Note on certain s-dimensional sets.

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Let P be an s-dimensional set (in the sense of Hausdorff measure)[1]) situated in n-dimensional Euclidean space. It is clear that, if $s<n-1$, the set of lines through a fixed point p of P, which intersect P in a point distinct from p, form, with $n-1$ of the coordinate axes, a set of angles, which considered as a set of points in $n-1$ dimensional space has dimension $\leqslant s$. From the more general results of F. Roger if $s>n-1$, then at almost all points of P, this set of directions includes the whole solid angle at p[2]). It is not immediately obvious whether, for $s<n-1$, all the lines which intersect P in at least 2 points, form, with the same $n-1$ coordinate axes a set of angles, whose dimension has an upper bound depending on s or not. The following example shows that this is not the case.

A free vector is said to be included in a point set P if among the points of P are two, p_1 and p_2, such that one possible position of the vector coincides with the segment p_1p_2. A vector is free if the initial point may be taken to be any point of the space.

There is a 0-dimensional set which includes every free vector of the space.

This includes the result stated above.

Let q_i be an increasing sequence of positive integers such that

$$q_1=10 \quad \text{and} \quad (q_1\cdot q_2\cdot\ldots\cdot q_{r-1}\cdot r)^{nr}<q_r \quad \text{for} \quad r=2,3,\ldots$$

[1]) For the definition of Hausdorff measure see A. S. Besicovitch, Mathematische Annalen **101** (1929), pp. 161-193.

[2]) F. Roger, *Les propriétés tangentielles des ensembles euclidiens de points*, Acta Mathematica **69** (1938), pp. 99-133.

Every positive number can be expressed as

$$p_0+\sum_1^\infty \frac{p_r}{q_1\ldots q_r}$$

where p_0 is a positive integer and where p_r are integers satisfying $0\leqslant p_r<q_r$.

The set of numbers $p_0+\sum_1^\infty \frac{p_{2r}}{q_1\cdot q_2\cdot\ldots\cdot q_{2r}}$ is called A.

The set of numbers $\sum_1^\infty \frac{p_{2r-1}}{q_1\cdot q_2\cdot\ldots\cdot q_{2r-1}}$ is called B.

The set of all points whose n coordinates are members of A is called α, similarly β is defined from B. The sets α and β are defined in the space

(1) $$x_1\geqslant 0, \quad x_2\geqslant 0, \quad \ldots, \quad x_n\geqslant 0.$$

Let geometrically congruent sets be defined in each of the spaces obtained by reversing any number of the inequalities in (1). (These sets are to be similarly situated, in the space defined by the inequalities corresponding to (1), as are α and β in the space defined by (1)). The sum of these sets, say P, has the properties required.

Proof. Let X be a vector with components along the coordinate axes $X_1, X_2, \ldots, X_n$. Certain of these will be positive or zero, and others will be negative, but by the definition of P as a sum of sets, it is sufficient to consider only the case where they are all positive or zero.

Write $X_i=a_i+b_i \quad a_i\in A, \; b_i\in B$.

The point $(-a_1,-a_2,\ldots,-a_n)$ belongs to the set congruent to α in the space $x_1\leqslant 0, \; x_2\leqslant 0,\ldots, \; x_n\leqslant 0$.

$(b_1,b_2,\ldots,b_n)$ belongs to β.

Hence they both belong to P and the segment joining them is a position of the vector X.

That α is 0-dimensional follows from the fact that the points of α inside the unit n-dimensional cube, can be covered by $(q_2q_4\cdots q_{2r-2})^n$ n-dimensional boxes of side-length $\frac{1}{q_{2r-1}}$. Similarly β is 0-dimensional.

Given a set of free vectors, V, let the initial point of each be taken to be the origin. Let the set of points formed by their end points be B.

By the dimension or measure of V, is meant, the dimension or measure of B.

The example above may be adjusted to show that, for any s, $0 \leqslant s \leqslant n$, there is an s-set A, in n-dimensional Euclidean space such that the set of vectors it contains has dimension n. The following example shows that the other extreme case is also possible.

If $0 \leqslant s \leqslant n$, there is an s-set such that the set of vectors it contains has dimension s.

The cases $s=0$ or n are trivial. Suppose $0<s<n$.

Let r be a positive integer and define $m(r)$ by

$$r^n\left(\frac{\sqrt{n}}{m(r)}\right)^s=1.$$

As $r\to\infty$, $\dfrac{r}{m(r)}\to 0$. Choose $r=l_1$, so large that $\dfrac{1}{m(l_1)}<\dfrac{1}{10 l_1(2\sqrt{n})}$ and so that $l_1>10$. Write $m(l_1)=m_1$.

Define l_2 to be $=l_1^2$ and $m_2=m(l_1\cdot l_2)$ and generally l_r and m_r having been defined, $l_{r+1}=l_r^2$ and $m_{r+1}=m(l_{r+1}\cdot l_r\cdot l_{r-1}\dots l_1)$. In what follows $\alpha_i,\beta_j,\gamma_k,\dots,\lambda_p$ denote integers, which may be positive, negative or zero.

Let Q_1 be the set of points whose coordinates are $\left(\dfrac{\alpha_1}{l_1},\dfrac{\alpha_2}{l_1},\dots,\dfrac{\alpha_n}{l_1}\right)$.

Let R_1 be the set of n-dimensional boxes, whose centres are at points of Q_1 and whose sides are of length $\dfrac{1}{m_1}$.

Let Q_2 be the set of points whose coordinates are

$$\left(\frac{\alpha_1}{l_1}+\frac{\beta_1}{2m_1l_2},\frac{\alpha_2}{l_1}+\frac{\beta_2}{2m_1l_2},\dots,\frac{\alpha_n}{l_1}+\frac{\beta_n}{2m_1l_2}\right),$$

where $-l_2\leqslant\beta_i\leqslant l_2$, $i=1,2,\dots,n$.

Let R_2 be the set of n dimensional boxes, whose centres are at points of Q_2 and whose sides are of length $\dfrac{1}{m_2}$.

Generally when Q_r, R_r have been defined, let Q_{r+1} be the set of points whose coordinates are

$$\left(q_1+\frac{\lambda_1}{2m_rl_{r+1}},q_2+\frac{\lambda_2}{2m_rl_{r+1}},\dots,q_n+\frac{\lambda_n}{2m_rl_{r+1}}\right),$$

where $(q_1,q_2,\dots,q_n)$ is a point of Q_r and $-l_{r+1}\leqslant\lambda_i\leqslant l_{r+1}$; $i=1,2,\dots,n$.

Let R_{r+1} be the set of n-dimensional boxes whose centres are at points of Q_{r+1} and whose sides are of length $\dfrac{1}{m_{r+1}}$.

Let $P=\prod\limits_{i=1}^{\infty}R_i$.

P is an s-dimensional set and the set of vectors that it contains is an s-dimensional set.

Proof. That P is s-dimensional is immediate. The set of vectors contained in a lattice (to which the origin belongs) is the same lattice. With each point of Q_1 associate a sphere whose centre is the point and whose diameter is $\dfrac{\sqrt{n}}{m_1}$. Call the points of this set of spheres, C_1, and the points of the set of concentric spheres of radius $\dfrac{2\sqrt{n}}{m_1}$, C_2. Then the set of vectors of C_1 is contained in C_2, from which the second property of P follows.