Let $E^{(n+1)}$ denote the $(n+1)$-fold Einhängung of H. Freudenthal. Then we have the

Theorem 6.3. *If g is a mapping of S^n onto itself with degree b, then for each $a \in \pi^p(S^m)$ we always have*

$$J_g(a) = bE^{(n+1)}(a). \tag{6.31}$$

$$_gJ(a) = (-1)^{(n+1)(m+p)} bE^{(n+1)}(a). \tag{6.32}$$

Proof. Let g_0 denote the identity mapping of S^n. Because of (4.3), we may suppose $n > 0$. Let g_0, g represent $\beta_0, \beta \in \pi^n(S^n)$ respectively, then $\beta = b\beta_0$. Hence we have $J_g(a) = a^\vee \beta = b(a^\vee \beta_0)$. Let a be represented by f: $S^p \to S^m$, then $a^\vee \beta_0$ is represented by $f^\vee g_0 = f^\vee \psi_0^\vee \psi_1^\vee \dots ^\vee \psi_n$, where ψ_i, $(i = 0, 1, \dots, n)$, denotes the identity mapping of S^0. Hence, by (4.31), $f^\vee g_0$ represents $E^{(n+1)}(a)$, and (6.31) is proved. (6.32) can be proved by successive use of (4.32).

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Squares are normal.

By

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1. Introduction. Two plane sets are *finitely equivalent* if and only if they can be split respectively into sets $\alpha_1, \alpha_2, \dots, \alpha_m$ and into sets $\alpha_1', \alpha_2', \dots, \alpha_m'$ in such a way that the corresponding subdivisions are congruent. A plane set S is *paradoxical* if and only if it can be split into two sets each of which is finitely equivalent to S. A plane set which is not paradoxical is *normal.* It has been known [1]) for some time that squares and a variety of other plane sets are normal. However, all known verifications of the normality of squares so far published depend in an essential way on the axiom of choice. By making use of appropriate known devices for establishing the existence of certain linear functionals we find it is indeed possible to show, without the axiom of choice, that any bounded plane set with inner points is normal.

Nowhere in the sequel directly or indirectly do we employ the axiom of choice.

If G is a group then those members of G of the form $bab^{-1}a^{-1}$ are *commutators*; the smallest subgroup of G containing the set of all commutators is the *commutator subgroup* of G.

If by forming successive commutator subgroups of G we reach, in a finite number of steps, the subgroup consisting of the identity then G is a *solvable* group.

[1]) S. Banach and A. Tarski, *Sur la décomposition des ensembles de points en parties respectivement congruentes,* Fund. Math. **6** (1924), pp. 244-277. See also the abstract of a paper of. Z. Waraszkiewicz, *Sur l'équivalence de deux carrés,* Ann. Soc. Polon. Math. **19** (1947), p. 239 (meeting of the Society of Oct. 19, 1945).

A function (or transformation) F with domain and range in vector spaces is *linear* if and only if

$$F(ax+by)=aF(x)+bF(y)$$

whenever x and y are in the domain of F, and a and b are finite real numbers.

A function whose range is a subset of the finite real numbers is a *functional*.

If E is a vector space and P is contained in E then the *span* of P consists of all the finite linear combinations of members of P.

2. Linear Functionals. Throughout this section we suppose that E is a vector space and that G is a group, with respect to superposition, of linear transformations which univalently map E into itself. We suppose further that G is solvable and that p is such a convex functional with domain E that:

$$p(x+y)\leqslant p(x)+p(y)$$

whenever x and y are members of E,

$$p(tx)=tp(x)$$

whenever x is in E and $0\leqslant t<\infty$,

$$p(g(x))=p(x)$$

whenever x is in E and g is in G.

An examination of Banach's proof[2]) of the Hahn-Banach theorem followed by a perusal of sections 1 and 2 of a paper[3]) by Agnew and myself should convince the reader that an effective proof is possible of the following

2.1. ***Theorem.*** *If M is a vector subspace of E, P is a countable subset of M, the span of P is M, H is a subgroup of G, each member of H maps M into itself; then there is such a linear functional f with domain M that*

$$f(g(x))=f(x)\leqslant p(x)$$

whenever g is in H and x is in M.

[2]) S. Banach, *Théorie des opérations linéaires,* Monogr. Matemat. **1**, Warsaw (1932), pp. 27-29.

[3] R. P. Agnew and A. P. Morse, *Extensions of linear functionals, with applications to limits, integrals, measures, and densities,* Ann. of Math. **39** (1938), pp. 20-30.

From this follows readily the

2.2. ***Theorem.*** *If N is a countable subset of E, K is a countable subset of G, then there is such a vector subspace M of E, such a subgroup H of G, and such a linear functional f with domain M that: N is contained in M, K is contained in H,*

$$f(g(x))=f(x)\leqslant p(x)$$

whenever g is in H and x is in M.

Proof. Let H be the smallest subgroup of G which contains the set K. Clearly H is countable. Next let P consist of those members of E of the form $g(x)$ where x is in N and g is in H. Clearly P is countable. Now let M be the span of P. After checking that M is a vector subspace of E and that each member of H is a linear transformation which maps M into itself we complete the proof by reference to 2.1.

3. Normal Sets. Theorem 2.2 makes possible an effective proof of

3.1. ***Theorem.*** *Each bounded plane set with inner points is normal.*

Proof. Suppose R is the plane and that S is a bounded plane set with inner points. In order to show S is normal we suppose S is paradoxical and proceed to a contradiction.

Since S is paradoxical it can be split into two sets A and B each of which is finitely equivalent to S. Now A and S can be split respectively into sets $\alpha_1,\alpha_2,\ldots,\alpha_m$ and into sets $\alpha'_1,\alpha'_2,\ldots,\alpha'_m$ in such a way that the corresponding subdivisions are congruent. Similarly B and S can be split respectively into sets $\beta_1,\beta_2,\ldots,\beta_n$ and into sets $\beta'_1,\beta'_2,\ldots,\beta'_n$ in such a way that again the corresponding subdivisions are congruent.

Let E consist of such bounded functionals x on R that the set of points z for which $|x(z)|>0$ is a bounded subset of the plane. With addition and scalar multiplication defined in the obvious manner E is a vector space.

Let G consist of such functions g on E to E that

$$\{g(x)\}(z)=x(T(z))$$

for some distance preserving transformation T of R into R, each x in E, and each z in R. Clearly G is a group with respect to superposition of linear transformations which univalently map E into E. The commutator subgroup of G is an abelian translational subgroup of G. Accordingly G is solvable.

The *characteristic function* of a set A contained in R, is, it is agreed, the function φ on R such that, for each z in R, $\varphi(z)$ is 1 or 0 according to whether z is in or is not in A.

Let N consist of those members of E each of which is the characteristic function either of the set α_j where j is some integer for which $1 \leqslant j \leqslant m$, or of the set β_k where k is some integer for which $1 \leqslant k \leqslant n$. Clearly N is a finite subset of E.

Let K be such a finite subset of G that: corresponding to each integer j for which $1 \leqslant j \leqslant m$ there is a member of K which transforms the characteristic function of α_j into the characteristic function of α'_j; corresponding to each integer k for which $1 \leqslant k \leqslant n$ there is a member of K which transforms the characteristic function of β_k into the characteristic function of β'_k. Clearly there is such a K.

Recall that S is a set with inner points and choose a positive finite number δ which is the radius of some circle whose interior is contained in S.

Let p be such a functional on E that for each x in E

$$p(x) = \sup_{g \text{ in } G} \sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} [\{g(x)\}(\mu\delta, \nu\delta)].$$

A check reveals that p satisfies the suppositions of section 2.

Now ascertain M, H, and f in accordance with 2.2; and let ξ, ξ', ξ'' be the respective characteristic functions of S, A, B.

Evidently every circle of radius δ has within it some point of the form $(\mu\delta, \nu\delta)$ where μ and ν are integers. This, 2.2, and our choice of δ and of ξ tell us

$$f(-\xi) \leqslant p(-\xi) \leqslant -1, \quad f(\xi) \geqslant 1;$$

on the contrary 2.2 alone tells us

$$f(\xi) = f(\xi') + f(\xi'') = 2f(\xi), \quad f(\xi) = 0.$$

3.2. ***Remarks.*** By using a different p and E it is no doubt possible to effectively show that plane sets of finite outer Lebesgue measure and positive inner measure are also normal.

Still another choice of p and E makes it easy to see that the whole plane is normal. On the other hand, Mazurkiewicz and Sierpiński have effectively constructed[4]) a paradoxical proper subset of the plane.

The notions introduced in the first paragraph of this paper can be generalized by considering, instead of plane sets, the subsets of an arbitrary set R and, instead of ordinary geometrical congruence, the congruence under any group G of univalent transformations of R into itself. We can thus speak of subsets of R which are normal under G. If the group G is solvable then our methods furnish an effective proof that R is normal under G. If G is abelian then, as was effectively shown[5]) by Lindenbaum and Tarski, every nonvacuous subset of R is normal under G.

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[4]) See W. Sierpiński's recent paper, *Sur un ensenble plan qui se décompose en $2^{\aleph_0}$ ensembles disjoints superposables avec lui*, Fund. Math. **34** (1947), pp. 9-14.

[5]) See A. Tarski's forthcoming book, *Cardinal Algebras*, Oxford University Press, New. York, § 16.

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