A system which can define its own truth.

By

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Tarski has shown 1) that for a certain class of logical systems $S$, the following holds:

It is impossible to define in $S$ the class of Gödel-numbers of true statements of $S$.

The essence of his proof consists of the following version of the Epimenides. Let $Fmla$ be the class of Gödel-numbers of meaningful statements of $S$; then we can define the class of false statements of $S$ as follows

$$x \in Fals \iff x \in Fmla \cdot \sim x \in True$$

where $True$ is the class of Gödel-numbers of true statements of $S$.

Let $Nom(y, z)$ say that $y$ is the Gödel-number of the numeral designating $x$, and let $Subst(z, y, z)$ say that $z$ is the Gödel-number of the result of writing the expression whose Gödel-number is $y$ for all free occurrences of $\alpha$ in the expression whose Gödel-number $z$ $x$. Let $n$ be the numeral designating the Gödel-number of

Ep. 1 $(Ey)(Ex)(Nom(y, n) \cdot Subst(z, y, n) \cdot x \in Fals)$.

Then the formula

Ep. 2 $(Ey)(Ex)(Nom(y, n) \cdot Subst(z, y, n) \cdot x \in Fals)$

says that the result of writing $n$ for all free occurrences of $\alpha$ in Ep. 1 is false. But this result is Ep. 2 itself; i.e. Ep. 2 affirms its own falsehood, an evident contradiction 2).

1) A. Tarski, *Pojecie prawdy w językach nauk dedukcyjnych*, Warszawa, 1933.

2) We have $(Ey)(Ex)(Nom(y, n) \cdot Subst(z, y, n) \cdot x \in Fals)$ (Ep. 2 is true); but by Tarski's schema for truth (see Tarski, op. cit.), also $(Ey)(Ex)(Nom(y, n) \cdot Subst(z, y, n) \cdot x \in Fals)\Rightarrow (Ep. 2 \text{ is true})$; the contradiction follows by the theory of deduction.

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It is obvious that this proof of Tarski's depends upon $S$'s containing a certain amount of conceptual apparatus; in particular it depends upon $S$'s containing negation. The question has hitherto remained undecided, whether any system, even without negation, can define its own truth. The purpose of this paper is to answer this question in the affirmative.

Róza Péter 3) has constructed a number-theoretic function $\Phi$ such that for every primitive recursive function $f$ of two arguments there is a number $x$ such that

$$f(y, x) = \Phi(x, y, x)$$

for all $y$ and $x$. Further, it is evident from the definition of this function that it is general recursive.

Let $S\alpha$ be a system consisting of the recursion equations for $\Phi$ and everything which can be deduced from them by the use of extensionality and substitution of constants for variables.

Let $S\beta$ be the class of all formulæ of $S\alpha$ which contain no free variables.

Let $S\gamma$ be a system consisting of all formulæ of $S\beta$ and everything which can be obtained from them by means of the rule:

From $\ldots \alpha \ldots$, where $\alpha$ is a numeral, infer $(Ex)(\ldots \varphi \ldots)$, where $\varphi$ is a variable not occurring in $\ldots \alpha \ldots$.

We shall show that $S\gamma$ can define its own truth.

It is evident that $S\gamma$ is a system, i.e. that the class of Gödel-numbers of theorems of $S\gamma$ forms the range of values of a general recursive function, say $\sigma$. Further $S\beta$, and hence $S\gamma$, is clearly complete and consistent, in the sense that all true formulæ expressible in the notation of $S\beta$ and $S\gamma$ and no others, are provable in $S\beta$ and $S\gamma$ respectively. Hence the class of true statements of $S\beta$ coincides with the class of theorems of $S\gamma$.

Rosser 4) has shown that the range of values of every general recursive function coincides with the range of values of some primitive


recursive function. Hence we may suppose a primitive recursive. We have

\[ \pi \text{ is the Gödel-number of a true statement of } S_4 = (E \varphi (x = \varphi(y))) = (E \varphi (x = \beta(y,0))) = (E \varphi (x = \sigma(m,y,0))) \]

for some primitive recursive \( \beta \) and for some \( m \) and this is clearly expressible in \( S_4 \) hence \( S_4 \) can define its own truth.

Q. E. D.

A Proof of the Completeness Theorem of Gödel.

By H. Rasiowa (Warszawa) and R. Sikorski (Warszawa).

In this paper we shall give a new proof of the following well-known theorem of Gödel\(^1\):

(*) If a formula \( \alpha \) of the functional calculus is valid in the domain of positive integers, then \( \alpha \) is provable.

Three ideas play an essential part in our proof: Mostowski's algebraic interpretation of a formula \( \alpha \) as a functional the values of which belong to a Boolean algebra; Lindenbaum's construction of a Boolean algebra from formulas of the functional calculus; and a theorem on the existence of prime ideals in Boolean algebras, the proof of which is topological and uses the well-known category method.

1. The functional calculus. By the functional calculus (of first order) we understand the system which can be briefly described as follows:

The symbols of the system are: **individual variables** \( x_1, x_2, \ldots \); **functional variables** \( F_1, F_2, \ldots \) with \( k \) arguments \( (k = 1, 2, \ldots) \); and **constants**. The constants are: the negation sign \( \neg \); the disjunction sign \( \lor \); the existential quantifier \( \exists \); and the brackets.

\[ F^0_1(x_1, \ldots, x_k) \] is a (elementary) formula of this system; if \( \alpha \) and \( \beta \) are formulae, then \( \alpha + \beta, \alpha' \) and \( \exists \alpha \) are also formulae.