locally contractible. Consequently P_∞ is also an example of an irreducible 2-dimensional locally contractible compactum. Every locally contractible closed proper subset of P_∞ is of the dimension $\leqslant 1$.

If we submit the space E_3 to a transformation consisting in the identification of all points of the set R, we obtain the space E_3^* homeomorphic to E_3, and the image P_∞^* of P_∞ is a locally connected compactum cutting E_3^* into two regions Γ_∞^* and Λ_∞^* and being their common boundary. It is easy to see that P_∞^* is an absolute neighborhood retract being a closed Cantor-surface and containing no 2-dimensional absolute retract.

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Linear functionals on spaces of continuous functions.

By

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1. Introduction. The present paper is concerned with the problems of classifying, representing, and approximating to linear functionals defined on spaces of real-valued continuous functions. Let X be any topological space; let $\mathfrak{C}(X,R)$ denote the set of all continuous real-valued functions defined on X; let $\mathfrak{C}^*(X,R)$ denote the set of all bounded functions in $\mathfrak{C}(X,R)$. We shall denote the real numbers throughout the present paper by the symbol R. A real-valued function I defined on $\mathfrak{C}(X,R)$ (or $\mathfrak{C}^*(X,R)$) is said to be a linear functional if $I(\alpha f+\beta g)=\alpha I(f)+\beta I(g)$ for all $f, g \in \mathfrak{C}(X,R)$ (or $\mathfrak{C}^*(X,R)$) and all $\alpha, \beta \in R$. We employ the usual definitions of sum, scalar multiplication, product, and positivity in $\mathfrak{C}(X,R)$ and $\mathfrak{C}^*(X,R)$. A linear functional I is said to be positive if it is not the zero-functional and if it is non-negative for positive functions. A linear functional is said to be bounded if it carries bounded sets of functions into bounded sets of real numbers.

In $\mathfrak{C}(X,R)$, there are at least four interesting topologies. They have been widely studied, and are described, for example, in [7], pp. 48-49. It is of some interest to consider the linear functionals on $\mathfrak{C}(X,R)$ which are continuous under these four topologies for $\mathfrak{C}(X,R)$. We shall say that a linear functional is p-, k-, u-, or m-continuous if it is a continuous mapping of $\mathfrak{C}(X,R)$ into R under the p-, k-, u-, or m-topology, respectively.

Representation of linear functionals by means of integrals, which forms the central theme of the present paper, has been studied by a number of writers during the past four decades. (We limit ourselves to linear functionals defined on spaces of continuous real-valued functions). The first theorem of this kind appears to be due to F. Riesz [15], who showed that any u-continuous linear functional on the space $\mathfrak{C}([0,1],R)$ [1]) can be represented as $\int_0^1 f(x)\,d\alpha(x)$, where $\alpha(x)$ is a function of bounded variation on $[0,1]$ and the integral is the ordinary Riemann-Stieltjes integral. Riesz's result was extended by J. Radon [14] to arbitrary closed bounded subsets of n-dimensional Euclidean space.

A further generalization has been made by S. Kakutani. (See [9], pp. 1008-1009). Let X be a compact Hausdorff space (throughout the present paper, we use the term *compact* to mean the same as *bicompact* in the sense of [2]). Let I be any positive linear functional on the space $\mathfrak{C}(X,R)$ such that $I(1)=1$. Then there exists a Carathéodory outer measure σ on X for which $\sigma(X)=1$ and for which all Borel sets are measurable, such that $I(f)=\int_X f(x)\,d\sigma$ for all $f\in\mathfrak{C}(X,R)$.

A. Markov [13] and A. D. Aleksandrov [1] have written on integral representations of linear functionals over spaces of functions which are defined on sets of various kinds. Their work deals only with bounded functions and may be reduced in part to the work of other writers.

Studies involving unbounded continuous functions appear to be very rare. We note the theorem of Mackey [12] and the remarks of Wehausen [21] and Sirvint [17]. We shall discuss the results obtained by these writers in connection with our own work in §§ 3 and 4.

The aim of the present paper is to obtain the best possible representation theorem for linear functionals on spaces $\mathfrak{C}(X,R)$, where we exclude any requirement of compactness in the space X, and accordingly deal with spaces $\mathfrak{C}(X,R)$ which may contain unbounded functions in great profusion. The absence of compactness necessitates certain modifications in the constructions employed and in the final representation theorem. Furthermore, in the case of compact Hausdorff spaces, the k-, u-, and m-topologies for $\mathfrak{C}(X,R)$ coincide, so that questions of continuity in the four senses described above can arise only in the absence of compactness in X. Throughout

[1]) The symbol $[\alpha,\beta]$ denotes the interval $\alpha\leqslant x\leqslant\beta$ in R; $(\alpha,\beta]$ the interval $\alpha<x\leqslant\beta$; $[\alpha,\beta)$ and (α,β) similarly.

most of the paper, we shall limit ourselves to the consideration of completely regular topological spaces (which we denote by the term CR-spaces), since, as we shall show, the representation problem in the case of more general spaces can easily be reduced to the corresponding problem for CR-spaces [2]).

In dealing with measures, we shall employ the following notation and definitions. For an arbitrary topological space X and an arbitrary function $\varphi\in\mathfrak{C}(X,R)$, let

$$P(\varphi)=E[x;x\in X,\varphi(x)>0] \quad\text{and}\quad Z(\varphi)=E[x;x\in X,\varphi(x)=0].$$

Let $\mathscr{P}(X)$ denote the family of all subsets $P(\varphi)$ in X and $\mathscr{Z}(X)$ the family of all subsets $Z(\varphi)$, where φ runs through all members of $\mathfrak{C}(X,R)$. The smallest family containing $\mathscr{P}(X)$ and closed under the formation of complements and of countable unions is called the family of Baire sets in X and is denoted by the symbol $\overline{\mathscr{P}}(X)$. The analogous family $\overline{\mathscr{O}}(X)$, obtained by starting with the family $\mathscr{O}(X)$ of open subsets of X, is called the family of Borel sets in X. If X is a normal space, $\mathscr{P}(X)$ consists exactly of the open F_σ's in X. We note that the relation $\mathscr{P}(X)\subset\mathscr{O}(X)\neq P(X)$ obtains for a large class of topological spaces X.

By a *Baire measure*, we shall mean a non-infinite real-valued function γ defined on the family $\overline{\mathscr{P}}(X)$ such that $\gamma(\sum_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\gamma(A_n)$ for every family $\{A_n\}_{n=1}^{\infty}\subset\overline{\mathscr{P}}(X)$ such that $A_n\cap A_m=0$ for all $n\neq m$. The term *Borel measure* is defined analogously. The theory of integrals for functions which are Baire or Borel measurable is well-known. (See, for example, [16], Chapters I and II).

Let p be any point of the topological space X and let μ_p be the measure function which assigns the value 1 to all sets containing p and the value 0 to all sets not containing p. The symbol $\sum_{i=1}^{m}\alpha_i\mu_{p_i}$ has an obvious meaning ($\alpha_i\in R$), as does $\sum_{i=1}^{\infty}\alpha_i\mu_{p_i}$ ($\alpha_i\in R$).

If p is any point in the topological space X, let M_p be the ring-homomorphism of $\mathfrak{C}(X,R)$ which carries $f\in\mathfrak{C}(X,R)$ into $f(p)$, for all $f\in\mathfrak{C}(X,R)$.

Finally, we shall denote by the symbol N_α the discrete space of cardinal number $\aleph_\alpha$.

[2]) The writer's thanks are due to Professors Paul R. Halmos and John C. Oxtoby for discussions concerning the subject matter of the present paper.

2. Bounded functionals and their representations. It is well-known that any bounded linear functional on $\mathfrak{C}(X,R)$ can be written as the difference of two non-negative linear functionals (see, for example, [5], p. 115, Theorem 7.19). We therefore restrict ourselves throughout the present section to the consideration of positive linear functionals. We first make a preliminary remark.

Theorem 1. *Let X be any CR-space and let I be a non-negative linear functional on $\mathfrak{C}(X,R)$. If I vanishes for all functions in $\mathfrak{C}^*(X,R)$ then I vanishes identically.*

If X admits no unbounded real-valued continuous functions, then there is nothing to prove. Thus, let f be an unbounded, positive function in $\mathfrak{C}(X,R)$. There is no loss of generality in assuming that $\inf_{x \in X} f(x)=0$. Let functions f_n $(n=1,2,3,\ldots)$ be defined as follows. Select a sequence $\{t_n\}_{n=1}^{\infty}$ of positive real numbers such that $t_n<t_{n+1}$, $f(x_n)=t_n$ for some $x_n \in X$ $(n=1,2,3,\ldots)$ and $\lim_{n\to\infty} t_n=+\infty$. Set $t_0=0$. Now, in the set $E[x; x \in X, f(x)\leqslant t_{n+1}]$, let $f_n=0$. In the set $E[x; x\in X, t_{n-1}<f(x)\leqslant t_n]$, let $f_n=f-t_{n-1}$. In the set $E[x; x\in X, t_n<f(x)]$, let $f_n=t_n-t_{n-1}$. The functions f_n are evidently continuous, and it is plain that the infinite series $\sum_{n=1}^{\infty} f_n$ converges everywhere in X to the function f. Now let $h_n=t_n f_n$, and let $h=\sum_{n=1}^{\infty} h_n$. This infinite series evidently converges throughout X, and the limit function h is continuous. It is easily verified that

$$t_n(f-f_1-f_2-\ldots-f_n)\leqslant h-h_1-h_2-\ldots-h_n,$$

for every n. If the functional I vanishes throughout $\mathfrak{C}^*(X,R)$, then we infer from the preceding inequality that $t_n I(f)\leqslant I(h)$ for all n. This implies that $I(f)=0$. If φ is any function in $\mathfrak{C}(X,R)$, we have $\varphi=\max(\varphi,0)+\min(\varphi,0)$, and plainly $I(\varphi)=0$.

Theorem 2. *Let I be any positive linear functional on $\mathfrak{C}(X,R)$. Then there is a positive real number a such that $aI(1)=1$.*

If $I(1)=0$, then I vanishes for every bounded function in $\mathfrak{C}(X,R)$, and therefore, by Theorem 1, vanishes identically. Hence $I(1)$ must be positive, and a may be taken as $1/I(1)$.

In considering bounded linear functionals, therefore, we may restrict ourselves to functionals I which are positive and for which $I(1)=1$.

Throughout the remainder of the present §, we take X to be an arbitrary but fixed CR-space. We proceed to the construction of a measure on X corresponding to a given linear functional on $\mathfrak{C}(X,R)$ (having the properties prescribed above).

Let G be any set in $\mathfrak{P}(X)$. We define the measure $\gamma(G)$ as $\sup I(f)$, where f runs through the set of all functions in $\mathfrak{C}(X,R)$ such that $0\leqslant f\leqslant \varphi_G$. (The characteristic function of a set A is denoted by φ_A).

Theorem 3. *The set-function γ has the following properties:*
(1) $G\subset H$ implies that $\gamma(G)\leqslant\gamma(H)$;
(2) $0\leqslant\gamma(G)\leqslant 1$ for all $G \in P(X)$;
(3) $\gamma(X)=1$;
(4) $\gamma(0)=0$;
G and H being arbitrary sets in $\mathfrak{P}(X)$.

These statements are obviously true.

Theorem 4. *$\gamma(G\cup H)\leqslant\gamma(G)+\gamma(H)$ for all sets G and H in $\mathfrak{P}(X)$.*

If G or H is void, there is nothing to prove. Let G and H both be non-void, and let f be a function in $\mathfrak{C}(X,R)$ such that $0\leqslant f\leqslant\varphi_{G\cup H}$ and such that $\gamma(G\cup H)-\varepsilon/3<I(f)$, ε being an arbitrary positive real number. Let $A=E[x; x\in X, f(x)\geqslant\varepsilon/3]\cap G'$. It is plain that $A \in \mathfrak{Z}(X)$, and that $A\subset H$. Let φ_1 be a function in $\mathfrak{C}(X,R)$ such that $\mathfrak{Z}(\varphi_1)=A$, and let φ_2 be a function in $\mathfrak{C}(X,R)$ such that $\mathfrak{Z}(\varphi_2)=H'$. The function $h=f\cdot\varphi_2^2/(\varphi_1^2+\varphi_2^2)$ is clearly continuous, is equal to f on A and 0 on H', and satisfies the inequalities $0\leqslant h\leqslant f$ throughout X. Next, let $\tilde{g}=f-h$. Setting $B=E[x; x\in X, g(x)\geqslant 2\varepsilon/3]$, we see that $B \in \mathfrak{Z}(X)$ and that $B\subset G$. Let ψ be a function in $\mathfrak{C}(X,R)$ such that $\psi=1$ on B and $\psi=0$ on G', while $0\leqslant\psi\leqslant 1$ everywhere (ψ can be obtained as h was obtained). Now, set $g=\psi\tilde{g}$. It is easily seen that $f\leqslant g+h+2\varepsilon/3$ throughout X, that $0\leqslant g\leqslant\varphi_G$, and that $0\leqslant h\leqslant\varphi_H$. Since I is positive, we have $\gamma(G\cup H)-\varepsilon/3<I(f)\leqslant I(g)+I(h)+$ $+I(2\varepsilon/3)\leqslant\gamma(G)+\gamma(H)+2\varepsilon/3$. Hence $\gamma(G\cup H)<\gamma(G)+\gamma(H)+\varepsilon$ for every positive ε, and the present theorem is proved.

Theorem 5. *If G, $H \in \mathfrak{P}(X)$ and $G\cap H=0$, then $\gamma(G\cup H)=$ $=\gamma(G)+\gamma(H)$.*

For, suppose that g and h are functions in $\mathfrak{C}(X,R)$ such that $0\leqslant g\leqslant\varphi_G$, $0\leqslant h\leqslant\varphi_H$, and $\gamma(G)+\gamma(H)-\varepsilon<I(g)+I(h)$. Then $0\leqslant g+h\leqslant\varphi_{G\cup H}$, and $\gamma(G)+\gamma(H)-\varepsilon<I(g+h)<\gamma(G\cup H)$. From this remark and Theorem 4 the present theorem follows immediately.

Theorem 6. *Let H be any set in $\mathfrak{P}(X)$ and let ε be any positive real number. Then there is a set J such that $J \in \mathfrak{P}(X)$, $J^{-}\subset H$, and $\gamma(H)-\varepsilon<\gamma(J)$.*

Let ψ be a function in $\mathfrak{C}(X,R)$ such that $0 \leqslant \psi \leqslant \varphi_H$ and $\gamma(H) - \varepsilon/2 < I(\psi)$, and let $J = E[x; x \in X, \varepsilon/2 < \psi(x)]$. It is clear that $J^- \subset H$. Now assume that $\gamma(J) \leqslant \gamma(H) - \varepsilon$. Since $\psi \leqslant \varepsilon/2$ on J', we have $I(\psi) - \varepsilon/2 = I(\psi - \varepsilon/2) \leqslant I(\max(\psi - \varepsilon/2, 0) \leqslant \gamma(J) \leqslant \gamma(H) - \varepsilon < I(\psi) - \varepsilon/2$; this is an obvious contradiction, and the theorem is proved.

Theorem 7. *Let $\{H_n\}_{n=1}^{\infty}$ be a family of sets in $\mathfrak{P}(X)$ such that $H_1 \supset H_2 \supset H_3 \supset \ldots \supset H_n \supset \ldots$ and $\prod_{n=1}^{\infty} H_n^- = 0$. Then $\lim_{n\to\infty} \gamma(H_n) = 0$.*

By Theorem 4, the sequence $\{\gamma(H_n)\}_{n=1}^{\infty}$ is decreasing and thus has a limit a, where $0 \leqslant a$. If $0 < a$, there exists, for every n, a function $f_n \in \mathfrak{C}(X,R)$ such that $0 \leqslant f_n \leqslant \varphi_{H_n}$ and $0 < a/2 < I(f_n)$. The function $\omega = \sum_{n=1}^{\infty} f_n$ is defined throughout X, and since $\prod_{n=1}^{\infty} H_n^- = 0$, every $p \in X$ is in some $H_{n_0}^{-\prime}$. In this open set, only a finite number of functions f_n are different from 0, and hence ω is continuous at p. (Note that ω need not be bounded). Now, $I(\omega) = I(f_1) + I(f_2) + I(f_3) + \ldots + I(f_k) + I(\sum_{i=n+1}^{\infty} f_i) > ka/2$, for every positive integer k. This is a palpable contradiction.

Theorem 8. *Let $\{H_n\}_{n=1}^{\infty}$ be a family of sets in $\mathfrak{P}(X)$ such that $H_1 \supset H_2 \supset \ldots \supset H_n \supset \ldots$ and $\prod_{n=1}^{\infty} H_n = 0$. Then $\lim_{n\to\infty} \gamma(H_n) = 0$.*

By Theorem 6, there is a set $G_n \in \mathfrak{P}(X)$ such that $G_n^- \subset H_n$ and $\gamma(H_n) - 1/n < \gamma(G_n)$. From Theorem 7, we infer that

$$\lim_{n\to\infty} \gamma(H_n) \leqslant \lim_{n\to\infty} \gamma(G_n) = 0.$$

Theorem 9. *Let $\{G_n\}_{n=1}^{\infty}$ be any sequence of sets in $\mathfrak{P}(X)$. Then $\gamma(\sum_{n=1}^{\infty} G_n) \leqslant \sum_{n=1}^{\infty} \gamma(G_n)$.*

Since every set G_n is in $\mathfrak{P}(X)$, it is clearly possible to find sets $\{K_{n,m}\}_{m=1}^{\infty}$ such that $K_{n,m} \in \mathfrak{Z}(X)$ $(m = 1,2,3,\ldots)$ and $\sum_{m=1}^{\infty} K_{n,m} = G_n$. Let $\sum_{n=1}^{\infty} G_n = G$. It is plain that $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K_{n,m} = G$. Hence, by appropriate re-numbering, we may write $G = \sum_{n=1}^{\infty} L_n$, where every L_n is some $K_{i,j}$. Let $M_k = L_1 \cup L_2 \cup \ldots \cup L_k$ $(k = 1,2,3,\ldots)$. Then $\{M_k' \cap G\}_{k=1}^{\infty}$ is a sequence of sets in $\mathfrak{P}(X)$ to which Theorem 8 applies; and consequently, for every $\varepsilon > 0$, there exists a k_0 such that $\gamma(M_{k_0}' \cap G) < \varepsilon$. Now, it can be shown, using only Theorems 3 and 4 and [16] Chapter II, that γ gives rise to an outer measure γ^* defined for all subsets of X which is finitely additive on measurable sets and for which the sets M_k are all measurable. Hence $\gamma(G) = \gamma^*(G) = \gamma^*(M_{k_0} \cap G) + \gamma(M_{k_0}' \cap G)$. Since every set M_k is contained in some set $G_1 \cup G_2 \cup \ldots \cup G_m$, we now have $\gamma(G) = \gamma(\sum_{n=1}^{\infty} G_n) < \gamma(\sum_{n=1}^{m} G_n) + \varepsilon \leqslant \sum_{n=1}^{m} \gamma(G_n) + \varepsilon \leqslant \sum_{n=1}^{\infty} \gamma(G_n) + \varepsilon$. From this result the present theorem follows immediately.

We now extend the measure-function γ to an outer measure γ^* defined for all subsets of X: for any $A \subset X$, let $\gamma^*(A) = \inf \gamma(G)$, where G runs through the family of all sets in $\mathfrak{P}(X)$ that contain A.

Theorem 10. *The outer measure γ^* has the following properties:*

(1) $0 \leqslant \gamma^*(A)$ *for all* $A \subset X$;
(2) $\gamma^*(A) \leqslant \gamma^*(B)$ *if* $A \subset B$;
(3) $\gamma^*(\sum_{n=1}^{\infty} A_n) \leqslant \sum_{n=1}^{\infty} \gamma^*(A_n)$ *for all* $\{A_1, A_2, \ldots, A_n, \ldots\} \subset X$;
(4) $\gamma^*(G) = \gamma(G)$ *for all* $G \in \mathfrak{P}(X)$.

These remarks are all immediate consequences of Theorems 3 and 9, and the definition of γ^*.

Theorem 11. *Every set in $\mathfrak{P}(X)$ is measurable with respect to the outer measure γ^*.*

The proof of this result is almost identical with the proof given by Kakutani for a corresponding theorem ([9], p. 1010), and will not be repeated here.

Theorem 12. *The outer measure γ^* is countably additive on the family $\overline{\mathfrak{P}}(X)$ of Baire sets in X.*

The family $\mathfrak{M}$ of measurable sets is closed under the formation of complements and countable unions ([16], pp. 45-46); and the measure γ^* is countably additive on the family of measurable sets ([16], p. 44). This observation completes the present proof.

Corollary. *If A and B are subsets of X such that for some $F \in \mathfrak{Z}(X)$, $A \subset F$ and $B \subset F'$, then $\gamma^*(A \cup B) = \gamma^*(A) + \gamma^*(B)$.*

This follows at once from the fact that F is measurable: $\gamma^*(A \cup B) = \gamma^*(F \cap (A \cup B)) + \gamma^*(F' \cap (A \cup B)) = \gamma^*(A) + \gamma^*(B)$.

In the sequel, we shall write γ^* as γ for sets in the family $\overline{\mathfrak{P}}(X)$.

The requirement that every continuous real function be integrable (γ) places very severe restrictions on the measure γ, as we now show.

Theorem 13. *Let f be any function in $\mathfrak{C}(X,R)$. Then there are real numbers a and β such that $\gamma(E[x; x \in X, a \leqslant f(x) \leqslant \beta]) = 1$.*

The proof is carried out by contradiction. By considering $\max(f,0)$ and $\min(f,0)$, we may limit ourselves to positive functions. If the theorem is false, we can find an $f \in \mathfrak{C}(X,R)$ such that $\gamma(E[x; x \in X, 0 \leqslant f(x) \leqslant r]) < 1$ for all $r > 0$. This clearly implies the existence of a sequence of real numbers

$$0 < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < \ldots,$$

where $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = +\infty$, with the property that, setting $A_n = E[x; x \in X, a_n \leqslant f(x) \leqslant b_n]$, we have $1/\gamma(A_n) = a_n > 0$. Let the function $g \in \mathfrak{C}(R,R)$ be defined as follows:

$g(t) = 0$ for $t \leqslant 0$;

$g(t) = (a_1/a_1)t$ for $0 \leqslant t \leqslant a_1$;

$g(t) = a_1$ for $a_1 \leqslant t \leqslant b_1$;

$g(t) = [(a_2 - a_1)/(a_2 - b_1)]\cdot(t - b_1) + a_1$ for $b_1 \leqslant t \leqslant a_2$;

. . . .

$g(t) = a_n$ for $a_n \leqslant t \leqslant b_n$;

$g(t) = [(a_{n+1} - a_n)/(a_{n+1} - b_n)]\cdot(t - b_n) + a_n$ for $b_n \leqslant t \leqslant a_{n+1}$;

. . . .

Then, the function $\varphi = g(f)$ assumes the values a_n throughout the set A_n, and we have $I(\varphi) \geqslant k$ for every positive integer k. This contradiction establishes the present theorem.

We now state our fundamental theorem concerning bounded linear functionals.

Theorem 14. *Let X be any completely regular space and let I be any bounded linear functional defined on $\mathfrak{C}(X,R)$. Then there exists a Baire measure γ on X such that every function in $\mathfrak{C}(X,R)$ is bounded except on a set of γ-measure 0 and such that $I(f) = \int_X f(x)\,d\gamma$ for all $f \in \mathfrak{C}(X,R)$.*

Writing I in the form $I_1 - I_2$, where I_1 and I_2 are non-negative linear functionals, we associate with I_i a Baire measure γ_i as in the preceding discussion ($i = 1,2$). It is an easy matter to verify that $I_i(f) = \int_X f(x)\,d\gamma_i$ for every $f \in \mathfrak{C}(X,R)$ and that, writing γ for $\gamma_1 - \gamma_2$, the present theorem holds. A reference to Theorem 13 shows that f must be bounded except on a set of measure 0. (For details of a similar proof, see, for example, [9], p. 1011).

The converse of Theorem 14 is obviously true.

We shall say that a Baire measure γ ($\gamma(X) \neq \pm\infty$) is regular if for every set $A \in \mathfrak{P}(X)$ and every $\varepsilon > 0$, there exists a set $G \in \mathfrak{P}(X)$ such that $G \supset A$ and $|\gamma(G) - \gamma(A)| < \varepsilon$. We note that the measure γ associated with a bounded linear functional is always regular.

Theorem 15. *If two regular Baire measures yield the same finite integral for all functions in $\mathfrak{C}(X,R)$, then they are identical.*

The difference $\gamma_1 - \gamma_2$ is a regular Baire measure, which we denote by the symbol δ, having the property that $\int_X f(x)\,d\delta = 0$ for all $f \in \mathfrak{C}(X,R)$. Let A be any set in $\mathfrak{P}(X)$. Then, δ being regular, there exist sets G and H in $\mathfrak{P}(X)$ such that $G \supset A$, $H \supset A'$, and $|\delta(G) - \delta(A)| < \varepsilon/2$, $|\delta(H) - \delta(A')| < \varepsilon/2$. It follows that $H' \subset A \subset G$ and that $|\delta(H') - \delta(G)| < \varepsilon$. Let g be a function in $\mathfrak{C}(X,R)$ such that $g = 1$ on H', $g = 0$ on G', and $0 \leqslant g \leqslant 1$ everywhere. Then

$$|\delta(A)| = \Big|\delta(A) - \int_X g(x)\,d\delta\Big| = \Big|\int_X (\varphi_A(x) - g(x))\,d\delta\Big| = \Big|\int_{G\cap H} g(x)\,d\delta\Big| < \varepsilon.$$

Hence, ε being an arbitrary positive real number, $\delta(A) = 0$, and γ_1 is identical with γ_2.

As noted above, any Baire measure γ satisfying the requirements of Theorem 13 yields a linear functional $\int_X f(x)\,d\gamma$ on the space $\mathfrak{C}(X,R)$. This linear functional, by the process set forth in Theorems 3-14, yields another Baire measure $\tilde{\gamma}$, which is necessarily regular. Theorem 15 asserts that γ and $\tilde{\gamma}$ are identical if and only if γ is regular. We also infer that with any Baire measure satisfying the requirement of Theorem 13, there may be associated a regular Baire measure yielding the same integral for functions in $\mathfrak{C}(X,R)$.

Remark 1. Previous writers (e. g., Kakutani [9] and Markov [13]) have considered, instead of our measure γ associated with the linear functional I, a measure δ obtained first for *all* open sets G in X by the definition $\delta(G) = \sup I(f)$, where f runs through all members of $\mathfrak{C}(X,R)$ such that $0 \leqslant f \leqslant \varphi_G$. For an arbitrary *CR*-space, this process yields an outer measure δ^* which is finitely additive on measurable sets. One may define the integral in the usual way and show that I is indeed the integral over X with respect to the measure δ^*. If one could prove that $\delta(\sum_{n=1}^{\infty} G_n) \leqslant \sum_{n=1}^{\infty}\delta(G_n)$ and that $\delta(G \cup H) = \delta(G) + \delta(H)$ ($\{G_n\}_{n=1}^{\infty} \subset \mathfrak{O}(X)$, G, $H \in \mathfrak{O}(X)$, and $G \cap H = 0$), he would have a stronger result than our Theorem 14, since all Borel sets would be measurable under δ^*. However, there exist non-normal completely regular spaces for which not all open sets are measurable (δ^*) and for which the inequality $\delta(\sum_{n=1}^{\infty} G_n) > \sum_{n=1}^{\infty}\delta(G_n)$ may obtain. Our example is the space T consisting of all pairs of ordinal numbers (α,β) ($\alpha \leqslant \Omega$, $\beta \leqslant \omega$), except for the pair (Ω,ω). A generic neighborhood of (α_0,β_0) is the set of all (α,β) such that $\alpha' < \alpha \leqslant \alpha_0$, $\beta' < \beta \leqslant \beta_0$, where $\alpha' < \alpha_0$ and $\beta' < \beta_0$. The linear functional $I(f)$ for $f \in \mathfrak{C}(T,R)$ is defined as $\lim_{\alpha\to\Omega} f(\alpha,\omega)$. This limit always exists, and produces, in fact, a ring-homomorphism of $\mathfrak{C}(T,R)$ onto R. We construct the corresponding δ and δ^*; we let $H=E[(\alpha,\beta); (\alpha,\beta)\in T, \alpha<\Omega]$; and we let $G_n=E[(\alpha,\beta); (\alpha,\beta)\in T, \beta=n]$ $(n=1,2,\ldots,\omega)$. Then, plainly, $\delta(H)=\delta^*(H')=0$ and $\delta(G_n)=0$. Thus $\delta(T)=1>(\delta(G_\omega)+\delta(G_1)+\delta(G_2)+\ldots)$ and H is non-measurable.

For compact H-spaces X, Kakutani's results [9] show that δ is countably additive on measurable sets and that all Borel sets are measurable. It follows from Theorem 14 that δ has these properties for all spaces such that every open set is in $\mathfrak{P}(X)$, e. g., for metric spaces. The question remains open for normal spaces as to whether δ is a Borel measure.

Remark 2. By constructing the measures γ and γ^* for the space T and linear functional I of Remark 1, we show that γ^* may admit non-Baire sets as measurable and that there may be open sets which are not measurable (γ^*). It is not difficult to see that a subset A of T is measurable (γ^*) if and only if $G_\omega \cap A$ is countable or has a countable complement in G_ω; whereas the Baire sets in T are those sets A such that $A\cap G_n$ is countable or has a countable complement in G_n for all $n=1,2,\ldots,\omega$. It is also easy to exhibit examples where the Baire sets are exactly the sets measurable (γ^*).

3. Confinement of measures. It is natural to inquire, in the light of Theorem 13, whether or not there is always a compact set $A\subset X$ such that $\gamma(A')=0$. For technical reasons, this cannot be hoped for, but we may expect that for non-pathological spaces, one can find a compact subset A such that $\gamma^*(A)=\gamma(X)$ and such that integration over X may be reduced to integration over A. This is indeed the case.

In an earlier paper, the writer has studied a class of CR-spaces called Q-spaces ([7], pp. 85-98). (The same class of spaces has been studied independently, from a different point of view, and with most fruitful results, by Dr. Leopoldo Nachbin). There it is shown that a CR-space is a Q-space if and only if every ring-homomorphism of $\mathfrak{C}(X,R)$ onto R is of the form M_p for some $p\in X$. We characterize Q-spaces from our present standpoint as follows.

Theorem 16. A CR-space X is a Q-space if and only if every Baire measure γ on X, assuming only the values 0 and 1, and such that $\gamma(X)=1$, coincides on all Baire sets with a measure μ_p.

First, suppose that X is not a Q-space. Then, according to [7], Theorem 50, p. 85, there exists a subfamily $\mathcal{M}$ of $\mathcal{Z}(X)$ such that: (1) $\mathcal{M}$ does not contain the void set; (2) $A,B\in\mathcal{M}$ imply that $A\cap B\in\mathcal{M}$; (3) $A\in\mathcal{M}$, $B\in\mathcal{Z}(X)$, and $B\supset A$ imply that $B\in\mathcal{M}$; (4) $B\in\mathcal{Z}(X)$ and B non $\in\mathcal{M}$ imply that $A\cap B=0$ for some $A\in\mathcal{M}$; (5) $\Pi_{A\in\mathcal{M}} A=0$; (6) no countable subfamily of $\mathcal{M}$ has void intersection. (It turns out that the functions in $\mathfrak{C}(X,R)$ which vanish on some set in $\mathcal{M}$ are just the functions which go into zero under a ring-homomorphism M of $\mathfrak{C}(X,R)$ onto R which is not of the form M_p). We define a Baire measure β on X as follows. For a set $G\in\mathfrak{P}(X)$, let $\beta(G)=1$ if and only if $G'\in\mathcal{M}$; otherwise, let $\beta(G)=0$. We extend β to an outer measure in the standard way; under this measure, it is clear that every set in $\mathfrak{P}(X)$ and hence every Baire set is measurable. It is furthermore easy to verify that $\int_X f(x)d\beta=M(f)\in R$, where M is the ring-homomorphism referred to above. Finally, given any $p\in X$, there is by (5) some $A\in\mathcal{M}$ such that p non $\in A$. Then $\mu_p(A)=0$ and $\beta(A)=1$. Hence β coincides with no measure μ_p on Baire sets.

Conversely, suppose that X is a Q-space, and let β be a Baire measure on X assuming only the values 0 and 1, with $\beta(X)=1$. Let f be any function in $\mathfrak{C}(X,R)$, and let $\{a_n\}_{n=-\infty}^{+\infty}$ be any strictly increasing set of real numbers such that $a_{n+1}-a_n$ is bounded and such that $\inf_{-\infty<n<+\infty} a_n \leqslant \inf_{x\in X} f(x)$ and $\sup_{-\infty<n<+\infty} a_n \geqslant \sup_{x\in X} f(x)$. Putting

$$A_n=E[x; x\in X, a_{n-1}\leqslant f(x)<a_n],$$

we find at once that every A_n but one, say A_{n_0}, has β-measure 0, since β is countably additive on Baire sets. We continue this argument in the natural manner and conclude that f is constant on a set $E\in\mathcal{Z}(X)$ such that $\beta(E)=1$. From this, it is apparent that $I(f)=\int_X f(x)d\beta$ is finite for all $f\in\mathfrak{C}(X,R)$ and that I is a ring-homomorphism of $\mathfrak{C}(X,R)$ onto R. Since X is a Q-space, there is a point $p\in X$ such that $I(f)=f(p_0)$ for every $f\in\mathfrak{C}(X,R)$. It is evident that the measure μ_p coincides with β for all Baire sets.

Remark 1. The outer measure β^* associated as in § 2 with the Baire measure β need not coincide with μ_p on all subsets of X. Consider $T_{\Omega+1}$, the space of all ordinal numbers $\leqslant\Omega$, in the order topology. Let $I(f)=f(\Omega)$. Then the Baire measure β associated with I by the process expounded in § 2, is defined only for Baire sets, is 0 for countable sets and 1 for sets containing some interval $(\alpha, \Omega]$, where $\alpha < \Omega$. (These sets exhaust the family of Baire sets). For the outer measure β^*, we have $\beta^*(\Omega) = \beta^*((1,\Omega)) = 1$, while $\mu_\Omega((1,\Omega)) = 0$.

Remark 2. Theorem 16 extends a theorem of Šmulian [18], stating that a ring-homomorphism of $\mathfrak{C}^*(X,R)$ onto R, where X is a normal space, is realized by an integral with respect to a finitely additive measure assuming only the values 0 and 1.

Theorem 17. *Let X be a CR-space, I a positive linear functional on $\mathfrak{C}(X,R)$, and γ and γ^* the measures associated with I as in Theorems* 10 *and* 14. *If X is a Q-space, there exists a compact subset F of X such that $\gamma(X) = \gamma^*(F)$. If X is not a Q-space, there exists a ring-homomorphism M of $\mathfrak{C}(X,R)$ onto R with associated Baire measure β such that $\beta^*(A) = 0$ for all compact subsets A of X.*

We first consider the case in which X is not a Q-space. Theorem 16 shows that a ring-homomorphism of $\mathfrak{C}(X,R)$ onto R can be found such that the family $\mathcal{M}$ of sets in $\mathcal{Z}(X)$ having measure 1 satisfy conditions (1)-(6) set forth in the proof of Theorem 16. Let K be any compact subset of X. In view of (5), if $p \in K$, there is a set $Z_p \in \mathcal{M}$ such that p non $\in Z_p$. The family $\{Z'_p\}_{p \in K}$ is thus an open covering of K, which admits a finite subcovering $\{Z'_{p_1}, Z'_{p_2}, \dots, Z'_{p_m}\}$. The set $W = \prod_{i=1}^{m} Z_{p_i}$ is a set of $\mathcal{M}$, by (2), and thus K is disjoint from the set $W \in \mathcal{M}$. It follows that $\beta^*(K) = 0$, if β^* is the outer measure associated with M.

Conversely, suppose that X is a Q-space. We may suppose that $\gamma(X) = 1$, by Theorem 2. In view of Theorem 13, we have, for every $f \in \mathfrak{C}(X,R)$, a pair of real numbers α and β such that, setting $A_f = E[x; x \in X, \alpha \leqslant f(x) \leqslant \beta]$, we have $\gamma(A_f) = 1$. It is obvious that $A_f \in \mathcal{Z}(X)$. Let $\mathcal{J}$ denote the family of Baire sets of γ-measure 1. It is obvious that $\mathcal{J}$ is closed under the formation of countable intersections and (within $\mathfrak{P}(X)$) arbitrary supersets, and does not contain the void set. Hence, $\mathcal{A} = \mathcal{J} \cap \mathcal{Z}(X)$ enjoys properties (1), (2), and (3) set forth in the proof of Theorem 16, and has the further important property that every function in $\mathfrak{C}(X,R)$ is bounded on some set in $\mathcal{A}$. By an application of Zorn's lemma (for a statement of which, see [19], p. 7), we infer that $\mathcal{A}$ can be imbedded in a family $\mathcal{A}_0 \subset \mathcal{Z}(X)$ such that $\mathcal{A}_0$ enjoys properties (1), (2), (3), and (4). An appeal to [7], page 85, Theorem 50, shows that $\mathcal{A}_0$ cannot enjoy properties (5) and (6) simultaneously. If (6) fails, then there is a countable subfamily $\{D_1, D_2, \dots, D_n, \dots\}$ of $\mathcal{A}_0$ such that $\prod_{n=1}^{\infty} D_n = 0$.

Let $B_k = D_1 \cap D_2 \cap \dots \cap D_k$ $(k = 1, 2, \dots)$ and let f_k be a function in $\mathfrak{C}(X,R)$ such that $0 \leqslant f_k \leqslant 1$, $f_k = 0$ on B_k, and $f_k > 0$ on B'_k $(k = 1, 2, \dots)$. Then $\varphi = (\sum_{k=1}^{\infty} 2^{-k} f_k)^{-1}$ is a function in $\mathfrak{C}(X,R)$ such that $\varphi \geqslant 2^k$ on B_k. The set A_φ is clearly contained in some B'_k and yet A_φ must, by definition, be in $\mathcal{A}_0$. By this contradiction, we see that (6) must obtain and that (5) must fail; i. e., $\Pi_{B \in \mathcal{A}_0} B \neq 0$. *A fortiori*, we have that $F = \Pi_{B \in \mathcal{A}} B \neq 0$. It is clear that F is a closed set, as the intersection of closed sets.

We shall now show that F is compact. (For this proof, we are indebted to Dr. Leopoldo Nachbin). Let $\mathcal{U}$ be the weakest uniform structure on X making all functions in $\mathfrak{C}(X,R)$ uniformly continuous. (For a discussion of uniformity see [6], p. 91 *et seq.*). Being a Q-space, X is complete in this uniform structure (see [7], p. 92); and as a closed subset of the complete space X, F is also complete in the structure $\mathcal{U}$. The structure $\mathcal{U}$ on F may be described as the weakest uniform structure under which all continuous real functions extensible continuously over X are uniformly continuous. These functions are all bounded on F (since F is contained in the intersection $\Pi_{f \in \mathfrak{C}(X,R)} A_f$) and hence F is totally bounded under the structure $\mathcal{U}$. A complete and totally bounded uniform structure, however, yields a compact topology, and therefore F is compact.

Next, consider the measure $\gamma^*(F)$, which is defined as $\inf \gamma(G)$, $G \supset F$ and $G \in \mathfrak{P}(X)$. If there is a set $G \supset F$ such that $G \in \mathfrak{P}(X)$ and $\gamma(G) < 1$, let C be any set in $\mathcal{A}$. It is clear that

$$1 = \gamma(C) = \gamma(C \cap G) + \gamma(C' \cap G) + \gamma(C \cap G') + \gamma(C' \cap G') = \gamma(C \cap G) + \gamma(C \cap G').$$

Hence, $\gamma(C \cap G') = 1 - \gamma(C \cap G) \geqslant 1 - \gamma(G) > 0$. It follows, since C was taken arbitrarily from $\mathcal{A}$, that $\{C \cap G'\}_{C \in \mathcal{A}}$ is a subfamily of $\mathcal{Z}(X)$ closed under the formation of countable intersections and not containing the void set. Just as in our definition of F above, we can now show that $\Pi_{C \in \mathcal{A}}(C \cap G') \neq 0$; and this proves that F is not the intersection of all sets of measure 1 in $\mathcal{Z}(X)$. This contradiction shows that $\gamma^*(F) = 1$, and proves the present theorem.

The example adduced in Remark 1 after Theorem 16 shows that F need not be measurable (γ^*). Here $F = \Omega$, and, as noted, $\gamma^*(F) = \gamma^*(F') = 1$. In spite of this fact, however, we can reduce integrals over X to integrals over F, as the following theorem shows.

Theorem 18. *Let X be a Q-space, and let I, γ, γ^*, and F be as in Theorem 17. Then the outer measure γ^*, restricted to subsets of F, makes every set in $\overline{\mathfrak{P}}(F)$ measurable; and furthermore, for every $f \in \mathfrak{C}(X,R)$, we have $\int_F f(x)\,d\gamma^* = \int_X f(x)\,d\gamma$.*

If F be measurable (γ^*), there is of course nothing to prove. We first observe that every function $\varphi \in \mathfrak{C}(F,R)$ can be continuously extended over X. Indeed, let p and q be two distinct points of F. (The case in which F consists of a single point need not detain us). Then there is a function $g \in \mathfrak{C}(X,R)$ such that $g(p) \neq g(q)$. The function g can be restricted to F, and provides a continuously extensible function assuming different values at p and q. It is clear that the set of functions in $\mathfrak{C}(F,R)$ which are continuously extensible over X form a closed subring of $\mathfrak{C}(F,R)$. Since this subring permits one to distinguish between arbitrary pairs of points, we may infer from the Stone-Weierstrass theorem (see, for example, [8]), that every function in $\mathfrak{C}(F,R)$ admits a continuous extension over X.

Next, suppose that $f \in \mathfrak{C}(X,R)$ and that $f=0$ on F. Then $E[x; x \in X, \varepsilon > f(x)]$ is a set in $\mathfrak{P}(X)$ containing F, for every $\varepsilon > 0$, and thus $I(f) = \int_X f(x)\,d\gamma = 0$. If $\varphi \in \mathfrak{C}(F,R)$ and if $\tilde{\varphi}$ and $\tilde{\tilde{\varphi}}$ are two continuous extensions of φ over X, it is plain that $I(\tilde{\varphi}) = I(\tilde{\tilde{\varphi}})$.

The functional I can thus be defined in a natural way on the space of functions $\mathfrak{C}(F,R)$: for any $f \in \mathfrak{C}(F,R)$, $I_F(f) = I(\bar{f})$, where $\bar{f}$ is any continuous extension of f over X. We denote the Baire measure and outer measure associated with I_F by γ_F and γ_F^*, respectively. We now prove that $\gamma_F(G) = \gamma^*(G)$, for every set $G \in \mathfrak{P}(F)$. Indeed, for every $\varepsilon > 0$, there exists a function $\psi \in \mathfrak{C}(F,R)$ such that $0 \leqslant \psi \leqslant \varphi_G$ and $\gamma_F(G) - \varepsilon/3 < I_F(\psi)$. Furthermore, there is a set $H \in \mathfrak{P}(X)$ such that $H \supset G$ and $\gamma(H) - \varepsilon/3 < \gamma^*(G)$. Let A be the set $E[x: x \in F, \psi(x) \geqslant \varepsilon/3]$. A is compact, as a closed subset of the compact space F. One may accordingly find a function $\tilde{\sigma} \in \mathfrak{C}(X,R)$ such that $\tilde{\sigma} = 1$ on A, $\tilde{\sigma} = 0$ on H', and $0 \leqslant \tilde{\sigma} \leqslant 1$ throughout X. (Since X is a CR-space, and H' is closed, one can find, for every $p \in A$, a function $\sigma_p \in \mathfrak{C}(X,R)$ such that $\sigma_p(p) = 2$ and $\sigma_p = 0$ on H'. There is a neighborhood $U(p)$ in which σ_p is greater than 1. A finite number of these neighborhoods, $U(p_1), U(p_2), \ldots, U(p_n)$, cover A, and the function $\tilde{\sigma} = \min(1; \max(\sigma_{p_1}, \sigma_{p_2}, \ldots, \sigma_{p_n}))$ has the required properties). Let $\tilde{\psi}$ be any continuous extension of ψ over X; let $\tilde{g} = \tilde{\sigma}\tilde{\psi}$; and let g be the function $\tilde{g}$ defined only on F. It is obvious that we have $\psi \leqslant g + \varepsilon/3$. The following relations then obtain: $\gamma_F(G) - \varepsilon/3 < I_F(\psi) \leqslant I_F(g + \varepsilon/3) = I_F(g) + \varepsilon/3 = I(\tilde{g}) + \varepsilon/3 \leqslant \gamma(H) + \varepsilon/3 < \gamma^*(G) + 2\varepsilon/3$. Hence we find that $\gamma_F(G) < \gamma^*(G) + \varepsilon$ for every $\varepsilon > 0$, and thus $\gamma_F(G) \leqslant \gamma^*(G)$. To prove the reverse inequality, we first observe that G is contained in a set $H_0 \in \mathfrak{P}(X)$ such that $F \cap H_0 = G$. For, G being in $\mathfrak{P}(F)$, there exists a function $g \in \mathfrak{C}(F,R)$ such that $P(f) = G$. For any continuous extension $\bar{f}$ of f over X, we may take $H_0 = P(\bar{f})$. Now, let $\tilde{\varrho}$ be a function in $\mathfrak{C}(X,R)$ such that $0 \leqslant \tilde{\varrho} \leqslant \varphi_{H_0}$ and $\gamma(H_0) - \varepsilon < I(\tilde{\varrho})$, ε being an arbitrary positive real number. Let ϱ denote the function $\tilde{\varrho}$ restricted to the domain F. Then $0 \leqslant \varrho \leqslant \varphi_G$, and we have the following relations:

$$\gamma_F(G) \geqslant I_F(\varrho) = I(\tilde{\varrho}) > \gamma(H_0) - \varepsilon \geqslant \gamma^*(G) - \varepsilon.$$

Thus $\gamma_F(G) > \gamma^*(G) - \varepsilon$; since ε is arbitrary, we have $\gamma_F(G) \geqslant \gamma^*(G)$, and finally, $\gamma_F(G) = \gamma^*(G)$.

It follows immediately that γ^* is countably additive on the family $\overline{\mathfrak{P}}(F)$, which is just the family of all sets $A \cap F$, where $A \in \overline{\mathfrak{P}}(X)$. The present theorem also follows at once.

Remark 1. Theorem 18 is a natural extension of the theorem of Mackey [12] which discusses positive linear functionals on the space $\mathfrak{C}(N_\alpha, R)$. He proves that every such functional I has a measure γ associated with it of the form $\gamma = \sum_{i=1}^{n} \alpha_i \mu_{p_i}$, $\alpha_i \in R$, if and only if N_α admits no measure countably additive on all subsets, vanishing for points, assuming only the values 0 and 1, and equal to 1 for N_α itself. Such spaces are just the spaces for which Ulam's theorem on two-valued measures is true [20]. The writer has stated ([7], p. 87, Theorem 52) that every space N_α is a Q-space. The proof is incorrect, and it can be shown without difficulty that N_α is a Q-space if and only if it admits no two-valued measure of the kind described above. By Theorem 18, for any such space N_α and any positive linear functional I on $\mathfrak{C}(N_\alpha, R)$ with associated Baire measure γ, there exists a compact set F (necessarily a Baire set in N_α) with $\gamma(F) = I(1) = \gamma(N_\alpha)$. This set F is necessarily finite, and Mackey's theorem is re-verified.

Remark 2. Mackey [12] further observes that even if N_α does admit a 2-valued measure of the kind specified above (the existence of cardinal numbers $\aleph_\alpha$ for which this is true being an open question), the measure γ associated with a positive linear functional I can be represented as $\sum_{i=1}^{m} \alpha_i \mu_{p_i} + \sum_{j=1}^{n} \beta_j \varrho_j$, where the ϱ_j are 2-valued measures of the kind described above, defined on subsets A_j of N_α which are disjoint from each other and from the set $\{p_1,p_2,\ldots,p_m\}$. If we imbed the space N_α in the Q-space νN_α (see [7], p. 88, Theorem 56), each of the measures ϱ_j corresponds to a point q_j in νN_α such that ϱ_j becomes μ_{q_j}. The set F of Theorems 17 and 18 becomes $\{p_1,p_2,\ldots,p_m,q_1,q_2,\ldots,q_n\}$ in νN_α.

Remark 3. Theorems 14, 17, and 18, embody as special cases the results of Banach ([4], pp. 10 and 50) and Wehausen ([21], p. 164) characterizing general continuous linear functionals defined on $\mathfrak{C}(N_0,R)$ and $\mathfrak{C}(R,R)$. The metric topologies imposed upon these function spaces coincide with the k-topologies, so that these authors are considering k-continuous functionals. As we shall show *infra*, k-continuity and boundedness are identical for these spaces, so that Banach's and Wehausen's observations are subsumed under ours. A generalization of the situation considered by them may be described as follows. Let X be any topological space such that $X=\sum_{n=1}^{\infty}K_n$, where K_n is a compact Hausdorff space $(n=1,2,\ldots)$. Let $P_n=K_1\cup K_2\cup\ldots\cup K_n$ $(n=1,2,\ldots)$, and let $\mathfrak{C}(X,R)$ be topologized with a metric ϱ:

$$\varrho(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|f-g|_n}{1+|f-g|_n},$$

where $|f-g|_n=\sup_{x\in P_n}|f(x)-g(x)|$.

This metric topology is identical with the k-topology for $\mathfrak{C}(X,R)$. Linear functionals on $\mathfrak{C}(X,R)$ continuous in this topology are exactly the bounded linear functionals, and, in view of Theorems 17 and 18, each of them may be represented by an integral $\int\limits_{P_n} f(x)d\gamma$, where γ is a Baire measure on P_n.

4. Continuity. We now discuss the various topologies for $\mathfrak{C}(X,R)$ and the functionals which are continuous in these topologies.

Theorem 19. *Every linear functional I on $\mathfrak{C}(X,R)$ which is continuous in the p- or k-topology for $\mathfrak{C}(X,R)$ is bounded.*

It suffices to prove this statement for k-continuous functionals, since every p-continuous functional is clearly k-continuous. For I to be k-continuous, there must be, for every positive real ε, a compact subset A of X and a $\delta>0$ such that $|f(x)|<\delta$ for all $x\in A$ implies that $|I(f)|<\varepsilon$. I being linear, this condition is also sufficient for continuity throughout $\mathfrak{C}(X,R)$. Let φ_1 and φ_2 be any two functions in $\mathfrak{C}(X,R)$ such that $\varphi_1\leqslant\varphi_2$; and let $\mathfrak{B}$ be the set of all functions f in $\mathfrak{C}(X,R)$ such that $\varphi_1\leqslant f\leqslant\varphi_2$. Since φ_1 and φ_2 are bounded on A, there exists a positive real number η such that $|\eta\varphi_i|<\delta$ on A $(i=1,2)$. Thus $|\eta f|<\delta$ on A for all $f\in\mathfrak{B}$, and accordingly we find that $|I(f)|<\varepsilon/\eta$ for every $f\in\mathfrak{B}$. Thus I carries bounded sets in $\mathfrak{C}(X,R)$ into bounded sets in R, and our theorem is established.

Theorem 20. *Any p-continuous linear functional I on $\mathfrak{C}(X,R)$ can be represented in the form $I=\sum_{i=1}^{n}a_iM_{p_i}$, where the a_i are real numbers. If X is a Q-space, the p-continuous linear functionals on $\mathfrak{C}(X,R)$ are just the linear combinations of ring-homomorphisms of $\mathfrak{C}(X,R)$ onto R. If X is not a Q-space, there exists a ring-homomorphism of $\mathfrak{C}(X,R)$ onto R which is not p-continuous.*

Consider first the case in which X is a Q-space and I is a positive linear functional such that $I(1)=1$. From Theorem 14, we infer that $I(f)=\int\limits_X f(x)\,d\gamma$, where γ is a non-negative Baire measure on $\overline{\mathfrak{P}}(X)$ such that $\gamma(X)=1$. By Theorem 18, we know that there is a compact set $F\subset X$ such that $\gamma^*(F)=1$ and $\int\limits_X f(x)\,d\gamma=\int\limits_F f(x)\,d\gamma^*$, for every $f\in\mathfrak{C}(X,R)$. It is obvious from the construction of F given in Theorem 17 that for $p\in F$, $G\in\mathfrak{P}(X)$, and $p\in G$, we have $0<\gamma(G)$. (If this were not the case, then for some $G\ni p$, $\gamma(G)=0$, $\gamma(G')=1$, and $F\subset G'$; thus p non $\in F$).

We now prove that F must be finite. If the contrary be true, then for every finite subset $B=\{q_1,q_2,\ldots,q_k\}$ of X, there is a point $p\in F\cap B'$, and there is certainly a positive function $g\in\mathfrak{C}(X,R)$ such that $g(p)=1$ and $g(q_i)=0$ $(i=1,2,\ldots,k)$. The set $H=E[x;x\in X, 1/2>g(x)]$ has the property that $0<\gamma(H)$, in view of our remark above. The function $g_n=(2n/\gamma(H))\cdot g$ is clearly in the p-neighborhood of 0 consisting of all functions $h\in\mathfrak{C}(X,R)$ such that $|h(q_i)|<\varepsilon$ $(i=1,2,\ldots,k)$. On the other hand, we have $I(g_n)\geqslant n$. Since $\{q_1,q_2,\ldots,q_k\}$ and ε are arbitrary, it follows that I is not p-continuous. This proves that F must be finite.

The most general measure on the finite set $F=\{p_1,p_2,\ldots,p_m\}$ is $\sum_{i=1}^{m}a_i\mu_{p_i}$. Theorem 18 shows that

$$\int\limits_X f(x)d\gamma=\sum_{i=1}^{m}a_if(p_i)=\sum_{i=1}^{m}a_iM_{p_i}(f),$$

where the a_i are non-negative and $\sum_{i=1}^{m}a_i=1$.

The case of a general p-continuous linear functional I (X a Q-space) is quickly reduced to the preceding case by writing $I=\alpha I_0-\beta I_1$, where I_0 and I_1 are positive linear functionals such that $I_0(1)=I_1(1)=1$. (It is easy to verify that the standard construction for I_0 and I_1 makes them p-continuous if I is p-continuous). Since the only ring-homomorphisms of X are of the form M_p, the second statement of the theorem is established.

To establish the remainder of the theorem, suppose that X is a non-Q-space imbedded in the space νX. Every function in $\mathfrak{C}(X,R)$ can be uniquely extended over νX so as to be in $\mathfrak{C}(\nu X,R)$, so that the rings $\mathfrak{C}(\nu X,R)$ and $\mathfrak{C}(X,R)$ are algebraically isomorphic. The p-topology in $\mathfrak{C}(X,R)$ may be strictly weaker than the p-topology in $\mathfrak{C}(\nu X,R)$, however. In any case, a linear functional I which is p-continuous on $\mathfrak{C}(X,R)$ is certainly p-continuous on $\mathfrak{C}(\nu X,R)$, and by our remarks above, $I=\sum_{i=1}^{n}\alpha_i M_{p_i}+\sum_{j=1}^{m}\beta_j M_{q_j}$, where $\{p_1,p_2,\ldots,p_n\}\subset X$ and $\{q_1,q_2,\ldots,q_m\}\subset\nu X\cap X'$. It is easy to see that all β_j must be zero. Suppose that some β_{j_0}, say β_1, is not zero. The point q_1 in $\nu X\cap X'$ corresponds to a ring-homomorphism of $\mathfrak{C}(X,R)$ not of the form M_p for $p\in X$. There is a function ϱ in $\mathfrak{C}(\nu X,R)$ such that $\varrho(q_1)=1$, $\varrho(q_2)=\ldots=\varrho(q_m)=0$, and ϱ also vanishes on an arbitrarily pre-assigned finite subset of X. It follows that I cannot be continuous in the p-topology for $\mathfrak{C}(X,R)$, and the first statement of the theorem is verified. The final statement of the theorem follows at once from our remark on q_1. If there is a point $q_1\in\nu X\cap X'$, then the ring-homomorphism of $\mathfrak{C}(X,R)$ corresponding to q_1 is not p-continuous.

An entirely analogous situation exists with respect to k-continuous linear functionals.

Theorem 21. *Let X be any CR-space. Then every k-continuous linear functional on $\mathfrak{C}(X,R)$ can be represented in the form $I(f)=\int_F f(x)\,d\gamma^*$, where F is a compact subset of X and γ^* is an outer measure on X which is countably additive on $\overline{\mathfrak{P}}(F)$.*

Let I be positive and let $I(1)=1$. If I is k-continuous, it is bounded (Theorem 19) and hence has a representation as the integral with respect to a Baire measure γ (Theorem 14). If for every compact subset F of X, we have $\gamma^*(F)<1$, then it can be shown, as in the proof of Theorem 20, that I cannot be k-continuous. Observing that the case of a general k-continuous I can be reduced to the case just treated, we may assert that the present theorem is valid.

Theorem 22. *If X is a Q-space, then a linear functional I on $\mathfrak{C}(X,R)$ is k-continuous if and only if it is bounded. If X is not a Q-space, there exists a bounded linear functional (indeed, a ring-homomorphism onto R) on $\mathfrak{C}(X,R)$ which is not k-continuous.*

The first statement of this theorem follows at once from Theorems 21 and 18. To prove the second, let X be a non-Q-space, $\mathcal{M}$ a subfamily of $\mathfrak{Z}(X)$ as described in the proof of Theorem 16, M the ring-homomorphism of $\mathfrak{C}(X,R)$ corresponding to $\mathcal{M}$, and β and β^* the measures corresponding to M. If K is any compact subset of X, then $\beta^*(K)=0$, as noted in the proof of Theorem 17. Let G be any set in $\mathfrak{P}(X)$ such that $G\supset K$ and $\beta(G)=0$. As observed in the proof of Theorem 18, there is a function $\sigma\in\mathfrak{C}(X,R)$ such that $\sigma=0$ on K, $\sigma=1$ on G', and $0\leqslant\sigma\leqslant 1$ throughout X. Then σ belongs to the k-neighborhood of 0 which consists of all functions $h\in\mathfrak{C}(X,R)$ with $|h|<\varepsilon$ on K, but $M(\sigma)=\int_X\sigma(x)\,d\beta=1$. Hence the ring-homomorphism M is not k-continuous.

We next consider the roles played by u- and m-continuity.

Theorem 23. *If the CR-space X admits any unbounded continuous real-valued functions, then $\mathfrak{C}(X,R)$ admits u-continuous linear functionals which are unbounded and hence not k-continuous. Furthermore, $\mathfrak{C}(X,R)$ admits m-continuous linear functionals which are not u-continuous.*

These assertions are verified by appropriate selections of Hamel bases in $\mathfrak{C}(X,R)$. (For a discussion of Hamel bases, see, for example, [11], pp. 157-158). It is easy to see that there exists a Hamel basis for $\mathfrak{C}(X,R)$ over R of the form $\mathfrak{H}^*\cup\mathfrak{H}$, where $\mathfrak{H}^*\subset\mathfrak{C}^*(X,R)$ and $\mathfrak{H}\cap\mathfrak{C}^*(X,R)=0$. $\mathfrak{H}$ can furthermore be chosen in such a way that there are functions $\varphi_1\leqslant\varphi_2$ in $\mathfrak{H}$ and a sequence $\{f_n\}_{n=1}^{\infty}\subset\mathfrak{H}$ such that $\varphi_1\leqslant f_n\leqslant\varphi_2$ for all n. Let $\bar{I}$ be any bounded linear functional on $\mathfrak{C}^*(X,R)$. It is obvious that such functionals exist in great profusion. (Indeed, every point $w\in\beta X\cap X'$ provides a non-trivial functional of this kind, and there are at least $2^{2^{\aleph_0}}$ such points). It is clear that $\bar{I}$ determines and is determined by the set of numbers $\{\bar{I}(h)\}_{h\in\mathfrak{H}^*}$. Now define $I(\varphi_1)$ as 0, $I(\varphi_2)$ as 1, and $I(f_n)$ as n ($n=1,2,\ldots$). Assign arbitrary values to $I(g)$ for all other $g\in\mathfrak{H}$, and set $I(g)=\bar{I}(g)$ for $g\in\mathfrak{H}^*$. Let ψ be any element of $\mathfrak{C}(X,R)$. Then ψ has a unique representation of the form $\psi=\sum_{i=1}^{k}\alpha_i h_i$, where $\{h_1,h_2,\ldots,h_k\}\subset\mathfrak{H}^*\cup\mathfrak{H}$;

and $I(h)=\sum_{i=1}^{k}a_i I(h_i)$ is a linear functional defined on $\mathfrak{C}(X,R)$ which agrees with $\bar{I}$ on $\mathfrak{C}^*(X,R)$ and is therefore u-continuous. From the construction, however, it is clear that I is unbounded and therefore not k-continuous.

A large number of m-continuous linear functionals which are not u-continuous are also at hand. We may, for example, choose a Hamel basis for $\mathfrak{C}(X,R)$ and values of I for that basis which send an entire m-neighborhood of zero into zero, at the same time assigning the value n to a function which is less than $1/n$ in absolute value everywhere. We omit the details of this construction.

Remark 1. From the preceding theorem, it is apparent that integral representations simply fail to exist for a large class of continuous linear functionals on $\mathfrak{C}(X,R)$. It is true, of course, that $\mathfrak{C}(X,R)$ is not a topological linear space under the m- or the u-topology if X admits any unbounded real-valued continuous functions; and this fact makes Theorem 23 less surprising than it might otherwise be.

Remark 2. G. Sirvint [17] has stated that any k-continuous linear functional I on the space $\mathfrak{C}(N_\alpha,R)$ has the form $I=\sum_{i=1}^{k}\beta_i M_{p_i}$, where the β_i are arbitrary real numbers. This result is a special case of Theorem 21, and is closely related to Mackey's theorem [12].

5. The space of linear functionals. Let X be any CR-space, and let $\boldsymbol{B}$ be the space of all bounded linear functionals defined on $\mathfrak{C}(X,R)$; we define addition and multiplication by real numbers within $\boldsymbol{B}$ in the usual way. A number of different topologies can be imposed on $\boldsymbol{B}$, which we shall now discuss. For an arbitrary finite subset $\{f_1,f_2,\dots,f_k\}$ of $\mathfrak{C}(X,R)$ and an arbitrary positive real number ε, let $\boldsymbol{U}(f_1,f_2,\dots,f_k;\varepsilon)$ be the set of all functionals $I\in\boldsymbol{B}$ such that $|I(f_i)|<\varepsilon$ $(i=1,2,\dots,k)$. As $\{f_1,f_2,\dots,f_k\}$ and ε assume all possible values, the corresponding sets $\boldsymbol{U}$ describe a complete family of neighborhoods of zero in $\boldsymbol{B}$. Neighborhoods of other functionals are defined by translation, and the resulting topology for $\boldsymbol{B}$ is the ordinary weak topology as customarily defined for Banach spaces. We shall call this topology the weak topology in $\boldsymbol{B}$.

Next, Arens [3] has observed that if one has an appropriate family $\mathcal{B}$ of bounded sets in $\mathfrak{C}(X,R)$, one can define a topology (the b-topology associated with this family of bounded sets) on $\boldsymbol{B}$ in the following manner. A generic neighborhood $(\mathfrak{A}/a)$ of zero in $\boldsymbol{B}$ consists of all $I\in\boldsymbol{B}$ such that $\sup_{f\in\mathfrak{A}}|I(f)|<a$, where $\mathfrak{A}\in\mathcal{B}$ and a is a positive real number. The p-, k-, u-, and m-topologies for $\mathfrak{C}(X,R)$ give rise to four families of bounded sets in $\mathfrak{C}(X,R)$, when we apply the usual definition of boundedness with respect to a given topology ([21], p. 158).

$\mathcal{B}_p$ consists of all sets $\mathfrak{A}\subset\mathfrak{C}(X,R)$ such that $\sup_{f\in\mathfrak{A}}|f(p)|<\infty$ for all $p\in X$.

$\mathcal{B}_k$ consists of all sets $\mathfrak{A}\subset\mathfrak{C}(X,R)$ such that $\sup_{f\in\mathfrak{A}}(\sup_{x\in K}|f(x)|)<\infty$ for every compact subset K of X.

$\mathcal{B}_u$ consists of all sets $\mathfrak{A}\subset\mathfrak{C}(X,R)$ such that $\sup_{f\in\mathfrak{A}}(\sup_{x\in X}|f(x)|)<\infty$.

We shall have no need to discuss the topology for $\boldsymbol{B}$ associated with the m-topology on $\mathfrak{C}(X,R)$ and hence omit a description of $\mathcal{B}_m$.

The topologies on $\boldsymbol{B}$ which we obtain by applying Arens's definition of the b-topology to the families $\mathcal{B}_p$, $\mathcal{B}_k$, and $\mathcal{B}_u$, are designated as the b_p-, b_K-, and b_u-topologies, respectively. We are particularly interested in the b_u- and b_k-topologies.

Theorem 24. *In the b_u-topology, $\boldsymbol{B}$ is a normed linear space, where $\|L\|=\sup_{-1\leqslant f\leqslant+1}|L(f)|$. In general, $\boldsymbol{B}$ is incomplete in the metric induced by this norm.*

A generic neighborhood of zero, $(\mathfrak{A}/a)$, in the b_u-topology consists of all $I\in\boldsymbol{B}$ such that $\sup_{f\in\mathfrak{A}}|I(f)|<a$, $\mathfrak{A}$ being any set in $\mathfrak{C}(X,R)$ whose elements all satisfy an inequality $-\beta\leqslant f\leqslant+\beta$ $(\beta>0)$. If $(\mathfrak{A}/a)$ is any b_u-neighborhood of zero, then, and if $\|I\|<a/\beta$, it follows that $I\in(\mathfrak{A}/a)$. It is obvious that every neighborhood of zero in the norm topology is a b_u-neighborhood of zero. Hence the topologies are identical. Verification of the usual norm properties for $\|I\|$ is very easy and is accordingly passed over.

To prove that $\boldsymbol{B}$ need not be complete, consider the space $[0,1)$, and let L_n be the functional on $\mathfrak{C}([0,1),R)$ such that $L_n(f)=\int_0^{1-\frac{1}{n}}f(x)\,dx$ $(n=2,3,\dots)$. It is clear that $\|L_n-L_m\|=|n^{-1}-m^{-1}|$, so that the sequence $\{L_n\}_{n=2}^{\infty}$ is a Cauchy sequence in $\boldsymbol{B}$. The limit functional L_0, however, is defined by the relation $L_0(f)=\int_0^1 f(x)\,dx$. It is obvious that L_0 is a linear functional on $\mathfrak{C}^*([0,1),R)$ which cannot be extended over $\mathfrak{C}([0,1),R)$ so as to be bounded. Hence $\boldsymbol{B}$ is incomplete in this case.

On the other hand, we find the following result for the b_k-topology.

Theorem 25. *Let X be a Q-space, and let $\boldsymbol{B}$ be the space of bounded linear functionals on $\mathfrak{C}(X,R)$. Under the b_k-topology, $\boldsymbol{B}$ is a complete, partially ordered, locally convex, topological linear space.*

If X is a Q-space, then $\boldsymbol{B}$ is exactly the set of k-continuous linear functionals on $\mathfrak{C}(X,R)$ (Theorem 22); and every such functional can be represented as an integral with respect to a Baire measure on a compact subset of X. Certainly $\boldsymbol{B}$ is a linear space over R under the usual definitions of addition and multiplication by real numbers; if I and J are bounded and α and β are real numbers, $\alpha I+\beta J$ is also bounded. Now consider the family $\mathscr{B}_k$. A generic set in this family is of the form $\{K,\alpha_K\}$ ($\{K\}$ being the family of all compact subsets of X and α_K being an arbitrary positive real number); $\{K,\alpha_K\}$ consists of all $f\in\mathfrak{C}(X,R)$ such that $\sup_{x\in K}|f(x)|\leqslant\alpha_K$ for all $K\in\{K\}$. A generic neighborhood of 0 in the b_k-topology for $\boldsymbol{B}$ consists of all I such that $\sup|I(f)|<1$, where f runs through all functions in a set $\{K,\alpha_K\}$. Denote this neighborhood by the symbol $\boldsymbol{V}(\{K,\alpha_K\})$. It is plain that if $I,J\in\boldsymbol{V}(\{K,\alpha_K/2\})$, then $I-J\in\boldsymbol{V}(\{K,\alpha_K\})$. To prove that multiplication by real numbers is continuous, it suffices to observe that if $\boldsymbol{V}(\{K,\alpha_K\})$ is any neighborhood of 0 and I is any element of $\boldsymbol{B}$, there is a positive real number δ_0 such that $|\delta|<\delta_0$ implies that $\delta I\in\boldsymbol{V}(\{K,\alpha_K\})$. If I is a positive functional, there exists a compact subset K_0 of X and a non-negative Baire measure γ^* on K_0 such that $I(1)=\gamma^*(K_0)>0$ and $I(f)=\int_{K_0}f(x)\,d\gamma^*$ (Theorem 18). Let $\delta_0=1/(2\alpha_{K_0}\gamma^*(K_0))$. This value of δ_0 clearly has the required property. Subtraction being a continuous operation, this result holds for an arbitrary I which is the difference of two non-negative functionals. Thus $\boldsymbol{B}$ is a topological linear space. The partial ordering in $\boldsymbol{B}$ is defined in the usual way, and it is clear that $\boldsymbol{B}$ is locally convex.

It remains only to prove that $\boldsymbol{B}$ is complete in the uniform structure defined by the neighborhoods $\boldsymbol{V}(\{K,\alpha_K\})$. To accomplish this, consider any Cauchy filter $\mathscr{F}$ of sets in $\boldsymbol{B}$. (See [6], pp. 99-102). For every neighborhood $\boldsymbol{V}(\{K,\alpha_K\})$ of 0 in $\boldsymbol{B}$, there is a set $\boldsymbol{F}\in\mathscr{F}$ such that if $I,J\in\boldsymbol{F}$, then $I-J\in\boldsymbol{V}(\{K,\alpha_K\})$. Let $I_{\boldsymbol{F}}$ be any functional in $\boldsymbol{F}$, and let g be a fixed element of $\mathfrak{C}(X,R)$. The set of numbers $\{I_{\boldsymbol{F}}(g)\}_{\boldsymbol{F}\in\mathscr{F}}$ is clearly a directed set under the relation

of inclusion in $\mathscr{F}$. It is plain that $\{I_{\boldsymbol{F}}(g)\}_{\boldsymbol{F}\in\mathscr{F}}$ admits a Moore-Smith limit, which we denote by $I_0(g)$. One may verify without difficulty that I_0 is linear and bounded and that the filter $\mathscr{F}$ converges to I_0. Hence $\boldsymbol{B}$ is complete.

If X is a non-Q-space, the description of $\boldsymbol{B}$ in the b_k-topology presents a number of technical difficulties, and we accordingly dismiss this subcase.

It is well-known that the unit sphere in the conjugate space of a Banach space is compact in the weak topology. This property fails in our present situation, if the norm introduced in Theorem 24 is used to describe the unit sphere in $\boldsymbol{B}$. Let X be the space N_0, identified with the positive integers: $N_0=\{1,2,3,\ldots,n,\ldots\}$. Let $\psi\in\mathfrak{C}(N_0,R)$ have the values $\psi(n)=n$. Introducing the weak topology in $\boldsymbol{B}$ is equivalent to imbedding $\boldsymbol{B}$ in the Cartesian product $P_{f\in\mathfrak{C}(X,R)}R_f$ ($R_f=R$) where $I\in\boldsymbol{B}$ is mapped into the element $\{I(f)\}_{f\in\mathfrak{C}(X,R)}$ in the Cartesian product. In our example, the set $\{M_n\}_{n=1}^{\infty}$ goes into a set of points in $P_{f\in\mathfrak{C}(N_0,R)}R_f$ whose „co-ordinates" on the „axis" corresponding to the function ψ have the values n $(n=1,2,\ldots)$. This shows that the unit sphere cannot be compact.

We obtain a yet weaker topology for $\boldsymbol{B}$ by defining $\boldsymbol{W}(f_1,f_2,\ldots f_n;\varepsilon)$ as the set of all $I\in\boldsymbol{B}$ such that $|I(f_i)|<\varepsilon$ $(i=1,2,\ldots,n)$, where $f_1,f_2,\ldots,f_n$ are elements of $\mathfrak{C}^*(X,R)$ and ε is an arbitrary positive real number. Theorem 1 shows that the topology obtained by using these sets as a complete family of neighborhoods of 0 is a T_1-topology. We may call it the *weak topology for $\boldsymbol{B}$. Since $\boldsymbol{B}$ is obviously a topological group under this topology, it follows that $\boldsymbol{B}$ is a CR-space under the *weak topology. In this case, we have:

Theorem 26. *Let $\boldsymbol{S}[\alpha]$ be the set of all functionals I in $\boldsymbol{B}$ such that $\|I\|\leqslant\alpha$ (α is any positive real number). Then $\boldsymbol{S}[\alpha]$ is a compact set in the *weak topology of $\boldsymbol{B}$.*

We observe that $|I(f)|\leqslant\alpha\cdot\sup_{x\in X}|f(x)|$, and that the imbedding of $\boldsymbol{B}$ into $P_{f\in\mathfrak{C}^*(X,R)}R_f$ carries $\boldsymbol{S}[\alpha]$ into a subset of a Cartesian product of closed finite intervals. It is simple to show, just as in the case of a Banach space, that the image of $\boldsymbol{S}[\alpha]$ in this Cartesian product is closed and is hence compact.

We make a final observation concerning the structure of the unit sphere in $\boldsymbol{B}$.

Theorem 27. The linear functionals on $\mathfrak{C}(X,R)$ which are ring-homomorphisms or the negatives of ring-homomorphisms are exactly the extreme points of the set $\boldsymbol{S}[1]$ in $\boldsymbol{B}$. If X is a Q-space, then the functionals $\pm M_p$ are the extreme points of $\boldsymbol{S}[1]$.

To avoid needless complexities, we first consider the case in which X is a Q-space. Let M be any ring-homomorphism of $\mathfrak{C}(X,R)$ onto R. Then $M=M_p$ for some $p \in X$. Assume that M_p is an interior point of a line segment in $\boldsymbol{S}[1]$: $M_p=aL_1+(1-a)L_2$, where $0<a<1$ and $L_1,L_2 \in \boldsymbol{S}[1]$. Write L_1 as P_1-N_1 and L_2 as P_2-N_2, where P_1, P_2, N_1, and N_2, are non-negative linear functionals, with corresponding Baire measures γ_1, γ_2, δ_1, and δ_2. Then we have $M_p=aP_1+(1-a)P_2-(aN_1+(1-a)N_2)$, and for any $f \in \mathfrak{C}(X,R)$, $f(p)=a\int_X f(x)\,d\gamma_1+(1-a)\int_X f(x)\,d\gamma_2-\Big(a\int_X f(x)\,d\delta_1+(1-a)\int_X f(x)\,d\delta_2\Big)$. Let A be any set in $\mathcal{Z}(X)$ not containing p, and let g be a function in $\mathfrak{C}(X,R)$ such that $g(p)=1$, $g=0$ on A, and $0\leqslant g\leqslant 1$ everywhere. Then we have

$$1=a\int_{A'} g(x)\,d\gamma_1+(1-a)\int_{A'} g(x)\,d\gamma_2-\Big(a\int_{A'} g(x)\,d\delta_1+(1-a)\int_{A'} g(x)\,d\delta_2\Big)\leqslant$$
$$\leqslant a\gamma_1(A')+(1-a)\gamma_2(A').$$

If it were true that $\gamma_1(A')<1$, we should have

$$1\leqslant a\gamma_1(A')+(1-a)\gamma_2(A')<a+(1-a)=1.$$

From this evident contradiction we infer that $\gamma_1^*(p)=\gamma_2^*(p)=1$. This implies that $P_1=P_2=M_p$ and that $N_1=N_2=0$. Thus M_p is not an interior point of any line segment in $\boldsymbol{S}[1]$. The case of a functional $-M_p$ is immediately reducible to this case.

Now suppose that $L \in \boldsymbol{B}$ and that L is not of the form $\pm M_p$. If $L=0$, then L is obviously not an extreme point of $\boldsymbol{S}[1]$. If $0<|L(1)|<1$, then $L=a(a^{-1}L)+(1-a)0$, where $a=|L(1)|$. If $L(1)=-1$, we consider the functional $-L$. Thus we have reduced our problem to functionals $L \in \boldsymbol{B}$ such that $L(1)=1$. L not being of the form M_p, Theorem 16 implies that the Baire measure γ associated with L has the property that $0<|\gamma(G)|<1$ for some $G \in \mathcal{P}(X)$. Suppose that $\gamma(G)=a>0$. Let $K_1(f)=a^{-1}\int_G f(x)d\gamma$ and let $K_2(f)=(1-a)^{-1}\int_{G'} f(x)\,d\gamma$. It is then clear that $L=aK_1+(1-a)K_2$. If the inequalities $-1<a<0$ obtain, then we have $\gamma(G')=1-\gamma(G)>1$; and thus there exists

a set $H \in \mathcal{P}(X)$ such that $H\supset G'$ and $1<\gamma(H)$. By definition of γ, there must be a function $\varrho \in \mathfrak{C}(X,R)$ such that $0\leqslant\varrho\leqslant\varphi_H$ and $1<I(\varrho)$; that is, I non $\in \boldsymbol{S}[1]$. Hence L is in no case an extreme point of $\boldsymbol{S}[1]$.

If X is a CR-space which is not a Q-space, we imbed X in $\boldsymbol{v}X$ (see [7], p. 88, Theorem 56) and proceed as before.

We now turn to weak limits of the linear subspace spanned by ring-homomorphisms.

Theorem 28. Any functional in $\boldsymbol{B}$ can be approximated arbitrarily closely in the weak topology for $\boldsymbol{B}$ by linear combinations of ring-homomorphisms; and if X is a Q-space, by linear combinations of functionals M_p.

Let X be a Q-space and let L be any functional in $\boldsymbol{B}$. By Theorem 18, $L(f)=\int_F f(x)\,d\gamma^*$, where F is a compact subset of X and γ^* is a Baire measure on F. The weak neighborhood of L defined by $f_1,f_2,\dots,f_m \in \mathfrak{C}(X,R)$ and $\varepsilon>0$ consists of all $I \in \boldsymbol{B}$ such that $|I(f_i)-L(f_i)|<\varepsilon$ $(i=1,2,\dots,m)$. For each i, there exists a division of F into pair-wise disjoint sets $A_1,A_2,\dots,A_{n_i} \in \overline{\mathcal{P}}(F)$ such that $|I(f_i)-\sum_{k=1}^{n_i}f(x_k)\gamma^*(A_k)|<\varepsilon$, where x_k is an arbitrary element of A_k $(k=1,2,\dots,n_i)$. Let $B_1,B_2,\dots,B_s$ be the family of pair-wise disjoint sets (all necessarily in $\overline{\mathcal{P}}(F)$) formed by taking all possible intersections of sets A_k and their complements. Let p_k be any point in B_k $(k=1,2,\dots,s)$. Then, clearly, the functional $\sum_{k=1}^{s}\gamma^*(B_k)M_{p_k}$ is in the $(f_1,f_2,\dots,f_m;\,\varepsilon)$ neighborhood of I. If X is not a Q-space, we imbed X in $\boldsymbol{v}X$ and apply the preceding result.

The difference between the weak topology and the b_p-, b_k-, and b_u-topologies, is illustrated by the following result.

Theorem 29. Let $\boldsymbol{K}$ denote the set of all linear combinations of ring-homomorphisms in $\boldsymbol{B}$, and let X be a Q-space. The closure of $\boldsymbol{K}$ in the b_k- and b_u-topologies consists exactly of the functionals $\sum_{n=1}^{\infty}a_nM_{p_n}$, where $\{p_n\}_{n=1}^{\infty}$ is a countable compact subset of X and $\sum_{n=1}^{\infty}|a_n|$ converges.

Suppose that I is a positive functional in $\boldsymbol{B}$ such that in every b_u-neighborhood of I there are functionals of the form $\sum_{i=1}^{m}a_iM_{p_i}$. Thus, for every positive integer k, there are points $p_1,p_2,\dots,p_{n_k} \in X$ and numbers $a_1,\dots,a_{n_k} \in R$ such that

$$\sup_{-1\leqslant f\leqslant 1}\Big|I(f)-\sum_{i=1}^{n_k}a_i(f(p_i))\Big|<k^{-1}.$$

This implies that if $f(p_1)=f(p_2)=\ldots=f(p_{n_k})=0$ and $0\leqslant f\leqslant 1$, then $|I(f)|<k^{-1}$. The Baire measure γ associated with I thus has the property that $\gamma(G)<k^{-1}$ for every set $G\in\mathfrak{P}(X)$ such that G is disjoint from the finite set $p_1,p_2,\ldots,p_{n_k}$. It is plain that the set $B=\sum_{k=1}^{\infty}\{p_1,p_2,\ldots,p_{n_k}\}$ is countable and has the property that $\gamma^*(B)=\gamma(X)=I(1)$. It is plain also that the compact set F described in Theorem 18 is contained in B and is therefore countable. Write the set F as $\{q_n\}_{n=1}^{\infty}$. The most general measure μ on F has the form $\mu(q_n)=a_n$, $\sum_{n=1}^{\infty}|a_n|<\infty$, and we see that the b_u-closure of $\boldsymbol{K}$ contains only functionals of the form $\sum_{n=1}^{\infty}a_nM_{p_n}$. It follows at once that the same is true of the b_k-closure of $\boldsymbol{K}$.

Conversely, consider any functional $I_0=\sum_{n=1}^{\infty}\beta_nM_{p_n}$, where $F=\{p_n\}_{n=1}^{\infty}$ is a compact subset of X and $\sum_{n=1}^{\infty}|\beta_n|<+\infty$. We shall show that the b_k-closure of $\boldsymbol{K}$ contains I_0. A generic b_k-neighborhood of I_0 consists of all functionals I such that $\sup|I(f)-I_0(f)|<1$, where f runs through a set $\mathfrak{A}=\{K,a_K\}$. F being compact, we have $\sup_{1\leqslant n<\infty}|f(p_n)|\leqslant a=a_F$. Thus, if $f\in\mathfrak{A}$, we have for every positive integer m, $|\sum_{n=m}^{\infty}\beta_nf(p_n)|<a\sum_{n=m}^{\infty}|\beta_n|$, and this number may be made arbitrarily small for sufficiently large values of m. Hence the number $|I_0(f)-\sum_{n=1}^{m-1}\beta_nM_{p_n}(f)|$ can be made arbitrarily small for all $f\in\mathfrak{A}$ by taking m large enough, and we see that the b_k-closure of $\boldsymbol{K}$ contains all functionals $\sum_{n=1}^{\infty}\beta_nM_{p_n}$ of the kind described. Thus, the present theorem is established.

Remark. It is not difficult to see, by means of appropriate examples, that the b_p-closure of $\boldsymbol{K}$ may be a proper subset of the set of all functionals of the kind described in Theorem 29.

6. Relations with other problems. Let X be a completely regular space and let I be a positive linear functional defined on the space $\mathfrak{C}^*(X,R)$. Then the process elaborated in § 2 can be carried out with the functional I, and one obtains a measure-function δ^* defined for all subsets of X such that

(1) $\delta^*(A\cup B)\leqslant\delta^*(A)+\delta^*(B)$ for all $A,B\subset X$;
(2) $0\leqslant\delta^*(A)$ for all $A\subset X$;
(3) $\delta^*(X)=I(1)$.

It is impossible to establish the property $\delta^*(\sum_{n=1}^{\infty}A_n)\leqslant\sum_{n=1}^{\infty}\delta^*(A_n)$ (N. B. The proof carried out in § 2 breaks down at Theorem 7, if we attempt to carry it over to the present situation). Under the usual definition of measurability, the function δ^* can be proved to be finitely additive on measurable sets; the family of measurable

sets is closed under the formation of complements and finite unions and contains the family $\mathfrak{P}(X)$. Countable additivity cannot be hoped for, as the following example shows. Let p be any point in the space $\beta N_0\cap N_0'$. Then every function in $\mathfrak{C}^*(N_0,R)$ can be continuously extended over p, and the functional $M_p(f)=f(p)$ is certainly a positive linear functional on $\mathfrak{C}^*(N_0,R)$. The measure ω on N_0 corresponding to the functional M_p vanishes for points and hence cannot be countably additive. In spite of this defect, however, the measure δ^* for general X yields an integral such that $I(f)=\int_X f(x)d\delta^*$ for all functions in $\mathfrak{C}^*(X,R)$.

A. Markov [13] has considered the problem of finding an integral representation for an arbitrary positive linear functional M ($M(1)=1$) defined on the space of all bounded continuous real-valued functions on a space X which satisfies the axiom of normality but need not satisfy any separation axiom at all. He proves that every such functional has an integral representation: $M(f)=\int_X f(x)d\mu$, where μ is an outer measure on X such that

$$\mu(A\cup B)\leqslant\mu(A)+\mu(B);\quad \mu(A\cup B)=\mu(A)+\mu(B)\ \text{ if } A^-\cap B=0;$$
$$\mu(X)=1;\quad \mu(A)=\inf\mu(G),$$

where G runs through the family of all open sets containing A. Markov's theory can be generalized and reduced to the problem treated in the preceding paragraph. Let X be any topological space. In X, identify all points $p,q,\ldots$ such that all functions in $\mathfrak{C}^*(X,R)$ have the same values at $p,q,\ldots$ Denote by $\tilde{X}$ the resulting space of subsets of X, with points $\tilde{p}=E[q;q\in X$, every $f\in\mathfrak{C}(X,R)$ is constant on $q]$. Topologize $\tilde{X}$ by setting $\tilde{U}_{f,\varepsilon}(p)=E[\tilde{q};\tilde{q}\in\tilde{X},|f(\tilde{q})-f(\tilde{p})|<\varepsilon]$; as f runs through all elements of $\mathfrak{C}(X,R)$ and ε runs through all positive real numbers, the neighborhoods $\tilde{U}_{f,\varepsilon}(\tilde{p})$ describe a complete family of neighborhoods of $\tilde{p}$. The space $\tilde{X}$ is clearly completely regular and is a continuous image of the space X under a mapping Φ. The space $\mathfrak{C}^*(\tilde{X},R)$ is identifiable with the space $\mathfrak{C}^*(X,R)$. The theory sketched above can be applied to any bounded linear functional I on $\mathfrak{C}^*(\tilde{X},R)$, to obtain a measure $\tilde{\delta}^*$ which is finitely additive for measurable sets and such that $I(f)=\int_{\tilde{X}}f(\tilde{x})d\tilde{\delta}^*$ for all $f\in\mathfrak{C}^*(\tilde{X},R)$. The measure δ^* on X such that $\delta^*(A)=\tilde{\delta}^*(\Phi(A))$ provides a similar integral representation for the functional I on $\mathfrak{C}^*(X,R)$.

In exactly the same way, we can use the results of § 2 to obtain an integral representation for a bounded linear functional defined on the space $\mathfrak{C}(X,R)$, where X is an arbitrary topological space.

A. D. Aleksandrov [1] has considered integral representations of positive linear functionals defined on spaces of continuous functions $\mathfrak{C}^*(T,R)$, where T is a space satisfying all of the axioms for a normal space, except that the union of an uncountable family of open sets need not be open. His theory, while closely related to the theory sketched in the first paragraph of the present work, cannot be immediately reduced to it. We note that the space $\mathfrak{C}^*(T,R)$ satisfies all requirements for an M-space (see [9]) and hence may be identified with the space of all real-valued continuous functions on a certain compact Hausdorff space Y, a certain subspace of which maps continuously onto T. A bounded linear functional on $\mathfrak{C}^*(T,R)$ yields the same functional on $\mathfrak{C}(Y,R)$, and with it a measure function on Y. There is, however, no obvious way of transferring the measure to the space T.

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