Soient:

$$(21)\quad \begin{cases} h_0(x)=m\cdot u_0(x)+j\cdot v_0(x) \\ h_1(x)=m[u_1(x)+a_0]+j[v_1(x)+b_0] \\ \dots\dots\dots\dots\dots\dots \\ h_n(x)=m[u_n(x)+a_0+\dots+a_{n-1}]+j[v_n(x)+b_0+\dots+b_{n-1}]. \end{cases}$$

Il vient en vertu de (19) et (20):

$$(22)\qquad h_n(x)=h_0(x),$$

d'où la formule (5) en raison de (18) et (21). Il existe donc une fonction $h\in(\mathcal{E}^2)^{\mathcal{X}}$ satisfaisant à la condition (6).

Nous allons démontrer que $f^m\cdot g^j\sim 1$, à savoir que

$$(23)\qquad f^m(x)\cdot g^j(x)=e^{2\pi i h(x)}.$$

Posons $x\in B_k$. Il vient d'après (17) et (21):

$$f^m(x)\cdot g^j(x)=e^{2\pi i[mu_k(x)+jv_k(x)]}=e^{2\pi i h_k(x)},$$

d'où l'égalité (23) en vertu de (6).

On derivates of discontinuous functions.

By

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Stefan Mazurkiewicz has shown that *there exists a function $f(x)$ continuous on the right (therefore of class 1) and such that everywhere $f'_+(x)=+\infty$* [1]. The object of this note is to prove the following:

Theorem I. *There exists a function $f(x)$ continuous on the right, but discontinuous at an everywhere dense set such that everywhere $f'_+(x)=0$.*

Proof. In order to prove the above we construct a simple example of a monotone function $f(x)$ having the properties stated in theorem I.

Let x in (0,1) be expressed in the scale of 2 as

$$(1)\qquad x=\frac{a_1}{2}+\frac{a_2}{2^2}+\dots+\frac{a_n}{2^n}+\dots$$

with infinitely many $a_n=0$. This means that whenever x has two representations

$$(\mathrm{a})\qquad x=\frac{a_1}{2}+\frac{a_2}{2^2}+\dots+\frac{a_m}{2^m}\qquad (a_m\neq 0)$$

$$(\mathrm{b})\qquad =\frac{a_1}{2}+\frac{a_2}{2^2}+\dots+\frac{0}{2^m}+\frac{1}{2^{m+1}}+\frac{1}{2^{m+2}}+\dots,$$

we choose the form (a), i. e. the ending representation. There is, therefore, always an infinite number of zeros in the representation of x, which is unique in the form (1).

[1]) See S. Mazurkiewicz, Fund. Math. **23** (1934), pp. 9-10 and A. N. Singh Fund. Math. **33** (1945), pp. 106-107.

We now define the function $f(x)$ for x in $(0,1)$ expressed in the form (1) as

$$f(x)=\frac{a_1}{3}+\frac{a_2}{3^2}+\ldots+\frac{a_n}{3^n}+\ldots$$

Evidently the function $f(x)$ is for $0\leqslant x<1$ non diminishing. To prove that $f'_+(x)=0$ for $0\leqslant x<1$ it would be therefore sufficient to show that for any number x such that $0\leqslant x<1$ there is an infinite series of numbers $x_1,x_2,\ldots$ such that

$$x<x_n<1,\ \lim_{n=\infty} x_n=x \quad\text{and}\quad \lim_{n=\infty}\frac{f(x_n)-f(x)}{x_n-x}=0.$$

Let then (1) be the dyadic development of x containing infinitely many figures 0. Let n be a given natural number. Then there is a natural number k_n such that $a_{n+k_n}=0$.

Putting $x_n=x+\frac{1}{2^{n+k_n}}$ we shall obtain of course $x<x_n<1$ and $\lim\limits_{n=\infty} x_n=x$.

One can also see easily from the definition of the function $f(x)$ that $f(x_n)=f(x)+\frac{1}{3^{n+k_n}}$; one has then

$$\frac{f(x_n)-f(x)}{x_n-x}=\left(\frac{2}{3}\right)^{n+k_n}$$

hence

$$\lim_{n=\infty}\frac{f(x_n)-f(x)}{x_n-x}=0 \qquad \text{q. e. d.}$$

The formula $f'_+(x)=0$ for $0\leqslant x<1$ is therefore established.

For x defined by the development (a), where $m=n$, one has of course for $k=1,2,\ldots$:

$$x-\frac{1}{2^{n+k}}=\frac{a_1}{2}+\frac{a_2}{2^2}+\ldots+\frac{a_{n-1}}{2^{n-1}}+\frac{0}{2^n}+\frac{1}{2^{n-1}}+\frac{1}{2^{n+2}}+\ldots+\frac{1}{2^{n+k}};$$

hence

$$f(x)=\frac{a_1}{3}+\frac{a_2}{3^2}+\ldots+\frac{a_n}{3^n},$$

$$f\left(x-\frac{1}{2^{n+k}}\right)=\frac{a_1}{3}+\frac{a_2}{3}+\ldots+\frac{a_{n-1}}{3^{n-1}}+\frac{0}{3^n}+\frac{1}{3^{n+1}}+\frac{1}{3^{n+2}}+\ldots+\frac{1}{3^{n+k}}.$$

Having $a_n\neq 0$, therefore $a_n=1$:

$$f(x)-f\left(x-\frac{1}{2^{n+k}}\right)=\frac{1}{3^n}-\frac{1}{3^{n+1}}-\frac{1}{3^{n+2}}-\ldots-\frac{1}{3^{n+k}}=$$

$$=\frac{1}{3^n}\left[1-\frac{1}{2}\left(1-\frac{1}{3^k}\right)\right]>\frac{1}{2\cdot 3^n}\quad\text{for}\quad k=1,2,\ldots,$$

and being $\lim\limits_{k=\infty}\left(x-\frac{1}{2^{n+k}}\right)=x$ this proves that the fraction $f(x)$ is discontinuous at the point x. It is therefore discontinuous at any point x, $0<x<1$ of which abscissa is a finite dyadic fraction, then at a set of points dense in the interval $(0,1)$.

The theorem 1 is therefore proved.

The question whether there can exist a function discontinuous at an everywhere dense set such that its right (left) hand differential coefficient is zero everywhere is answered by the following

Theorem II. *If $f(x)$ has a finite right (left) hand differential coefficient everywhere, its discontinuities (if any) form a non-dense set.*

Proof. Suppose that $f(x)$ has in (a,b) an everywhere dense set of discontinuities. There exists then a point b_1 of discontinuity such that the oscillation of $f(x)$ at b_1, $\omega(b_1)>0$, and $a<b_1<b$. Then, there exists a point a_1 such that $a<a_1<b_1$ and $b_1-a_1<\min[1,\omega(b_1)]$. The points of discontinuity being everywhere dense, there exists a point b_2 in the interval (a_1,b_1) such that $\omega(b_2)>0$ and a point a_2 such that $a_1<a_2<b_2$, and $b_2-a_2<\min[\frac{1}{2},\frac{1}{2}\omega(b_2)]$. Reasoning in this manner we get an infinite set of intervals (a_n,b_n) $(n=1,2,\ldots)$ such that:

(i) (a_n,b_n) is contained in (a_{n-1},b_{n-1}),

(ii) $\omega(b_n)>0$,

(iii) $b_n-a_n<\min\left[\frac{1}{n},\frac{1}{n}\omega(b_n)\right]$,

(iv) $\lim\limits_{n=\infty} a_n=\lim\limits_{n=\infty} b_n=x_0$ (say).

It follows from the definition of $\omega(b_n)$ that there exist two points x_n and x'_n in the interior of the interval $(x_0,b_n+b_n-a_n)$ such that

(1) $$f(x_n)-f(x'_n)>\tfrac{1}{2}\omega(b_n).$$

Having

$$a_n < x_0 < x_n < b_n + (b_n - a_n)$$

and

$$a_n < x_0 < x'_n < b_n + (b_n - a_n),$$

we have:

$$0 < x_n - x_0 < 2(b_n - a_n),$$

$$0 < x'_n - x_0 < 2(b_n - a_n),$$

$$\lim_{n=\infty} x_n = \lim_{n=\infty} x'_n = x_0.$$

Now, $x_n > x_0$, $x'_n > x_0$ and $f'_+(x)$ exists and is finite by hypothesis, therefore there is a number A such that

$$\left|\frac{f(x_n)-f(x_0)}{x_n - x_0}\right| < A \quad \text{and} \quad \left|\frac{f(x'_n)-f(x_0)}{x'_n - x_0}\right| < A \quad \text{for } n=1,2,3,\ldots,$$

whence

$$|f(x_n)-f(x_0)| < A(x_n - x_0) < 2A(b_n - a_n)$$

and

$$|f(x'_n)-f(x_0)| < 2A(b_n - a_n).$$

But

$$b_n - a_n < \frac{1}{n}\,\omega(b_n),$$

hence

$$|f(x_n)-f(x'_n)| < 4A(b_n - a_n) < \frac{4A\,\omega(b_n)}{n}.$$

Using (1) we have

$$\omega(b_n) < \frac{8A}{n}\,\omega(b_n), \quad \text{and} \quad \omega(b_n) > 0,$$

therefore $8A > n$ $(n=1,2,3,\ldots)$, which is impossible.

The function $f(x)$ cannot, therefore, have an everywhere dense set of points of discontinuity and theorem II is thus proved.

It may be noted that the existence of $f'_+(x)$ as a finite number involves the continuity of $f(x)$ on the right, so that the points of discontinuity of $f(x)$ are enumerable.

It is to be mentioned that a function of a real variable $f(x)$ such that for any real x one has $f'_+(x) = 0$, and, more generally, such that for any real x one has $f'_+(x) \neq -\infty$ is a function of a class $\leqslant 2$ of Baire.

Indeed, one can see easily that if for a function $f(x)$ of a real variable one has $f'_+(x_0) \neq -\infty$, then the function $f(x)$ is semicontinuous lower at the right hand at the point x_0, i. e., for any $\varepsilon > 0$ there is a $\delta > 0$ such that $f(x_0+h) > f(x_0) - \varepsilon$ for $0 < h < \delta$. On the other side, one can easily show that every function of a real variable everywhere semicontinuous lower at the right hand is of a class $\leqslant 2$ of Baire.

Hence, there is a function $f(x)$ of the class 2 of Baire such that $f'_+(x) = 0$ for every real x.

Indeed, let P be a perfect non dense set of Cantor, and let H be the set of all left extremities of intervals contiguous to P. Putting (for $0 \leqslant x < 1$) $f(x) = 0$ for $x \in P - H$ and $f(x) = 1$ for any other x such that $0 \leqslant x < 1$ one can easily see that (for $0 \leqslant x < 1$) $f'_+(x) = 0$. The function $f(x)$ is therefore of a class $\leqslant 2$. Hence, the set H being dense in the perfect set P, the function $f(x)$ is everywhere discontinuous at the set P; hence, according to the well known theorem of Baire the function cannot be of a class $\leqslant 1$; it is therefore of the class 2.