recursive function. Hence we may suppose a primitive recursive. We have

$$x \text{ is the Gödel-number of a true statement of } S_3 \equiv (Ey)\,(x=a(y)) \equiv (Ey)\,(x=\beta(y,0)) \equiv (Ey)\,(x=\Phi(m,y,0))$$

for some primitive recursive β and for some m and this is clearly expressible in S_3; hence S_3 can define its own truth.

Q. E. D.

A Proof of the Completeness Theorem of Gödel.

By

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In this paper we shall give a new proof of the following well-known theorem of Gödel[1]):

(*) *If a formula a of the functional calculus is valid in the domain of positive integers, then a is provable.*

Three ideas play an essential part in our proof: Mostowski's algebraic interpretation of a formula a as a functional the values of which belong to a Boolean algebra; Lindenbaum's construction of a Boolean algebra from formulas of the functional calculus; and a theorem on the existence of prime ideals in Boolean algebras, the proof of which is topological and uses the well-known category method.

1. ***The functional calculus.*** By the *functional calculus* (of first order) we understand the system which can be briefly described as follows:

The symbols of the system are: *individual variables* $x_1, x_2, \ldots$; *functional variables* $F_1^k, F_2^k, \ldots$ with k arguments ($k=1,2,\ldots$); and *constants*. The constants are: the negation sign $'$, the disjunction sign $+$, the existential quantifier $\sum_{x_k}$, and the brackets.

$F_j^k(x_{i_1},\ldots,x_{i_k})$ is a (elementary) *formula* of this system; if a and β are formulae, then $a+\beta$, a' and $\sum_{x_k} a$ are also formulae.

[1]) K. Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, Monatshefte für Mathematik und Physik **37** (1930), pp. 349-360. See also D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Band II, Berlin 1939; and L. Henkin, *The completeness of the first-order functional calculus*, Journal of Symbolic Logic **14** (1949), pp. 159-166.

We shall assume that the notion of free and bound individual variable is familiar. The following formulae

A 1. $(\alpha+\alpha)\to\alpha$,

A 2. $\alpha\to(\alpha+\beta)$,

A 3. $(\alpha\to\gamma)\to((\beta+\alpha)\to(\gamma+\beta))$,

where α,β,γ are arbitrary formulae and $\alpha\to\beta$ is the abbreviation for $\alpha'+\beta$, are the *axioms* of the system [2]). The rules of inference are: modus ponens (α and $\alpha\to\beta$ give β), the rule of substitution for free individual variables, and the two well-known rules for $\sum_{x_k}$ (from $\alpha\to\beta$ follows $\sum_{x_k}\alpha\to\beta$, when x_k is not free in β; and from $\sum_{x_k}\alpha\to\beta$ follows $\alpha\to\beta$).

A formula α is said to be *provable* if it can be obtained from the axioms by the above rules of inference.

2. *Tarski's definition of the satisfiability.* The set of all positive integers will always be denoted by I. The symbols $\{f_n\}$, $\{g_n\}$ will denote enumerable sequences of positive integers. $\{K_n^m\}$ will denote a double sequence $(m,n=1,2,3,...)$ of sets such that elements of a set K_n^m are m-element sequences of positive integers.

The definition of the satisfiability is inductive [3]). Two sequences $\{f_n\}$ and $\{K_n^m\}$ satisfy an elementary formula $F_j^k(x_{i_1},...,x_{i_k})$ if $\{f_{i_1},f_{i_2},...,f_{i_k}\}\in K_j^k$. Two sequences $\{f_n\}$ and $\{K_n^m\}$ satisfy a formula α' if they do not satisfy the formula α. Two sequences $\{f_n\}$ and $\{K_n^m\}$ satisfy a formula $\alpha+\beta$ if they satisfy either α or β. Two sequences $\{f_n\}$ and $\{K_n^m\}$ satisfy a formula $\sum_{x_i}\alpha$, if there exists a sequence $\{g_n\}$ such that $\{g_n\}$ and $\{K_n^m\}$ satisfy α and $g_n=f_n$ for $n\neq i$.

A formula α is *satisfiable* if there exist two sequences $\{f_n\}$ and $\{K_n^m\}$ which satisfy α. A formula α is *valid in* I if all sequences $\{f_n\}$, $\{K_n^m\}$ satisfy α.

3. *Mostowski's functionals* Φ_α. We assume the definition of a Boolean algebra B as known. The Boolean sum (join) and the complement of elements $a,b\in B$ will be denoted by $a+b$ and a' respectively. If $a+b=b$, we shall write $a\subset b$. The sum [4]) of elements $a_x\in B$, where x runs through an abstract set X, will be denoted by $\sum_x a_x$ (or, more precisely, by $\sum_{x\in X}a_x$) whenever it exists.

[2]) See H. Rasiowa, *Sur certain système d'axiomes du calcul des propositions*, Norsk Mathematisk Tidsskrift **31** (1949), pp. 1-3.

[3]) A. Tarski, *Pojęcie prawdy w językach nauk dedukcyjnych*, Prace Towarzystwa Naukowego Warszawskiego, Wydział III, 1933, pp. 1-116.

The letter B_0 will always denote the two-element Boolean algebra. The elements of B_0 are 0 and 1.

$\mathfrak{F}^k$ will denote the set of all k-argument functions φ^k (called (I,B_0) *functions*), whose arguments run over I and whose values belong to B_0.

We shall say that

$$\Phi=\Phi(x_{i_1},...,x_{i_n},F_{j_1}^{k_1},...,F_{j_m}^{k_m}) \tag{1}$$

is a (I,B_0) *functional* [5]) if Φ is a function whose values belong to B_0 and which has n arguments x_{i_p} running over I, and m arguments $F_{j_p}^{k_p}$ running over $\mathfrak{F}^{k_p}$ respectively.

Every formula

$$\alpha=\alpha(x_{i_1},...,x_{i_n},F_{j_1}^{k_1},...,F_{j_m}^{k_m}) \tag{2}$$

from the functional calculus with n individual variables x_{i_p} and with m functional variables $F_{i_p}^{k_p}$ can be interpreted [5]) as an (I,B_0) functional if

a) the individual variables x_{i_p} are interpreted as variables running over I;

b) the functional variables $F_{j_p}^{k_p}$ are interpreted as variables running over $\mathfrak{F}^{k_p}$ respectively;

c) the operations $+$, $'$, and $\sum_{x_i}$ are interpreted as the Boolean operations [6]) in B_0.

The (I,B_0) functional obtained in this way from a formula α will be denoted by Φ_α.

4. *An algebraic interpretation of the satisfiability.* The following lemmas establish the relation between the satisfiability and the functionals Φ_α.

[4]) An element $a\in B$ is said to be the *sum* of elements a_x $(x\in X)$ provided that 1° $a_x\subset a$ for every $x\in X$, and 2° if $a_x\subset b\in B$ for every $x\in X$, then $a\subset b$.

[5]) See A. Mostowski, *Proofs of non-deducibility in intuitionistic functional calculus*, The Journal of Symbolic Logic **13** (1948), pp. 204-207.

[6]) Obviously $\sum_{x_i}$ is then interpreted as the symbol of Boolean sum $\sum_{x_i\in I}$.

(i) *For every formula α from the functional calculus, the (I, B_0) functional Φ_α assumes the value* 1 (0) *if and only if α (α') is satisfiable.*

If K^m is a set of m-element sequences of positive integers, then $c_{K^m} \epsilon \mathfrak{F}^m$ will denote the characteristic function of K^m, i. e.

$$c_{K^m}(\mathfrak{s}) = 1 \epsilon B_0 \quad \text{if} \quad \mathfrak{s} \epsilon K^m, \quad \text{and} \quad c_{K^m}(\mathfrak{s}) = 0 \epsilon B_0 \quad \text{if} \quad \mathfrak{s} \text{ non } \epsilon K^m.$$

It can be easily proved by induction that two sequences $\{f_n\}$ and $\{K_n^m\}$ satisfy a formula α of the form (2) if and only if

$$\Phi_\alpha(f_{i_1}, \ldots, f_{i_n}, c_{K_{j_1}^{k_1}}, \ldots, c_{K_{j_m}^{k_m}}) = 1 \epsilon B_0.$$

This proves (i). It follows directly from (i) that

(ii) *A formula α is valid in the set I of all positive integers if and only if the (I, B_0) functional Φ_α is identically equal to $1 \epsilon B_0$.*

5. *A theorem on the existence of prime ideals in Boolean algebras.* Let B be a Boolean algebra. The set of all prime ideals of B will be denoted by $\mathcal{S}$. For every $a \epsilon B$ let $S(a)$ denote the set of all prime ideals $\mathfrak{p}$ of B such that a non $\epsilon\, \mathfrak{p}$, and let $\mathfrak{S}$ be the class of all sets $S(a)$ where $a \epsilon B$. By definition:

(3) $$\mathfrak{p} \epsilon S(a) \quad \text{if and only if} \quad a \text{ non } \epsilon\, \mathfrak{p}.$$

We shall consider the set $\mathcal{S}$ as a topological space with $\mathfrak{S}$ as the class of neighbourhoods. As Stone[7]) has proved, $\mathcal{S}$ is a totally disconnected bicompact Hausdorff space, and the mapping $S = S(a)$ is an isomorphism of B on the field $\mathfrak{S}$ of all both open and closed subsets of $\mathcal{S}$.

Let $\mathfrak{p}$ be a prime ideal of B. Then the quotient algebra $B/\mathfrak{p}$ is the two-element Boolean algebra. The element of $B/\mathfrak{p}$ which is determined by an element $a \epsilon B$ will be denoted by $[a]$[8]). By definition:

(4) $$[a] = 1 \epsilon B/\mathfrak{p} \quad \text{if} \quad a \text{ non } \epsilon\, \mathfrak{p}; \quad [a] = 0 \epsilon B/\mathfrak{p} \quad \text{if} \quad a \epsilon\, \mathfrak{p}.$$

Suppose an element $a \epsilon B$ is the sum of a class of elements $a_x \epsilon B$ where x runs over an abstract set X. In symbols:

(5) $$a = \sum_{x \epsilon X} a_x \quad \text{in } B.$$

[7]) M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Am. Math. Soc. **41** (1937), pp. 375-481. See p. 378.

[8]) The Boolean operations in $B/\mathfrak{p}$ are defined by the equalities:

$$[a] + [b] = [a+b], \quad [a]' = [b'].$$

We shall say that the ideal $\mathfrak{p}$ *preserves* the sum (5) if

$$[a] = \sum_{x \epsilon X} [a_x] \quad \text{in } B/\mathfrak{p}.$$

Obviously the ideal $\mathfrak{p}$ does not preserve the sum (5) if and only if $[a] = 1$ and $[a_x] = 0$ for every $x \epsilon X$, i. e., on account of (4) and (3), if

$$\mathfrak{p} \epsilon S(a) - \sum_{x \epsilon X} S(a_x).$$

Hence,

(iii) *The set of all prime ideals which do not preserve a sum* (5) *is nowhere dense in the space $\mathcal{S}$.*

In fact, the set $S(a) - \sum_{x \epsilon X} S(a_x)$ is closed. Suppose its interior is not empty. Then there exists an element $a_0 \neq 0$ ($a_0 \epsilon B$) such that $S(a_0) \subset S(a) - \sum_{x \epsilon X} S(a_x)$, i. e. $S(a_x) \subset S(a) - S(a_0) = S(a - a_0)$ for every $x \epsilon X$. The mapping $S = S(b)$ being an isomorphism, we infer that $a_x \subset a - a_0 \neq a$ in contradiction[9]) to (5).

(iv) *Let a_0, a_n, $a_{n,x}$ ($x \epsilon X_n$ where X_n is an arbitrary abstract set, $n = 1, 2, \ldots$) be elements of B such that*

a) $a_n = \sum_{x \epsilon X_n} a_{n,x}$ *in* B $(n = 1, 2, \ldots)$;

b) a_0 *is not the unit of* B.

Then there exists a prime ideal $\mathfrak{p}$ preserving all the sums a) *and such that $a_0 \epsilon\, \mathfrak{p}$.*

Let P be the set of all prime ideals which preserve all the sums a). By (iii) the set $\mathcal{S} - P$ is of the first category in $\mathcal{S}$. The space $\mathcal{S}$ being bicompact, we infer that the set P is dense in $\mathcal{S}$. By b) the open set $S(a_0') = \mathcal{S} - S(a_0)$ is not empty. Consequently $P \cdot S(a_0') \neq 0$. Every prime ideal $\mathfrak{p} \epsilon P \cdot S(a_0')$ satisfies the thesis of the theorem (iv).

6. *Lindenbaum's algebra* B^*. Two formulae β, γ (from the functional calculus) are said to be *equivalent* if the formulae $\beta \rightarrow \gamma$ and $\gamma \rightarrow \beta$ are provable. The class of all formulae γ equivalent to a formula β will be denoted by $E(\beta)$.

The set of all classes $E(\alpha)$ is a Boolean algebra denoted by B^*. The Boolean operations in B^* are defined by the equalities:

(6) $$E(\beta) + E(\gamma) = E(\beta + \gamma);$$

(7) $$E(\beta)' = E(\beta').$$

[9]) See footnote 4.

It is easy to show that

(v) *$E(\beta)\subset E(\gamma)$ if and only if $\beta\to\gamma$ is provable,*
(vi) *The unit of B^* is the class of all provable formulae.*

For every formula β, let $\beta\binom{x_p}{x_k}$ be the formula which we obtain from β in the following way:

We choose an integer l such that β contains neither the individual variable x_l nor the quantifier $\sum\limits_{x_l}$. We replace every bound variable x_p by the variable x_l, and every quantifier $\sum\limits_{x_p}$ by $\sum\limits_{x_l}$. Further, we replace every free variable x_k by x_p.

The formula $\beta\binom{x_p}{x_k}$ defined in such a way is not uniquely determined; but the element $E\left(\beta\binom{x_p}{x_k}\right)\in B^*$ is uniquely determined, since it does not depend on the choice of the integer l.

Using this notation we shall demonstrate that

(vii) *For every formula β,*

(8) $$\sum_{p\in I} E\left(\beta\binom{x_p}{x_k}\right)=E\left(\sum_{x_k}\beta\right).$$

In fact, the provable formula $\beta\binom{x_p}{x_k}\to\sum\limits_{x_k}\beta$ implies by (v) that

$$E\left(\beta\binom{x_p}{x_k}\right)\subset E\left(\sum_{x_k}\beta\right)\quad\text{for}\quad p=1,2,\ldots$$

Suppose a formula γ satisfies the inclusion

$$E\left(\beta\binom{x_p}{x_k}\right)\subset E(\gamma)\quad\text{for}\quad p=1,2,\ldots$$

The formula

(9) $$\beta\binom{x_p}{x_k}\to\gamma$$

is thus provable for $p=1,2,\ldots$ (see (v)). Let q be an integer such that x_q is not free in γ. Then, by (9), the formula $\sum\limits_{x_q}\beta\binom{x_q}{x_k}\to\gamma$ is also provable; hence, by (v),

$$E\left(\sum_{x_k}\beta\right)=E\left(\sum_{x_q}\beta\binom{x_q}{x_k}\right)\subset E(\gamma),$$

which proves (vii).

7. *The proof of Gödel's theorem.* By (ii), in order to prove Gödel's theorem (*), it is sufficient to show that

(**) *If a formula α is not provable, then the (I,B_0) functional Φ_α assumes the value 0 (the zero element of B_0).*

Suppose the formula α is not provable. Let $\mathfrak{p}^*$ be a prime ideal of B^* preserving all sums (8) and such that $E(\alpha)\in\mathfrak{p}^*$. The existence of such an ideal follows from (iv), (vi), and from the fact that the set of all sums of the form (8) is enumerable[10].

Then $B_0=B^*/\mathfrak{p}^*$ is the two-element Boolean algebra, and

(10) $$[E(\alpha)]=0 \qquad \text{(since } E(\alpha)\in\mathfrak{p}^*);$$
(11) $$[E(\beta)]+[E(\gamma)]=[E(\beta+\gamma)] \qquad \text{(by (6))[11]};$$
(12) $$[E(\beta)]'=[E(\beta')] \qquad \text{(by (7))[11]};$$
(13) $$\sum_{p\in I}\left[E\left(\beta\binom{x_p}{x_k}\right)\right]=\left[E\left(\sum_{x_k}\beta\right)\right]$$

(on account of (vii), since $\mathfrak{p}^*$ preserves all the sums (8)).

Let $\varphi_j^k\in\mathfrak{F}^k$ (for $k,j=1,2,\ldots$) be an (I,B_0) function defined by the equality

(14) $$\varphi_j^k(p_1,p_2,\ldots,p_k)=[E(F_j^k(x_{p_1},x_{p_2},\ldots,x_{p_k}))],$$

where $\{p_1,p_2,\ldots,p_k\}$ is any k-element sequence of positive integers.

Let Φ_β^0 denote (for each formula β) the value of the (I,B_0) functional Φ_β for the following values of its arguments:

$$x_i=i\quad\text{and}\quad F_j^k=\varphi_j^k.$$

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[10]) In the case of the Boolean algebra B^*, the space $\mathcal{S}$ constructed in § 5 is Cantor's discontinuous set.

[11]) See footnote 8.

The equalities (11-14) imply that $\Phi_\beta^0 = [E(\beta)]$ for every formula β. The easy proof (by induction on the length of β) is omitted.

In particular $\Phi_\alpha^0 = [E(\alpha)] = 0 \in B_0$ by (10), which proves (**).

8. *Generalizations.* By the same method Gödel's theorem can be proved for the functional calculus with the sign of equality $=$. The axioms of this systems are the axioms A 1-3 and

A 4. $x_k = x_k$.

A 5. $(x_k = x_l) \to \left(\alpha \to \alpha\binom{x_k}{x_l}\right)$.

The algebraic interpretation of the formula $x_k = x_l$ is $\psi(x_k, x_l)$ where $\psi \in \mathfrak{F}^2$ is an (I, B_0) function defined by the conditions:

$$\psi(m,n) = 1 \in B_0 \quad \text{if} \quad m = n; \qquad \psi(m,n) = 0 \in B_0 \quad \text{if} \quad m \neq n.$$

A method similar to that of our proof may be used for the two-valued sentential calculus.

Note also that the condition that I is the set of all positive integers is not essential in sections 2, 3 and 4. I may be an arbitrary non-void abstract set.

Le dernier théorème de Fermat pour les nombres ordinaux.

Par

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Le dernier théorème de Fermat n'est pas vrai pour les nombres ordinaux. En effet, on a le

Théorème 1. *Quel que soit le nombre ordinal μ, il existe trois nombres ordinaux distincts α, β et γ dont chacun est plus grand que μ et tels qu'on a*

$$\alpha^n + \beta^n = \gamma^n \quad \textit{pour} \quad n = 1, 2, 3 \ldots \tag{1}$$

Le théorème 1 est une conséquence immédiate de la formule

$$(\omega^\xi)^n + (\omega^\xi \cdot 2)^n = (\omega^\xi \cdot 3)^n \tag{2}$$

qui, comme on le vérifie sans peine, est vraie pour tout nombre ordinal $\xi > 0$ et pour tout nombre naturel n (pour le voir, il suffit de remarquer qu'on a pour tout nombre ordinal positif ξ et pour k et n naturels $(\omega^\xi k)^n = \omega^{\xi n} k$).

Les termes à gauche de la formule (2) sont commutables; si l'on voulait avoir des termes non commutables, on pourrait remplacer la formule (2) par la formule

$$(\Omega^\xi \omega)^n + (\Omega^\xi)^n = (\Omega^\xi(\omega + 1))^n$$

qui a lieu pour tout nombre ordinal $\xi > 0$ et pour tout nombre naturel n.

Citons encore sans démonstration les solutions suivantes de l'équation (1) pour n naturel donné (où les nombres ordinaux α, β et γ dépendent de n et dont on ne peut pas déduire le théorème 1):

$$(\omega^{n+1})^n + (\omega^n)^n = (\omega^{n+1} + \omega)^n \quad \text{pour} \quad n = 1, 2, 3, \ldots$$

et

$$[\lambda(\lambda+1)^{n-1}]^n + [(\lambda+1)^{n-1}]^n = [(\lambda+1)^n]^n$$

quel que soit le nombre naturel n et le nombre ordinal λ de deuxième espèce.