ω_μ-additive ideal of $\boldsymbol{K}$, also possesses this property. Consequently every such quotient algebra $\boldsymbol{K}/\boldsymbol{I}$ is isomorphic to an ω_μ-additive field of sets.

An instance of an ω_μ-complete Boolean algebra with the property (b) is the field of all both open and closed subsets of D^0_μ.

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On an Irreducible 2-dimensional Absolute Retract.

By

Karol Borsuk (Warszawa).

In 1934 Mazurkiewicz and the author of the present paper[1]) constructed in the Euclidean 3-dimensional space E_3 an absolute retract[2]) which cannot be split into finite sum of proper subcontinua having the 1-dimensional Betti number vanishing.

The purpose of this paper is to give an example of an absolute retract P_∞ (lying also in E_3) which is a 2-dimensional Cantor-surface[3]), such that every proper 2-dimensional closed subset of it has the infinite 1-dimensional Betti number. In particular P_∞ contains no 2-dimensional proper subset being an absolute retract.

1. Irreducible cuttings. A subcompactum C of the 3-dimensional Euclidean space E_3 is said to be an *irreducible cutting* of E_3 provided that $E-C$ is not connected, but for every closed proper subset A of C the set $E-A$ is connected. Any irreducible cutting of E_3 is a 2-dimensional Cantor-surface.

It is known[4]) that irreducible cuttings of E_3 can be characterized as compacta $C\subset E_3$ such that[5])

(1) $$p^2(C)>0,$$

(2) $$\text{if} \quad A=\bar{A}\subset C \quad \text{and} \quad A\neq C, \quad \text{then} \quad p^2(A)=0.$$

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[1]) K. Borsuk and S. Mazurkiewicz, *Sur les rétractes absolus indécomposables*, Comptes Rendus de l'Académie des Sciences **199** (Paris, 1934), p. 110-112.

[2]) A subset A of a space M is called a *retract* of M, if there exists a continuous mapping f (called a *retraction*) of M onto A such that $f(x)=x$ for every $x\in A$. A compactum A is said to be an *absolute retract* provided it is a retract of every space $M\supset A$.

[3]) A compact 2-dimensional space is called a *Cantor-surface* if it cannot be disconnected by any subset of dimension 0. See P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fund. Math. **7** (1925), p. 122, 123.

[4]) See P. Alexandroff and H. Hopf, *Topologie I*, Berlin, Springer 1935, p. 391.

[5]) $p^k(C)$ denotes the k-dimensional Betti number of the compactum C.

Suppose now that the irreducible cutting C of E_3 is locally connected and that

(3) $$p^1(C)=0.$$

Let A be a proper subcontinuum of C, and G a component of $C-A$. Since C is locally connected, the component G is open in C [6]. Let us observe that

(4) $$\overline{G}-G \quad \text{is a continuum.}$$

First we note that, by (3), the set C is unicoherent [7], i. e. if we split C into two continua, their common part is also a continuum. But the set $\overline{G}$ is a continuum, and so is the set $C-G$ obtained from A by addition of all components of $C-A$ different from G. We infer, by the relation $C=\overline{G}+(C-G)$, that the set $\overline{G}\cdot(C-G)=\overline{G}-G$ is a continuum.

Let B be a closed subset of the continuum A such, that $p^1(B)=0$, and $B\cdot\overline{C-A}=0$. Let us show that

(5) $$A-B \quad \text{is connected.}$$

Otherwise A would be decomposable into two closed proper subsets A_1, A_2 such that

$$A_1\cdot A_2=B.$$

By (4) and $B\cdot\overline{C-A}=0$, for every component G of $C-A$ holds at least one of the two following inclusions:

$$\overline{G}-G\subset A_1-B \quad \text{or} \quad \overline{G}-G\subset A_2-B.$$

Adding to A_1 all components of $C-A$ satisfying the first of these inclusions, we obtain [8] (with regard to local connectedness of C) a closed subset A_1^* of C. Similarly, adding to A_2 all components of $C-A$ satisfying the second of these inclusions, we obtain a closed subset A_2^* of C.

Since $A_1\neq A\neq A_2$, we infer that

$$A_1^*\subsetneqq C, \quad \text{and} \quad A_2^*\subsetneqq C.$$

[6] C. Kuratowski, *Une définition topologique de la ligne de Jordan*, Fund. Math. **1** (1920), p. 43.

[7] E. Čech, *Sur les continus Péaniens unicohérents*, Fund. Math. **20** (1933), p. 232.

[8] C. Kuratowski, *Sur les continus de Jordan et le théorème de M. Brouwer*, Fund. Math. **8** (1926), p. 140.

It follows by (2) that

$$p^2(A_1^*)=p^2(A_2^*)=0.$$

Furthermore, we have $A_1^*+A_2^*=C$, and $A_1^*\cdot A_2^*=A_1\cdot A_2=B$. By $p^1(B)=0$ and the well-known formula of Menger-Vietoris-Čech [9] it follows that $p^2(C)=0$, in contradiction to (1).

Thus the relation (5) is proved.

From (5) it follows:

(6) Let A be a subcontinuum of an irreducible locally connected cutting C of E_3 such that $p^1(C)=0$. Then A cannot be disconnected by a finite sum of disjoint simple arcs lying in $A-\overline{C-A}$.

2. Polyhedral irreducible cuttings. If the irreducible cutting C of E_3 is a polytope [10], then E_3-C contains exactly two regions [11]. One of these regions has finite diameter; it will be called the *interior region* and denoted by Γ. The other region, with infinite diameter, will be called the *exterior region* and denoted by Λ. Since C is a Cantorian surface, every triangulation τ of C is homogeneously 2-dimensional i. e. every simplex of τ is either a triangle or a face of a triangle belonging to τ.

Let L denote the normal to the triangle $T=\sigma(a_0,a_1,a_2)\in\tau$ at the barycenter [12] b of T. There exist on L two points b' and b'' different from b and such that the interior of the tetraeder $\sigma(a_0,a_1,a_2,b')$ lies in the interior region Γ and the interior of the tetraeder $\sigma(a_0,a_1,a_2,b'')$ — in the exterior region Λ. We shall say that every point belonging to the interior of $\sigma(a_0,a_1,a_2,b')$ *lies on the interior side of the triangle* T (relative to the cutting C).

[9] E. Čech, *Théorie générale de l'homologie dans un espace quelconque*, Fund. Math. **19** (1932), p. 178.

[10] We shall consider polytopes in the elementary sense, as sets contained in some Euclidean space E_n and decomposable in a finite collection of simplexes. By a k-dimensional simplex with the linear independent vertices $a_0,a_1,\dots,a_k\in E_n$ we understand the irreducible convex subset of E_n containing $a_0,a_1,\dots,a_k$; it will be denoted by $\sigma(a_0,a_1,\dots,a_k)$. It is known (see P. Alexandroff and H. Hopf, l. c. p. 136) that every polytope can be triangulated, i. e. simplicially decomposed, i. e. decomposed in a finite collection of simplexes in such a manner that the common part of each two simplexes is the simplex determined by their common vertices. The collection of all simplexes (and their faces) of a triangulation of a polytope will be called a *geometrical complex*.

[11] See P. Alexandroff and H. Hopf, l. c. p. 393.

[12] By the *barycenter* of the simplex $\sigma(a_0,a_1,\dots,a_k)$ we understand the point $b(\sigma)=\frac{1}{k+1}(a_0+a_1+\dots+a_k)$.

A plane π is said to *lie on the interior side of the triangle* T if $\pi \cdot T = 0$ and π contains at least one point lying on the interior side of T.

Two triangles T' and T'' belonging to the triangulation $\tau(C)$ are said to *adjoin* if they have one 1-face J common, and for every two points b', b'' such that b' lies on the interior side of T' and b'' on the interior side of T'' there exists in every neighborhood of the center of J a point b such, that the polygonal line $\sigma(b,b')+\sigma(b,b'')$ lies in the interior region Γ. It can be seen without much difficulty that to every 1-face J of every triangle $T' \epsilon \tau$ there exists exactly one triangle $T'' \epsilon \tau$ such, that $T' \cdot T'' = J$ and the triangles T' and T'' adjoin.

Let T' and T'' be two adjoining triangles of the triangulations $\tau(C)$ and let J denote their common face. Let π' and π'' denote two half-planes containing respectively T' and T'' and such that J lies on their common edge. There exists exactly one plane π passing by J and such that π' and π'' lie symmetrically to π. This plane π will be called the *plane separating* the triangles T' and T''. The common edge of π' and π'' cuts π into two half-planes. Exactly one of them cuts every plane lying on the interior side of the triangles T' and T''. This half-plane will be called the *half-plane of the segment* J.

3. The zone of a geometrical subcomplex of a triangulation of an irreductible cutting. Let $\tau(C)$ be a triangulation of a polyhedral irreducible cutting C of the space E_3. For every simplex $\sigma \epsilon \tau(C)$ let us denote its barycenter by $b(\sigma)$. If σ is a triangle T, then we understand by the *inner normal* of σ the ray starting from $b(\sigma)$, perpendicular to T and containing at least one point lying on the interior side of σ. If σ is a segment J, then we understand by the *interior normal* of σ the ray starting from the center $b(\sigma)$, perpendicular to the segment J and lying in its half-plane.

Let us denote, for every $t \geqslant 0$ and every simplex $\sigma \epsilon \tau(C)$ $(\sigma \neq 0)$, by $b_t(\sigma)$ the point defined in the following manner:

1⁰ If $\dim \sigma = 0$, i. e. σ contains only one vertex a, then we put $b_t(\sigma) = a$.

2⁰ If $\dim \sigma > 0$, then $b_t(\sigma)$ denotes the point lying on the interior normal of σ in the distance t from $b(\sigma)$.

Let ε be a positive number. Denote by $\gamma_\varepsilon(\sigma)$ the *ε-zone of the simplex* σ, i. e. the set defined in the following manner:

(7) If $\sigma = \sigma(a_0)$, then $\gamma_\varepsilon(\sigma) = (a_0)$,

(8) If $\sigma = \sigma(a_0, a_1)$, then $\gamma_\varepsilon(\sigma) = \sigma(a_0, a_1, b_\varepsilon(\sigma))$,

(9) If $\sigma = \sigma(a_0, a_1, a_2)$, then
$$\gamma_\varepsilon(\sigma) = \sigma(a_0, a_1, a_2, b_\varepsilon(\sigma)) + \sigma(a_0, a_1, b_\varepsilon(\sigma(a_0, a_1)), b_\varepsilon(\sigma)) + \sigma(a_0, a_2, b_\varepsilon(\sigma(a_0, a_2)), b_\varepsilon(\sigma)) + \sigma(a_1, a_2, b_\varepsilon(\sigma(a_1, a_2)), b_\varepsilon(\sigma)).$$

Hence, in the case $\sigma = \sigma(a_0, a_1, a_2)$ the ε-zone $\gamma_\varepsilon(\sigma)$ is a geometrical complex consisting of 4 tetraeders. Those tetraeders and their faces will be called the *simplexes of the zone* $\gamma_\varepsilon(\sigma)$.

Clearly, if the diameter of σ is $\leqslant \eta$ then the diameter of $\gamma_\varepsilon(\sigma)$ is $\leqslant 2\varepsilon + \eta$.

Now consider any subcomplex K of the triangulation $\tau(C)$. By $|K|$ we denote the polytope composed of all simplexes of K. By the *ε-zone of* K we mean the polytope $\gamma_\varepsilon(K)$ being the sum of ε-zones of all simplexes constituting the complex K. We see at once that, for ε sufficiently small, the simplexes of those ε-zones constitute a complex. In particular, it holds if $\varepsilon < \frac{1}{2}\varrho(x,y)$ for every two points x, y belonging to two disjoint simplexes of the triangulation $\tau(C)$. The number ε satisfying the last inequality will be said to be *adequate to the triangulation* $\tau(C)$. Speaking of an ε-zone of a subcomplex K of $\tau(C)$ we shall allways suppose that ε is adequate to the triangulation $\tau(C)$.

Remark. For ε sufficiently small the ε-zone $\gamma_\varepsilon(K)$ lies evidently in an arbitrarily given neighborhood of the subcomplex K. Furthermore, it can be easily proved, that for every polytope W contained in the polytope $C+\Gamma$ and constituting a neighborhood in $C+\Gamma$ for every point $x \epsilon |K|$ different from all vertices of the triangulation $\tau(C)$, there exists a positive number ε_0 such that, for every $0 < \varepsilon < \varepsilon_0$, the ε-zone $\gamma_\varepsilon(K)$ lies in W.

Theorem. *If C is a polyhedral irreducible cutting of E_3 and K is a subcomplex of a triangulation $\tau(C)$ of C, then for every $\varepsilon > 0$ adequate to the triangulation $\tau(C)$ there exists a mapping $r_\varepsilon(x,t)$ retracting by deformation*[13] *the ε-zone $\gamma_\varepsilon(K)$ to K in such a manner that $r_\varepsilon(x,t) = x$ for every $x \epsilon |K|$ and $0 \leqslant t \leqslant 1$, and $r_\varepsilon(x,t) \epsilon \gamma_\varepsilon(\sigma)$ for every simplex σ of K, every $x \epsilon \gamma_\varepsilon(\sigma)$, and every $0 \leqslant t \leqslant 1$.*

[13]) The mapping $r(x,t)$ will be called a *retraction* of the set X to its subset X_0 *by deformation*, if it is defined and continuous in the Cartesian product of X and of the interval $0 \leqslant t \leqslant 1$ and is such that: 1⁰ $r(x,t) \epsilon X$ for every $x \epsilon X$ and $0 \leqslant t \leqslant 1$, 2⁰ $r(x,0) = x$ and $r(x,1) \epsilon X_0$ for every $x \epsilon X$, 3⁰ $r(x,1) = x$ for every $x \epsilon X_0$.

Proof. If $x \in \gamma_\varepsilon(K)$, then there exists exactly one simplex σ of K such that x belongs to $\gamma_\varepsilon(K)$ but does not belong to the ε-zone of any proper face of σ. If $\sigma=\sigma(a_0)$, then $\gamma_\varepsilon(\sigma)=\sigma(a_0)$ and $x=a_0$. Then we put

$$r_\varepsilon(x,t)=x \quad \text{for every} \quad 0\leqslant t\leqslant 1.$$

If $\sigma=\sigma(a_0,a_1)$, then by (8) it is $\gamma_\varepsilon(\sigma)=\sigma(a_0,a_1,b_\varepsilon(a_0,a_1))$. Hence

$$x=\lambda_0\cdot a_0+\lambda_1\cdot a_1+\lambda_2\cdot b_\varepsilon(a_0,a_1),$$

where $\lambda_0,\lambda_1,\lambda_2$, are positive numbers such that $\lambda_0+\lambda_1+\lambda_2=1$.

In this case we put

$$r_\varepsilon(x,t)=\lambda_0\cdot a_0+\lambda_1\cdot a_1+\lambda_2\cdot b_{(1-t)\varepsilon}(a_0,a_2).$$

If $\sigma=\sigma(a_0,a_1,a_2)$, then x lies in one of the four tetraeders appearing on the right side of the formula (9). If $x\in\sigma(a_0,a_1,a_2,b_\varepsilon(\sigma))$, then

$$x=\lambda_0\cdot a_0+\lambda_1\cdot a_1+\lambda_2\cdot a_2+\lambda_3\cdot b_\varepsilon(\sigma).$$

In this case we put

$$r_\varepsilon(x,t)=\lambda_0\cdot a_0+\lambda_1\cdot a_1+\lambda_2\cdot a_2+\lambda_3\cdot b_{(1-t)\varepsilon}(\sigma).$$

If $x\in\sigma(a_{i_0},a_{i_1},b_\varepsilon(\sigma(a_{i_0},a_{i_1})),b_\varepsilon(\sigma))$, where $0\leqslant i_0<i_1\leqslant 2$, then

$$x=\lambda_0\cdot a_{i_0}+\lambda_1\cdot a_{i_1}+\lambda_2\cdot b_\varepsilon(\sigma(a_{i_0},a_{i_1}))+\lambda_3\cdot b_\varepsilon(\sigma).$$

In this case we put

$$r_\varepsilon(x,t)=\lambda_0\cdot a_{i_0}+\lambda_1\cdot a_{i_1}+\lambda_2\cdot b_{(1-t)\varepsilon}(\sigma(a_{i_0},a_{i_1}))+\lambda_3\cdot b_{(1-t)\varepsilon}(\sigma).$$

We verify that the transformation $r_\varepsilon(x,t)$ defined in this manner is a retraction of the ε-zone $\gamma_\varepsilon(K)$ to $|K|$ by deformation, and that for every $0\leqslant t\leqslant 1$ and $x\in\sigma$ the point $r_\varepsilon(x,t)$ lies always in the set $\gamma_\varepsilon(\sigma)$.

Thus the proof of the theorem is finished.

In particular, if the polytope $|K|$ is contractible to a point (i. e. there exists a transformation $\varphi(x,t)$ retracting $|K|$ by deformation to a point), then putting

$$\psi(x,t)=r_\varepsilon(x,2t) \qquad \text{for} \quad 0\leqslant t\leqslant \tfrac{1}{2},$$
$$\psi(x,t)=\varphi(r_\varepsilon(x,1),\,2t-1) \quad \text{for} \quad \tfrac{1}{2}\leqslant t\leqslant 1$$

we obtain a transformation ψ retracting $\gamma_\varepsilon(K)$ by deformation to a point.

Using the theorem that a polytope is an absolute retract [14]) if and only if is contractible, we obtain the following

Corollary. *$|K|$ is an absolute retract if and only if $\gamma_\varepsilon(K)$ is an absolute retract.*

4. Smoothly connected subpolytopes of a polyhedral irreducible cutting. Let P be a homogeneously 2-dimensional subpolytope of a polyhedral irreducible cutting C of the space E_3. There exists a triangulation $\tau(C)$ of C such that P is representable as a subcomplex K of C. This subcomplex will be said *smoothly connected on* C if for every two triangles T and T' of it there eixsts in K a finite sequence of triangles

$$T=T_0,T_1,\ldots,T_k,T_{k+1}=T'$$

such that T_i and T_{i+1} adjoin for every $i=0,1,\ldots,k$. Obviously this property is independent from the choice of the triangulation $\tau(C)$; it depends only upon the polytopes P and C. Consequently, we can speak of the smooth connectivity of the polytope P lying on the polytope C.

5. Flat rosaries. Let C be a polyhedral irreducible cutting of the space E_3 with the 1-dimensional Betti number vanishing. Let P be a smoothly connected subpolytope of C. Consider a triangulation $\tau(C)$ of C such, that P constitutes a subcomplex K of $\tau(C)$. Let R denote the sum of all triangles of $\tau(C)$ not belonging to K. Hence $R=\overline{C-P}$.

From (6) (where we put $A=P$) and from the smooth connectivity of P we infer, that there exists for every triangle $T\in K$ a polygonal simple arc

$$L_T\subset P-R$$

such that:

1° L_T has as its starting point a_T' an interior point of the triangle T.

2° L_T has as its end point a_T'' an interior point of a triangle of K.

3° If T and T' are two different simplexes of K, then $L_T\cdot L_{T'}=0$, and $L_T\cdot T'\neq 0$.

[14]) See K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), p. 229.

4⁰ No vertex of K lies on L_T.

5⁰ If a is a point of L_T lying on a segment $J \in K$, then there exists a neighborhood U of a such that $U \cdot L_T$ is formed of two segments perpendicular to J and lying in two adjoining triangles of K.

If follows that there exists a natural number n such that L_T is decomposable into a sum of $n+2$ segments [15])

$$L_T = L_{T,0} + L_{T,1} + \ldots + L_{T,n+1}$$

having disjoint interiors and satisfying the two following conditions:

6⁰ The interior of every segment $L_{T,i}$ lies in the interior of one of the triangles of K.

7⁰ $L_{T,i}$ has as its end points $a_{T,i}$ and $a_{T,i+1}$ and $a_{T,0} = a'_T$, $a_{T,n+2} = a''_T$.

Let $b_{T,i}$ denote the centre of the segment $L_{T,i}$.

There exists a positive number a so small, that the distance of $b_{T,i}$ from every segment $L_{T',i'} \neq L_{T,i}$ (for every two triangles $T, T' \in K$ and every two indices i, i') is $> a\sqrt{2}$ and also the distance of $b_{T,i}$ from every segment belonging to K is $> a\sqrt{2}$. Let $Q_{T,i}$ denote the quadrat lying in P and having $b_{T,i}$ as its centre, a as the length of the sides and let the direction of one pair of its sides be parallel to $L_{T,i}$. The choice of the number a implies that the quadrates $Q_{T,i}$ are disjoint sets (see Fig. 1) lying in the interiors of the triangles of K, and that the common part of $Q_{T,i}$ with the polygonal line $\sum_T L_T$ is a sub-segment of $L_{T,i}$ having $b_{T,i}$ as its centre and a as its length. Let us denote the end points of this segment (ordered as they appear on the oriented segment $L_{T,i}$ from $a_{T,i}$ to $a_{T,i+1}$) by $a'_{T,i}$ and $a''_{T,i}$.

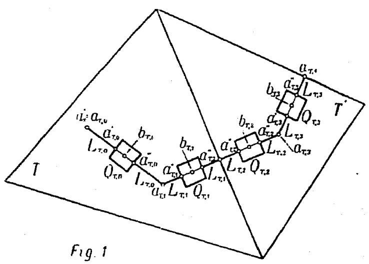

Fig. 1

[15]) Obviously we can assume that the number n is independent from T. This assumption is of no true importance for the sequel but simplifies the notations.

Let us put:

(10) $$M_{T,i} = Q_{T,i} + \overline{a''_{T,i} a_{T,i+1}} + \overline{a_{T,i+1} a'_{T,i+1}} + \dot{Q}_{T,i+1},$$

where $\dot{Q}_{T,i+1}$ denotes the boundary of the quadrate $Q_{T,i+1}$.

Putting

(11) $$M_T = \sum_{i=0}^{n} M_{T,i},$$

(12) $$M = \sum_T M_T,$$

we shall say that M is a *flat rosary* (corresponding to the triangulation τ) for the polytope P. The polytopes M_T will be called *components* of M, and the polytopes $M_{T,i}$ — links of M. The quadrates $Q_{T,i}$ will be called *quadrates* of the rosary M, their sides — *sides* of the rosary M, the segments of the form $\overline{b_{T,i} a_{T,i+1}}$ — *exit segments* and the segments of the form $\overline{a_{T,i+1} a'_{T,i+1}}$ — *entrance segments* of the rosary M. The sides, exit segments, and entrance segments will be jointly called *segments* of the rosary M and the quadrates and segments will be jointly called *elements* of the rosary M.

If the diameters of the simplexes of the triangulation τ are all $\leqslant \eta$ it will be said that the flat rosary M corresponding to the triangulation τ is a *flat η-rosary*. Clearly the diameters of all links of the η-rosary are $< 2\eta$.

6. Space rosary. Let ε be a positive number adequate to the triangulation $\tau(C)$ and supposed arbitrarily small. Consider an ε-zone $\gamma_\varepsilon(K)$ of the complex K and choose a positive number β so small that:

1. If E is an element of the flat rosary M contained in the triangle $T_0 \in K$, and x is a point lying on the interior side of T_0 in the distance $\leqslant 2\beta$ from E, then $x \in \gamma_\varepsilon(K)$. In particular, if E lies in the interior of T_0, then $x \in \gamma_\varepsilon(T_0)$.

2. If x, y are two points belonging to two disjoint segments of the rosary M, then $\varrho(x,y) > 2\beta$.

3. The length a of the sides of the quadrates of the rosary M is $> 4\beta$.

Consider for every quadrate $Q_{T,i}$ of the rosary M the pyramid $\Delta(Q_{T,i})$ lying on the interior side and having the height equal to β. It follows from 3. that the triangles being faces of this pyramid constitute with its base the angles $< 30^0$.

Let us observe that the intersection of the pyramid $\Delta(Q_{T,i})$ with the plane parallel to the base and lying on the interior side in the distance $\beta/2$ from the base is a quadrate, its sides having the lengths equal to $\alpha/2>2\beta$.

Now let us consider a side J of a quadrate $Q_{T,i}$ of the rosary M. Let $T_0\in K$ denote the triangle containing $Q_{T,i}$. Consider two planes π_1, and π_2 containing J and constituting with the normal to T_0 the angles 30^0. Futhermore let us draw through the centre $b_{T,i}$ of the quadrate $Q_{T,i}$ two planes π_3 and π_4 perpendicular to T_0 and passing through both end points of J. Finally let us denote by π_5 the plane parallel to T_0 and lying on the interior side of T_0 in the distance $\beta/2$ from T_0. Clearly there exists exactly one region bounded by planes $\pi_1,\pi_2,\dots,\pi_5$ such that its closure $\nabla(J)$ is a polytope its common part with K being the segment J. The polytope $\nabla(J)$ has the shape of a prism cut off obliquely; the normal profile of this prism is an equilateral triangle with the length of the sides equal to $\beta/\sqrt{3}<\beta$. The distance of all points of the polytope $\nabla(J)$ from the quadrate $Q_{T,i}$ is $<\beta$. From 1. we infer that they belong to $\gamma_\varepsilon(T_0)\subset\gamma_\varepsilon(K)$. Futhermore we see at once that $\gamma_\varepsilon(K)$ constitutes in the set $C+\Gamma$ a neighborhood of the set $\nabla(J)$. Finally let us remark that by 2. the sets $\nabla(J)$ and $\nabla(J')$ for disjoint sides J and J' are disjoint.

For every quadrate $Q_{T,i}$ with the sides J_1, J_2, J_3, J_4 let us denote by $\nabla(\dot{Q}_{T,i})$ the sum $\sum_{\nu=1}^{4}\nabla(J_\nu)$. Clearly $\nabla(\dot{Q}_{T,i})$ is a polytope homeomorphic to the anchor ring, and the boundary $\dot{Q}_{T,i}=\nabla(\dot{Q}_{T,i})\cdot C$ of the quadrate $Q_{T,i})$ corresponds to one of the parallels of the anchor ring.

We now consider two segments of the rosary M with a common end point $a_{T,i+1}$: an *exit segment* $J=\overline{b_{T,i},a_{T,i+1}}$ and an *entrance segment* $J'=\overline{a_{T,i+1},a'_{T,i+1}}$. Let us draw through J two planes π_1, and π_2 cutting the normal to the triangle $T_0\supset J$ at the angles 30^0. Similarly let us draw through J' two planes π'_1 and π'_2 cutting the normal to the triangle $T'_0\supset J'$ at the angles 30^0. Denote by π_3 the plane perpendicular to J and passing through $b_{T,i}$ and by π'_3 the plane perpendicular to J' and passing through $a'_{T,i+1}$. Furthermore we draw through $a_{T,i+1}$ a plane π_4 such that the rays starting from $a_{T,i+1}$ and containing respectively J and J' are symmetric to π_4; it is easy to see that if the triangles T_0 and T'_0 are different, the plane π_4 is the *plane separating* those triangles (defined in **2**).

Finally let us denote by π_5 and π'_5 two planes passing parallelly respectively to the triangles T_0 and T'_0 on their interior sides in the distance β from the segments J and J'. It is clear that the planes $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ constitute the boundary of exactly one region such that its closure $\nabla(J)$ is a polytope, and $\nabla(J)\cdot T_0=J$. By 1. the polytope $\nabla(J)$ lies in the zone $\gamma_\varepsilon(T_0)$, and the zone $\gamma_\varepsilon(K)$ constitutes a neighborhood of $\nabla(J)$ in the set $C+\Gamma$. Similarly the planes $\pi'_1, \pi'_2, \pi'_3, \pi'_4, \pi'_5$, constitute the boundary of one connected region such that its closure $\nabla(J')$ is a polytope, and $\nabla(J')\cdot T'_0=J'$. The zone $\gamma_\varepsilon(K)$ constitutes in $C+\Gamma$ a neighborhood of $\nabla(J')$.

Let us also consider two planes π_1^* and π_2^* passing through the exit segment $J=\overline{b_{T,i}a_{T,i+1}}$ at the angles 45^0 to the normal to the triangle T_0. Let π_3^* denote the plane perpendicular to J and passing through the centre $b_{T,i}$ of $Q_{T,i}$. The planes $\pi_1^*, \pi_2^*, \pi_3^*, \pi_4, \pi_5$ constitute the boundary of a polyhedral region. Let us denote its closure by $\vee(J)$. It is easy to see that the set $\nabla^*(\dot{Q}_{T,i})$ defined as the closure of the set

$$\nabla(\dot{Q}_{T,i})-\vee(J)$$

is a polytope homeomorphic to a figure obtained from the sphere by matching two different points of its surface. The boundary $\dot{Q}_{T,i}$ of the quadrate $Q_{T,i}$ constitutes the common part of the polytopes $\nabla^*(\dot{Q}_{T,i})$ and C.

The polytope $\nabla(J)+\nabla(J')$ constitutes a kind of a bar, with the triangular profile, joining the pyramid $\Delta(Q_{T,i})$ with the polytope $\nabla(\dot{Q}_{T,i+1})$ in such a manner that it adheres to the triangles T_0 and T'_0 along the segments J and J' and meets the polytope $\nabla^*(\dot{Q}_{T,i})$ only in the point $a''_{T,i}$.

We put:

$$N_{T,i}=\Delta(Q_{T,i})+\nabla(J)+\nabla(J')+\nabla^*(\dot{Q}_{T,i+1}), \tag{13}$$

$$N_T=\sum_{i=0}^{n}N_{T,i}, \quad N=\sum_{T}N_T. \tag{14}$$

The polytope N will be called the *space rosary* (corresponding to the triangulation τ and to the zone $\gamma_\varepsilon(K)$) for the polytope P. The polytopes N_T will be called *components* and the polytopes $N_{T,i}$ — the *links* of this rosary.

It is clear that the common part of the polytope C and the space rosary N is the flat rosary M. We shall say that M is the *base* of N. Besides

$$C\cdot N_{T,i}=M_{T,i}; \quad C\cdot N_T=M_T.$$

By our construction the rosary N lies in the zone $\gamma_\varepsilon(K)$ and moreover the zone $\gamma_\varepsilon(K)$ constitutes a neighborhood of N in the set $C+\Gamma$. Furthermore it is easy to see, that for every point $x\in N_{T,i}$ there exists exactly one point of $M_{T,i}$ next to x; let us denote it by $r_1(x)$. We define now a transformation $r_{T,i}(x,t)$ retracting the polytope $N_{T,i}$ to the polytope $M_{T,i}$ by deformation, putting $r_{T,i}(x,t)$ equal to the point of the segment $\overline{x\,r_1(x)}\subset N_{T,i}$ which divides this segment in the ratio $t:(1-t)$. The transformation $r_{T,i}(x,t)$ evidently satisfies the condition:

(15) $\quad \varrho(r_{T,i}(x,t),x)\leqslant\varepsilon \quad$ for every $\quad x\in N_{T,i}, \quad 0\leqslant t\leqslant 1.$

Now if we put

$$r_1(x,t)=x \quad \text{for} \quad x\in P \quad \text{and} \quad 0\leqslant t\leqslant 1,$$
$$r_1(x,t)=r_{T,i}(x,t) \quad \text{for} \quad x\in N_{T,i} \quad \text{and} \quad 0\leqslant t\leqslant 1,$$

we obtain a mapping $r_1(x,t)$ retracting the polytope $P+N$ to P by deformation. In view of the inequality (15) we obtain the following.

Theorem. *There exists a mapping $r_1(x,t)$ retracting by deformation the polytope $P+N$ to the polytope P in such a manner that*

$$\varrho(r_1(x,t),x)\leqslant\varepsilon \quad \textit{for every} \quad x\in P+N \quad \textit{and} \quad 0\leqslant t\leqslant 1.$$

Repeating the reasoning used at the end of **3** we have

Corollary 1. *In order that P should be an absolute retract it is necessary and sufficient that $P+N$ be an absolute retract.*

Finally, let us remark that if σ is a simplex of the complex K then the mapping $r_1(x,t)$ considered only for

$$x\in\gamma_\varepsilon(\sigma)\cdot(P+N)$$

is a retraction by deformation of the polytope $\gamma_\varepsilon(\sigma)\cdot(P+N)$ to the simplex σ. Thus we have

Corollary 2. *The set $(P+N)\cdot\gamma_\varepsilon(\sigma)$ is an absolute retract.*

By the construction every link $N_{T,i}$ of N lies in one or in two adjoined triangles of the triangulation τ. Hence the diameter of $N_{T,i}$ is $\leqslant 2(\varepsilon+\eta)$, where η denotes (as at the end of **5**) such a positive number that the diameters of all simplexes of the triangulation τ are less or equal to η.

The link $N_{T,i}$ is homeomorphic to the sphere in which two points belonging to the surface are identified. By this homeomorphism, a set homeomorphic to a cercle lying on the surface of the sphere beyond the identified points corresponds to the quadrate $Q_{T,i}$. It follows that the set

$$\dot{N}_{T,i}-[Q_{T,i}-\dot{Q}_{T,i}]$$

(where $\dot{N}_{T,i}$ denotes the boundary of the polytope $N_{T,i}$) is a retract by deformation of $N_{T,i}$. It means that there exists a continuous mapping $r'_{T,i}(x,t)$ defined for $x\in N_{T,i}$ and $0\leqslant t\leqslant 1$ such that

(16) $\quad r'_{T,i}(x,t)\in N_{T,i} \quad$ for $\quad x\in N_{T,i} \quad$ and $\quad 0\leqslant t\leqslant 1,$

(17) $\quad r'_{T,i}(x,0)=x \quad$ for $\quad x\in N_{T,i},$

(18) $\quad r'_{T,i}(x,1)\in\dot{N}_{T,i}-[Q_{T,i}-\dot{Q}_{T,i}] \quad$ for $\quad x\in N_{T,i},$

(19) $\quad r'_{T,i}(x,t)=x \quad$ for $\quad x\in\dot{N}_{T,i}-[Q_{T,i}-\dot{Q}_{T,i}] \quad$ and $\quad 0\leqslant t\leqslant 1.$

From the fact that the diameter of the link $N_{T,i}$ is $\leqslant 2(\varepsilon+\eta)$ we infer

(20) $\quad \varrho(r'_{T,i}(x,t),x)\leqslant 2(\varepsilon+\eta) \quad$ for $\quad x\in N_{T,i} \quad$ and $\quad 0\leqslant t\leqslant 1.$

Remark. It follows from the construction of the rosary N that if x is a point of $P+N$, not being a vertex of the complex K, then the zone $\gamma_\varepsilon(K)$ constitutes a neighborhood of x in the set $P+\Gamma$.

7. Subordinate polytope. Let $\dot{N}$ denote the boundary of the space rosary N. The polytope

$$P'=P+\dot{N}-\sum_T\sum_{i=0}^{n}[Q_{T,i}-\dot{Q}_{T,i}]$$

will be said to be the *subordinate polytope* to P (corresponding to the triangulation τ and to the zone $\gamma_\varepsilon(K)$). We see at once that the common part of P' and $R=\overline{C-P}$ is the same as the common part of P and R and that the polytope

$$C'=P'+R$$

is an irreducible cutting of the space E_3 with the interior region

$$\Gamma'=\Gamma-N$$

and the exterior region

$$A'=A+(N-\dot{N})+\sum_T\sum_{i=0}^{n}(Q_{T,i}-\dot{Q}_{T,i}).$$

Moreover it is clear that the polytope P' is smoothly connected on C' and that the 1-dimensional skeleton of P [16]) (corresponding to the triangulation τ) is a subpolytope of P'.

Theorem. *There exists a retraction by deformation $r_2(x,t)$ of the polytope $P+N$ to the subordinate polytope P' such that*

$$\varrho(r_2(x,t),x)\leqslant 2\cdot(\varepsilon+\eta) \quad \text{for} \quad x\in P+N \quad \text{and} \quad 0\leqslant t\leqslant 1.$$

Proof. Consider the mappings $r'_{T,i}(x,t)$ (defined et the end of **6**) retracting by deformation the sets $N_{T,i}$ to the sets $\dot N_{T,i}-[Q_{T,i}-\dot Q_{T,i}]$. By the equalities

$$P+N=P'+\sum_T\sum_{i=0}^{n}N_{T,i} \quad \text{and} \quad N_{T,i}\cdot P'=\dot N_{T,i}-[Q_{T,i}-\dot Q_{T,i}],$$

and the formulae (16), (17), (18), (19) and (20) we infer that putting

$$r_2(x,t)=x \quad \text{for} \quad x\in P' \quad \text{and} \quad 0\leqslant t\leqslant 1,$$

$$r_2(x,t)=r'_{T,i}(x,t) \quad \text{for} \quad x\in N_{T,i} \quad (i=0,1,\dots,n) \quad \text{and} \quad 0\leqslant t\leqslant 1.$$

we obtain the retraction required.

Using the same reasoning as at the end of **3**, we obtain from the last theorem the following

Corollary 1. *In order that P' should be an absolute retract it is necessary and sufficient that $P+N$ be an absolute retract.*

Combining this result with the corollary 1 of **6** we obtain

Corollary 2. *In order that P' should be an absolute retract it is necessary and sufficient that P be an absolute retract.*

8. Subordinate zone. Let C be a polyhedral irreducible cutting with $p^1(C)=0$. Consider a homogeneously 2-dimensional subpolytope P of C and the complementary polytope $R=\overline{C-P}$. Let τ denote the triangulation of the polytope C such that P is representable in the form of a subcomplex K of τ. As before, let us denote by η a positive number greater or equal to the diameters of all simplexes of the triangulation τ, and by ε a positive number adequate to the triangulation τ. In **7** we have constructed the subordinate polytope P' corresponding to the triangulation τ and to the zone $\gamma_\varepsilon(K)$. Let us denote by Γ' the interior and by Λ' the

[16]) By the 1-dimensional skeleton of $|K|$ we understand the polytope built of all 0- and 1-dimensional simplexes of K.

exterior region of E_3-C', where $C'=P'+R$. Now consider a triangulation τ' of the polytope C' such that P' is representable in the form of a subcomplex K' of τ', and so are the polytopes $P', P\cdot P'$ and $\overline{P'-P\cdot P}$. We can assume that the diameters of all simplexes of the triangulation τ' are $\leqslant\eta$, and that for the 1-dimensional skeleton of K, the triangulation τ' is a subdivision of the triangulation τ.

Theorem. *For every sufficiently small number $\varepsilon'>0$ there exists a retraction $r_4(x)$ of the ε-zone $\gamma_\varepsilon(K)$ to the ε'-zone $\gamma_{\varepsilon'}(K')$ satisfying the condition*

(21) $$\varrho(r_4(x),x)\leqslant 4\varepsilon+2\eta \quad \textit{for every} \quad x\in\gamma_\varepsilon(K).$$

Proof. Let σ' be a simplex of the triangulation τ'. There exists a triangle $T\in K$ such that the zone $\gamma_\varepsilon(T)$ constitutes a neighborhood in $C+\Gamma$ of every point $x\in\sigma'$ not being a vertex of τ. Hence, by the remark in **3**, we infer that there exists a positive number ε' so small that $\gamma_{\varepsilon'}(\sigma')\subset\gamma_\varepsilon(T)$. We may assume that the number ε' is so small, that the last inclusion holds for every $\sigma'\in K'$. Let us observe that, for every 1-dimensional simplex $J\in K$, if $\sigma'\subset\gamma_\varepsilon(J)$, then $\gamma_{\varepsilon'}(\sigma')\subset\gamma_\varepsilon(J)$. If J lies only on one triangle of K, then in a neighborhood of every interior point of J, the polytopes P and P' are locally identical and consequently $\sigma'\subset J$ and $\gamma_{\varepsilon'}(\sigma')\subset\gamma_\varepsilon(J)$. If J is a common side of two adjoined triangles T_1 and T_2 of P, then in a neighborhood of every interior point of J, the polytope $P\cdot\gamma_\varepsilon(T_2+T_2)=T_1+T_2$ is locally symmetrical to the plane π of the triangle $\gamma_\varepsilon(J)$. It follows by the construction of the subordinate polytope P', that the polytope $P'\cdot\gamma_\varepsilon(T_1+T_2)$ in a neighborhood of the triangle $\gamma_\varepsilon(J)$ is also locally symmetrical to the plane π. Consequently $\gamma_{\varepsilon'}(\sigma')\subset\pi$ and, for ξ' sufficiently small, $\gamma_{\varepsilon'}(\sigma')\subset\gamma_\varepsilon(J)$.

The polytope $P'\cdot\gamma_\varepsilon(J)$ is the sum of the segment J and of a finite number of triangles each of which has exactly one vertex on J. It is easily seen (Fig. 2) that the set

$$W(J)=\gamma_{\varepsilon'}(P'\cdot\gamma_\varepsilon(J))+N\cdot\gamma_\varepsilon(J)$$

Fig. 2

is an absolute retract.

Let $r_3(x)$ denote a mapping retracting $\gamma_\varepsilon(J)$ to $W(J)$. Thus the mapping r_3 is defined on the zone of the 1-dimensional skeleton of the complex K. If T is one of the triangles of K and $\dot T$ denotes its boundary, then the mapping $r_3(x)$ is defined on the zone $\gamma_\varepsilon(\dot{T})$ and it retracts this zone to the set

$$\gamma_{\varepsilon'}(P'\cdot\gamma_\varepsilon(\dot{T}))+N\cdot\gamma_\varepsilon(\dot{T})\subset\gamma_{\varepsilon'}(P'\cdot\gamma_\varepsilon(T))+N\cdot\gamma_\varepsilon(T).$$

But from the theorem in **3** it follows that the polytope $(P+N)\cdot\gamma_\varepsilon(T)$ is a retract by deformation of the set

$$\gamma_{\varepsilon'}(P'\cdot\gamma_\varepsilon(T))+N\cdot\gamma_\varepsilon(T).$$

By the corollary 2 in **6**, the set $(P+N)\cdot\gamma_\varepsilon(T)$ is an absolute retract. Consequently the set $\gamma_{\varepsilon'}(P'\cdot\gamma_\varepsilon(T))+N\cdot\gamma_\varepsilon(T)$ is an absolute retract. Hence, putting

$$r_3(x)=x \quad \text{for every} \quad x\in\gamma_{\varepsilon'}(P'\cdot\gamma_\varepsilon(T))+N\cdot\gamma_\varepsilon(T)$$

we obtain a mapping $r_3(x)$ which can be extended over the set $\gamma_\varepsilon(T)$ in such a manner, that its values lie in $\gamma_{\varepsilon'}(P'\cdot\gamma_\varepsilon(T))+N\cdot\gamma_\varepsilon(T)$.

If we extend r_3 in this manner over all zones $\gamma_\varepsilon(T)$, of the triangles $T\in K$, then we obtain a retraction $r_3(x)$ of the zone $\gamma_\varepsilon(K)$ to the set $\gamma_{\varepsilon'}(K')+N$. By this retraction every point $x\in\gamma_\varepsilon(T)$ will be mapped on the point lying in $\gamma_\varepsilon(T)$.

Now consider the mapping $r_2(x,t)$ defined in **7**. Putting

$$g(x)=x \quad \text{for} \quad x\in\gamma_{\varepsilon'}(K'),$$
$$g(x)=r_2(x,1) \quad \text{for} \quad x\in N,$$

and

$$r_4(x)=gr_3(x) \quad \text{for} \quad x\in\gamma_\varepsilon(K)$$

we obtain a retraction r_4 of $\gamma_\varepsilon(K)$ to $\gamma_{\varepsilon'}(K')$, such that for every point $x\in\gamma_\varepsilon(T)$ the point $r_4(x)$ belongs to $\gamma_\varepsilon(T)$ or to $\gamma_\varepsilon(T')$, where T' is a triangle adjoined to T. But the diameters of $\gamma_\varepsilon(T)$ and $\gamma_\varepsilon(T')$ are $\leqslant 2\varepsilon+\eta$ and $\gamma_\varepsilon(T)\cdot\gamma_\varepsilon(T')\neq 0$. Hence the inequality (21) holds.

The set $\gamma_{\varepsilon'}(K')$ will be called the *zone* of the polytope P' *subordinate to the zone* $\gamma_\varepsilon(K)$. Evidently:

$$\gamma_{\varepsilon'}(K')\subset\gamma_\varepsilon(K). \tag{22}$$

9. Sequences $\{P_k\}$ **and** $\{D_k\}$. Let H be a regular tetraeder lying in the space E_3 with the sides of the length 1. Let C denote the boundary of H, P one of its 2-dimensional faces and R the sum of all other faces. We shall denote by $\dot{R}$ the boundary of the polytope R or, which is the same, the boundary of the triangle P.

We shall define in E_3 two sequences of polytopes, $\{P_k\}$ and $\{D_k\}$, satisfying the following conditions:

1_k. $C_k=P_k+R$ is a polyhedral irreducible cutting of E_3 with $p^1(C_k)=0$. The interior region E_3-C_k will be denoted by Γ_k, and the exterior region — by Δ_k.

2_k. $P_k\cdot R=\dot{R}$.

3_k. P_k is smoothly connected on C_k.

4_k. D_k is the ε_k-zone of the polytope P_k by some η_k-triangulation τ_k of C_k, where $\eta_k\leqslant\frac{1}{2^{2k-1}}$ and ε_k is adequate to the triangulation τ_k and less than $\frac{1}{2^{2k}}$.

5_k. $D_{k+1}\subset D_k$ and there exists a retraction $r_k(x)$ of D_k to D_{k+1} such that $\varrho(x,r_k(x))\leqslant\frac{1}{2^{k-3}}$ for every $x\in D_k$.

The sequences $\{P_k\}$ and $\{D_k\}$ will be defined by induction. We put $P_1=P$ and denote by D_1 the $\frac{1}{4}$-zone of the polytope P_1, corresponding to the arbitrary triangulation τ_1 of C_1. By virtue of the corollary in **3** the polytope D_1 is an absolute retract.

Assume that the polytopes P_k and D_k and the triangulation τ_k satisfying the conditions 1_k, 2_k, 3_k, 4_k are already defined. We shall define the polytopes P_{k+1} and D_{k+1} in the following manner:

Let P_{k+1} denote the subordinate polytope to P_k corresponding to the triangulation τ_k and to the zone $\gamma_{\varepsilon_k}(\tau_k(P_k))$. By the remark made at the end of **6** the polytope D_k constitutes a neighborhood in $P_k+\Gamma_k$ for every point of P_{k+1} different from all vertices of the triangulation $\tau_k(P_k)$. Moreover there exists a positive number $\eta_{k+1}\leqslant\frac{1}{2^{2k+1}}$ and an η_{k+1}-triangulation τ_{k+1} of the polytope P_{k+1} such that:

1. in τ_{k+1} the polytopes $P_k\cdot P_{k+1}$ and $\overline{P_{k+1}-P_k\cdot P_k}$ are representable in the form of subcomplexes.

By the same reasoning as in **8** there exists a positive number $\varepsilon_{k+1}<\frac{1}{2^{2k+2}}$ adequate to the triangulation τ_{k+1} and such that

2. the zone $D_{k+1}=\gamma_{\varepsilon_{k+1}}(\tau_{k+1}(P_{k+1}))$ of P_{k+1} corresponding to the triangulation τ_{k+1} is a subset of D_k.

3. There exists a retraction $r_{k+1}(x)$ of D_k to D_{k+1} such that

$$\varrho(r_{k+1}(x),x)\leqslant 4\varepsilon_k+2\eta_k\leqslant 4\cdot\frac{1}{2^{2k}}+2\frac{1}{2^{2k-1}}\leqslant\frac{1}{2^{k-2}} \quad \text{for} \quad x\in D_k. \tag{23}$$

From **7** we infer that P_{k+1} satisfies the conditions 1_{k+1}, 2_{k+1}, and 3_{k+1}. The conditions 4_{k+1} and 5_k follow from the construction of the set D_{k+1}.

As we have already stated, the set D_1 is an absolute retract. By 5_1—5_k we conclude that

(24) $$D_{k+1} \text{ is an absolute retract.}$$

By the theorem in **3** and the condition 4_k there exists a retraction $r'_k(x)$ of D_k to P_k. This retraction maps every point $x \in D_k$ lying in the zone of a simplex σ of the complex $\tau_k(P_k)$ onto a point $r'_k(x) \in \sigma$. But the diameter of the zone of σ is $\leqslant 2\varepsilon_k + \eta_k < \frac{1}{2^{2k-2}}$. Hence the retraction $r'_k(x)$ satisfies the condition

(25) $$\varrho(x, r'_k(x)) < \frac{1}{2^{2k-2}} \quad \text{for every} \quad x \in D_k.$$

Remark 1. Putting $r'_k(x) = x$ for every $x \in R$ we extend the mapping $r'_k(x)$ over $D_k + R$ without a loss of continuity and of the condition (25).

Remark 2. By the construction of the polytope P_{k+1} the 1-dimensional skeleton of the complex $\tau_k(P_k)$ lies in P_{k+1} and $P_{k+1} \subset P + \Gamma_k = \overline{\Gamma}_k$. Hence $\Lambda_k \subset \Lambda_{k+1}$ and $\Gamma_{k+1} \subset \Gamma_k$.

10. Construction of the set P_∞. Now consider the sequence of the mappings $\{f_k\}$ defined on the polytope D_1 by the formula

$$f_k(x) = r_k r_{k-1} \dots r_2 r_1(x) \quad \text{for} \quad x \in D_1.$$

By 5_k the mapping $f_k(x)$ is a retraction of D_1 to the polytope D_{k+1}, and

$$\varrho(f_k(x), f_{k+1}(x)) \leqslant \frac{1}{2^{k-2}} \quad \text{for every} \quad x \in D_1.$$

It follows that the sequence $\{f_k(x)\}$ uniformly converges in D_1. Putting

$$f_\infty(x) = \lim_{k=\infty} f_k(x) \quad \text{for every} \quad x \in D_1$$

we get a continuous mapping f_∞ of D_1 onto a set

$$P_\infty = f_\infty(D_1) \subset D_k \quad \text{for every} \quad k = 1, 2, \dots$$

Since $r_k(x) = x$ for every $x \in P_\infty$ and $k = 1, 2, \dots$, also $f_\infty(x) = x$. It means that f_∞ is a retraction of the polytope D_1 to P_∞. Hence *P_∞ is an absolute retract.*

For $\nu < k_0$ and $x \in D_{k_0}$, $r_\nu(x) = x$. Consequently the mapping f_∞ can be defined in the set D_{k_0} as the limit of the mappings

$$r_k r_{k-1} \dots r_{k_0}(x).$$

By (23) we conclude that

(27) $$\varrho(f_\infty(x), x) \leqslant \sum_{k=k_0}^{\infty} \frac{1}{2^{k-2}} = \frac{1}{2^{k_0-3}} \quad \text{for} \quad x \in D_{k_0}.$$

By $P_{k_0} \subset D_{k_0}$ we infer that

(28) $$\varrho(x, P_\infty) \leqslant \frac{1}{2^{k_0-3}} \quad \text{for every} \quad x \in P_\infty.$$

Moreover, by (25) and by the inclusion $P_\infty \subset D_{k_0}$, we obtain

(29) $$\varrho(x, P_{k_0}) \leqslant \frac{1}{2^{k_0-2}} \quad \text{for every} \quad x \in P_\infty.$$

Both relations (28) and (29) imply that the absolute retract P_∞ is the limit of the sequence of absolute retracts $\{P_k\}$:

$$\lim_{k=\infty} P_k = P_\infty.$$

11. Elementary properties of P_∞.

Property 1. By the remark 2 at the end of **9** the 1-dimensional skeletons of all sets P_k lie in P_∞. On the other hand the polytope $\overline{P_{k+1} - P_k \cdot P_k}$ is an 1-dimensional subcomplex of the triangulation τ_{k+1}. Consequently

$$\overline{P_{k+1} - P_k} \cdot P_k \subset P_\infty \quad \text{for every} \quad k = 1, 2, \dots$$

Property 2. The sets P_∞ and R as absolute retracts are acyclic and $P_\infty \cdot R = \dot{R}$ is a simple closed curve. Hence, by the well-known Mayer-Vietoris-Čech formula

$$p^2(P_\infty + R) = 1.$$

Property 3. From the property 2 we conclude that $P_\infty + R$ cuts E_3 into exactly two regions. Let us show that the exterior region Λ_∞ is identical with the sum $\sum_{k=1}^{\infty} \Lambda_k$. By the remark 2 in **9** $\sum_{k=1}^{\infty} \Lambda_k \subset \Lambda_\infty$. If $x_0 \in \Lambda_\infty$, then there exists a simple arc $L \subset \Lambda_\infty$ joining x_0 with a point $x_1 \in \Lambda_1$. Then for k sufficiently great $L \subset E_3 - P_k$ and consequently $x_0 \in \Lambda_k$, which proves that $\Lambda_\infty \subset \sum_{k=1}^{\infty} \Lambda_k$.

On the other hand the set $\prod_{k=1}^{\infty}\Gamma_k$ constitutes the interior region Γ_∞ of $E_3-(P_\infty+R)$. For, by remark 2 in 9, $\prod_{k=1}^{\infty}\Gamma_k\subset\Gamma_\infty$. Moreover, if $x_0\,\bar{\epsilon}\,\prod_{k=1}^{\infty}\Gamma_k$, then $x_0\in E_3-\Gamma_k$ for some k. Hence $x_0\in P_k+A_k\subset P_\infty+A_\infty$.

If A is a closed proper subset of $P_\infty+R$, then E_3-A is connected. For, suppose on the contrary that A cuts E_3; then (by (25)), for k sufficiently great the set $r'_k(A)$ also cuts E_3 [17]. On the other hand, choosing a point $x_0\in P_\infty-A$, we have for sufficiently great k

$$r'_k(x_0)\in P_k-r'_k(A).$$

Hence $r'_k(A)$ is a proper subset of the irreducible cutting P_k+R and consequently it does not cut E_3. This contradiction shows that the supposition that A cuts E_3 was wrong.

Property 4. P_∞ is a 2-dimensional Cantor-surface.

For if P_∞ is not a Cantor-surface, then it contains a 0-dimensional closed set $A\subset P_\infty$ and two closed subsets P' and P'' of P_∞ such that

$$P=P'+P'' \quad\text{and}\quad A=P'\cdot P''.$$

At least one of the sets P' and P'' does not contain $\dot{R}$. Let us admit that there exists a point $x_0\in\dot{R}-P'$.

Putting $Q'=P'$; $Q''=P''+R$ we have

$$Q'\cdot Q''\subset\dot{R}+P'\cdot P'' \quad\text{and}\quad x_0\,\bar{\epsilon}\,Q'\cdot Q''.$$

Then $p^1(Q'\cdot Q'')=0$, and $p^2(Q')=p^2(Q'')=0$. By the formula of Mayer-Vietoris-Čech $p^2(Q'+Q'')=0$, contrary to $Q'+Q''=P_\infty+R$ and to the property 2.

12. 2-dimensional subsets of P_∞.

Lemma. *Let A be a 2-dimensional closed proper subset of P_∞. There exists a natural number k_0 such that for every $k\geqslant k_0$ there exists in the triangulation $\tau_k(P_k)$ a triangle T such that $T\cdot P_\infty\subset A$.*

[17]) See K. Borsuk and S. Ulam, *Über gewisse Invarianten der ε-Abbildungen*, Math. Annalen **108** (1933).

Proof. Suppose the contrary: that for every triangle T of the triangulation $\tau_k(P_k)$ there exists a point $a\in T\cdot P_\infty-A$. The 1-dimensional skeleton of P_k being a subset of P_∞, we have $a\in T-\dot{T}$, where $\dot{T}$ denotes the boundary of T. The zone $\gamma_{\varepsilon_k}(\dot{T})$ of $\dot{T}$ is composed of 3 triangles erected on 3 sides of the triangle T. It is seen at once that there exists a mapping $\varphi(x)$ retracting $\gamma_{\varepsilon_k}(\dot{T})$ to $\dot{T}$ and that all such retractions are homotopic.

Now consider a sphere S with the centre a and the radius so small, that S does not meet the sets A, $\gamma_{\varepsilon_k}(\dot{T})$ and $\Gamma_k-\gamma_{\varepsilon_k}(T)$, where Γ_k denotes the interior region of $E_3-(P_k+R)$. Then $\gamma_{\varepsilon_k}(\dot{T})$ does not cut the region Γ_k-S. Let a_1 be a point of the surface $\dot{S}$ of S lying in the interior of $\gamma_{\varepsilon_k}(T)$. Hence a_1 lies in Γ_k. Let a_2 be a point of the surface $\dot{S}$ lying in the exterior region A_k of the set $E_3-(P_k+R)$. Then the segment $L_0=\overline{a_1a_2}$ cuts T in a point belonging to the interior of T. Denote by a_0 the vertex of the tetraeder H opposite to the triangle P. Since the zone $\gamma_{\varepsilon_k}(\dot{T})$ does not cut the region Γ_k-S, there exists a simple arc L_1 joining a_0 with a_1 such that its interior lies in the set $\Gamma_k-\gamma_{\varepsilon_k}(\dot{T})-S$. Moreover, there exists a simple arc L_2 joining a_0 with a_2 such that its interior lies in A_k-S. The set

$$\Omega=L_0+L_1+L_2$$

is a simple closed curve and the absolute linking number [18]) of Ω and $\dot{T}$ is equal to 1. It follows [19]) that there exists a mapping $\psi(x)$ retracting $E_3-\Omega$ to $\dot{T}$. The mappings $\varphi(x)$ and $\psi(x)$, considered only on the zone $\gamma_{\varepsilon_k}(\dot{T})$ are homotopic. It follows [20]) that φ can be extended on $E_3-\Omega$, without loss of continuity, in such a manner that the values of the extended mapping φ_7 lie on $\dot{T}$. This means that φ_7 is a retraction of $E_3-\Omega$ to $\dot{T}$. In particular φ_7 is a retraction

[18]) By the *absolute linking* of two polygonal simply closed curves Ω_1 and Ω_2 we understand the absolute value of the linking coefficient of two 1-dimensional cycles obtained by coherent orientation off all segments constituting Ω_1 resp. Ω_2. See K. Borsuk and S. Eilenberg, *Über stetige Abbildungen der Teilmengen Euklidischer Räume auf die Kreislinie*, Fund. Math. **26** (1936), p. 215, footnote 17.

[19]) l. c., p. 215.

[20]) See K. Borsuk, *Sur un espace des transformations continues et ses applications topologiques*, Monatsh. f. Math. u. Phys. **38** (1931), p. 382.

of the set $A\cdot\gamma_{\varepsilon_k}(T)+\dot{T}$ to $\dot{T}$. But the diameter of the zone $\gamma_{\varepsilon_k}(T)$ is $\leqslant 2\varepsilon_k+\eta_k\leqslant\frac{1}{2^{2k-2}}$. Hence

$$\varrho(\varphi_T(x),x)\leqslant\frac{1}{2^{2k-2}}\quad\text{for every}\quad x\in A\cdot\gamma_{\varepsilon_k}(T)+\dot{T}.$$

Now consider the mapping $\varphi^*(x)$ defined in the set $A\cdot\gamma_{\varepsilon_k}(T)+\dot{T}$ (for every triangle T of the triangulation $\tau_k(P_k)$) by the formula

$$\varphi^*(x)=\varphi_T(x)\quad\text{for}\quad x\in A\cdot\gamma_{\varepsilon_k}(T)+\dot{T}.$$

We obtain a continuous mapping φ^* transforming A in the 1-dimensional skeleton of P_k and satisfying the condition

$$\varrho(\varphi^*(x),x)\leqslant\frac{1}{2^{2k-2}}\quad\text{for every}\quad x\in A.$$

But this is incompatible with the supposition $\dim A=2$. Thus the lemma is proved.

Theorem. *If A is a 2-dimensional proper closed subset of P_∞, then $p^1(A)=\infty$.*

Proof. It is sufficient to show that $p^1(A)\geqslant m$ for every natural number m. To prove it consider m disjoined closed 2-dimensional subsets $A_1,A_2,\ldots,A_m$ of A and a closed 2-dimensional subset A_0 of P_∞ contained in $P_\infty-A$. By the preceding lemma there exists a natural number k_0 such that for every $k>k_0$ there exists in the triangulation $\tau_k(P_k)$ such triangles $T_0,T_1,\ldots,T_m$ that $T_\nu\cdot P_\infty\subset A_\nu$ for $\nu=0,1,\ldots,m$.

The set P_{k+1} is the polytope subordinate to P_k corresponding to the triangulation τ_k and to the zone $\gamma_{\varepsilon_k}(\tau_k(P_k))$. By the construction of the subordinate polytope (see **5**, **6** and **7**) there exists for every triangle T_ν $(\nu=1,2,\ldots,m)$ a component

$$N_{T_\nu}=\sum_{i=0}^{n}N_{T_{\nu,i}}$$

of the space rosary N. The base of N_{T_ν} is the component

$$M_{T_\nu}=\sum_{i=1}^{n}M_{T_{\nu,i}}$$

of the rosary M. The set M_{T_ν} contains quadrates lying in each triangle belonging to the triangulation $\tau_k(P_k)$.

Consider the boundary $\dot{Q}_{T_\nu,0}$ of the first of the quadrates of M_{T_ν}. It lies on the triangle T_ν. We have

$$\dot{Q}_{T_\nu,0}\subset T_\nu\cdot P_\infty\subset A.$$

Among the quadrates $Q_{T_\nu,i}$ $(i=0,1,\ldots,n)$ of M_{T_ν} there exists one lying on $T_0\subset P_k-A$. We infer that there exists an index $0\leqslant i_\nu<n$ such that the boundary $\dot{Q}_{T_\nu,i_\nu}$ of the quadrate Q_{T_ν,i_ν} lies on A and the boundary $\dot{Q}_{T_\nu,i_\nu+1}$ of the quadrate $Q_{T_\nu,i_\nu+1}$ does not lie on A. Hence there exists a point

$$a_\nu\in\dot{Q}_{T_\nu,i_\nu+1}-A.$$

By the construction of the space rosary in every neighborhood of a_ν there exist points a'_ν, a''_ν such that a'_ν belongs to $\nabla^*(\dot{Q}_{T_\nu,i_\nu+1})$, the point a''_ν belongs to A_k, the segment

$$L'_\nu=\overline{a'_\nu a''_\nu}$$

does not cut the set A and the points a'_ν and a''_ν can be joined in $A_{k+1}\subset A_\infty$ by a simple arc L''_ν lying arbitrarily near the link $(\dot{N}_{T_\nu,i_\nu}-Q_{T_\nu,i_\nu})+\dot{Q}_{T_\nu,i_\nu}$ in such a manner, that the arcs L'_ν and L''_ν have disjoint interiors. We infer that the closed simple curve $\Omega_\nu=L'_\nu+L''_\nu\subset E_3-A$ has the absolute linking number with the curve $\dot{Q}_{T_\nu,i_\nu}\subset A$ equal to 1 and with each curve $\dot{Q}_{T_{\nu'},i_{\nu'}}$ $(\nu\neq\nu')$ equal to 0. It follows that if we give to each curve $\dot{Q}_{T_\nu,i_\nu}$ an orientation, we obtain in A a system of m linearly independent 1-dimensional cycles. Thus $p^1(A)\geqslant m$ and the theorem is proved.

The set P_∞ is an absolute retract, but no one of its 2-dimensional proper subsets is an absolute retract. Hence P_∞ *is an irreducible 2-dimensional absolute retract*. The only proper subsets of P_∞ being absolute retracts are dendrites [21]). In particular the circle cannot be topologically imbedded in P_∞ [22]). Moreover, for every 2-dimensional closed set $A\subsetneqq P_\infty$, $p^1(A)=\infty$. Hence [23]) A is not

[21]) A continuum M is said to be a *dendrite* if it is locally connected and contains no simple closed curve. Among the spaces of the dimension $\leqslant 1$ the class of the dendrites is the same as the class of the absolute retracts.

[22]) Another example of a 2-dimensional absolute retract which contains no homeomorphic image of the circle is given in K. Borsuk, *Sur les rétractes*, Fund. Math. **17** (1931), p. 164.

[23]) See S. Lefschetz, *On locally connected and related sets*, Annals of Math. **35** (1934), p. 118 and K. Borsuk, *Zur kombinatorischen Eigenschaften der Retrakte*, Fund. Math. **21** (1933), p. 97.

locally contractible. Consequently P_∞ is also an example of an irreducible 2-dimensional locally contractible compactum. Every locally contractible closed proper subset of P_∞ is of the dimension $\leqslant 1$.

If we submit the space E_3 to a transformation consisting in the identification of all points of the set R, we obtain the space E_3^* homeomorphic to E_3, and the image P_∞^* of P_∞ is a locally connected compactum cutting E_3^* into two regions Γ_∞^* and Λ_∞^* and being their common boundary. It is easy to see that P_∞^* is an absolute neighborhood retract being a closed Cantor-surface and containing no 2-dimensional absolute retract.

Państwowy Instytut Matematyczny.

Linear functionals on spaces of continuous functions.

By

Edwin Hewitt (Seattle, Washington, U.S.A.).

1. Introduction. The present paper is concerned with the problems of classifying, representing, and approximating to linear functionals defined on spaces of real-valued continuous functions. Let X be any topological space; let $\mathfrak{C}(X,R)$ denote the set of all continuous real-valued functions defined on X; let $\mathfrak{C}^*(X,R)$ denote the set of all bounded functions in $\mathfrak{C}(X,R)$. We shall denote the real numbers throughout the present paper by the symbol R. A real-valued function I defined on $\mathfrak{C}(X,R)$ (or $\mathfrak{C}^*(X,R)$) is said to be a linear functional if $I(\alpha f+\beta g)=\alpha I(f)+\beta I(g)$ for all $f, g \in \mathfrak{C}(X,R)$ (or $\mathfrak{C}^*(X,R)$) and all $\alpha, \beta \in R$. We employ the usual definitions of sum, scalar multiplication, product, and positivity in $\mathfrak{C}(X,R)$ and $\mathfrak{C}^*(X,R)$. A linear functional I is said to be positive if it is not the zero-functional and if it is non-negative for positive functions. A linear functional is said to be bounded if it carries bounded sets of functions into bounded sets of real numbers.

In $\mathfrak{C}(X,R)$, there are at least four interesting topologies. They have been widely studied, and are described, for example, in [7], pp. 48-49. It is of some interest to consider the linear functionals on $\mathfrak{C}(X,R)$ which are continuous under these four topologies for $\mathfrak{C}(X,R)$. We shall say that a linear functional is p-, k-, u-, or m-continuous if it is a continuous mapping of $\mathfrak{C}(X,R)$ into R under the p-, k-, u-, or m-topology, respectively.

Representation of linear functionals by means of integrals, which forms the central theme of the present paper, has been studied by a number of writers during the past four decades. (We limit ourselves to linear functionals defined on spaces of continuous