let $P$ denote the segment of the Y-axis and put

$$A = \sum_{n=1}^{\infty} M_n + P + \sum_{n=1}^{\infty} X_n, \quad B = A - X_*, \quad C = B - M_*.$$

Obviously $B$ is an open subset of $A$, and $C$ is an open subset of $B$.

2. The sets $A$ and $C$ are homeomorphic.

The desired homeomorphism is defined as follows:

$$h\left(\frac{1}{n}, y\right) = \left(\frac{1}{n + 1}, y^2\right) \quad \text{for} \quad n = 1, 2, \ldots,$$

$$h(0, y) = (0, y^2), \quad h(1_1) = 1_2.$$

the definition on the left side of $A$ being symmetric.

It is easy to see that

$$h(S_n) = S_{n+1}, \quad h(P) = P \quad \text{and} \quad h\left(\frac{1}{n}, y\right) = \left(\frac{1}{n + 1}, y_{n-1}\right).$$

It follows that $h(A) = C$.

3. The sets $A$ and $B$ are not homeomorphic.

Suppose they are. Let $f$ be a homeomorphism of $A$ onto $B$.

It is easy to see that

$$f(0,0) = (0,0), \quad f(0,1) = (0,1),$$

and that there exists a permutation $j_n$ of the sequence of all positive integers such that $f(M_n) = M_{j_n}$.

Write $m = j_n - n$. Clearly

$$f\left(\frac{1}{n}, y\right) = \frac{1}{n+1} P_{k-n}$$

for $k > n$.

the points $\left(\frac{1}{n}, P_k\right)$ and $\left(\frac{1}{n}, P_{k-n}\right)$ having the same order.

We prove now that the sequence $\{m_n\}$ is convergent. Let $\{m_n\}$ be any subsequence of $\{m_n\}$. The continuity of $f$ yields in view of (3) that

$$f\left(0, \frac{1}{2^n}\right) = \lim_{n \to \infty} f\left(\frac{1}{2^n}, \frac{1}{2}\right) = \lim_{n \to \infty} \left(\frac{1}{2^{n+1}}, y_{n}\right) = (0, \lim_{n \to \infty} y_{n})$$

provided the last limit exists.

Now, this limit exists in the following three cases:

$$\lim_{n \to \infty} y_{n} = \infty, \quad \lim_{n \to \infty} y_{n} = -\infty,$$

$$\lim_{n \to \infty} y_{n} = -\infty \quad \text{where} \quad -\infty < m < \infty.$$

In cases (5) and (6) we have by (4)

$$f(0, \frac{1}{2^n}) = (0,0) \quad \text{or} \quad f(0, \frac{1}{2^n}) = (0,1)$$

respectively. But this is impossible by (1) and (2), $f$ being a one-one correspondence.

Thus the cases (5) and (6) can be eliminated. Hence the sequence $\{m_n\}$ is bounded. Moreover it cannot contain two convergent subsequences with different limits, since (7) and (4) give

$$f(0, \frac{1}{2^n}) = (0, P_{-n})$$

Thus the sequence $\{m_n\}$ is convergent. Put

$$m = \lim_{n \to \infty} m_n.$$
The sequence \( \{s_n\} \) being a permutation of the sequence of all positive integers, it follows at once that \( m = 0 \). In other words: we have, for \( n \) sufficiently great, \( m = 0 \), hence \( f(m) = n \). Therefore

\begin{equation}
(8)
f(0, \frac{1}{2}) = (0, \frac{1}{2}).
\end{equation}

Applying the same argument to the left side of the set \( A \), it can be shown that there exists a sequence of integers \( \{t_n\} \) such that \( f(X_n) = Y_n \), and that the sequence \( \{t_n - n\} \) is convergent. Let \( g \) be its limit. We have then (cf. (4)): \( f(0, \frac{1}{2}) = (0, p_{t_n}) \) and therefore \( g = 0 \) (by (8)). It readily follows that \( \{t_n\} \) is again a permutation of all positive integers. But this is impossible since \( t_n \geq 2 \) in view of the definition of \( B \).

Set Theoretical Approach to the Disconnection Theory of the Euclidean Space.

By

Karol Borsuk (Warszawa).

1. Introduction. In 1931 I gave \(^1\) an elementary proof of the qualitative part of the known theorem of L. E. J. Brouwer \(^2\) asserting that if a compactum \( A \) disconnects the \((n+1)\)-dimensional Euclidean space \( E_{n+1} \) then so does every subset of \( E_{n+1} \) homeomorphic to \( A \). That elementary proof consists in the characterization of the continua \( J \subseteq E_{n+1} \), which do not disconnect \( E_{n+1} \) by the connectivity of the functional space of continuous transformations of \( A \) in the \( n \)-dimensional Euclidean sphere \( S_n \).

In 1935 S. Eilenberg \(^3\) showed how the continuous transformations of \( A \subseteq E_2 \) allow to prove also the invariance of the number of the regions in which \( A \) decomposes the Euclidean plane \( E_2 \). In his reasoning S. Eilenberg uses the fact that the continuous transformations of \( A \) in \( S_2 \) can be multiplied and thus constitute an Abelian group. A similar multiplication for arbitrary continuous transformations of \( A \) in \( S_n \) is for \( n \geq 1 \) impossible. However it is possible to define an operation of multiplication (homotopic multiplication) for some pairs of homotopy classes (called henceforth multiplicable classes) and obtain in such a manner a group having as elements the homotopy classes of continuous transformations