D'autre part, on vérifie facilement que chacune des sommes suivantes c_k donne, pour toutes les permutations de ses termes, k et seulement k valeurs distinctes:

$$\begin{aligned}
\sigma_{17} &= \omega^2+(\omega+1)+(\omega+2)+(\omega+3)+(\omega+4),\\
\sigma_{18} &= \omega^2+(\omega+1)+\omega\cdot 2+\omega\cdot 3+\omega\cdot 4,\\
\sigma_{19} &= \omega^2+\omega+(\omega\cdot 2+1)+\omega\cdot 3+\omega\cdot 5,\\
\sigma_{20} &= \omega^2+(\omega+1)+\omega\cdot 2+\omega\cdot 3+\omega\cdot 6,\\
\sigma_{21} &= \omega^2+(\omega+1)+\omega\cdot 2+\omega\cdot 3+\omega\cdot 7,\\
\sigma_{22} &= \omega^2+(\omega+1)+\omega\cdot 2+\omega\cdot 4+\omega\cdot 7,\\
\sigma_{23} &= \omega^2+(\omega+1)+\omega\cdot 2+\omega\cdot 4+\omega\cdot 8,\\
\sigma_{24} &= \omega^2+(\omega+1)+(\omega\cdot 2+1)+\omega\cdot 4+\omega\cdot 7,\\
\sigma_{25} &= \omega^2+(\omega+1)+(\omega\cdot 2+1)+\omega\cdot 4+\omega\cdot 8,\\
\sigma_{26} &= \omega^2+\omega+(\omega+1)+(\omega\cdot 2+2)+(\omega\cdot 3+3),\\
\sigma_{27} &= \omega^2+(\omega+1)+(\omega\cdot 2+2)+(\omega\cdot 2+3)+(\omega\cdot 3+4),\\
\sigma_{28} &= \omega^2+(\omega+1)+(\omega+2)+(\omega\cdot 2+3)+(\omega\cdot 4+4),\\
\sigma_{29} &= \omega^2+(\omega+1)+(\omega+2)+(\omega\cdot 3+3)+(\omega\cdot 5+4),\\
\sigma_{31} &= \omega^2+(\omega+1)+(\omega\cdot 2+2)+(\omega\cdot 3+3)+(\omega\cdot 4+4),\\
\sigma_{32} &= \omega^2+(\omega+1)+(\omega\cdot 2+2)+(\omega\cdot 3+3)+(\omega\cdot 6+4),\\
\sigma_{33} &= \omega^2+(\omega+1)+(\omega\cdot 2+2)+(\omega\cdot 4+3)+(\omega\cdot 8+4).
\end{aligned}$$

Il est ainsi établi que $E_5=N_{33}-\{30\}$.

Extensions of measure.

By

J. Łoś and E. Marczewski (Wrocław).

In measure theory one is especially interested in extension problems. This paper deals with the most elementary one, namely with extension of a measure from any field $\boldsymbol{M}$ of sets to the field determined by $\boldsymbol{M}$ and an arbitrary set Z which does not belong to $\boldsymbol{M}$[1]).

We get a simple effective method of extension (*canonical extensions*)[2]) which can be used in some cases (Theorems 1, 2 and 4[3])), in particular for each measure assuming only finite values[4]). We deal also with remained cases of infinite values (Theorem 3 and 5[5])).

Terminology and symbolism. We denote by $\boldsymbol{A}$ a fixed Boolean algebra (e. g. the class of all subsets of a set). An additive and complementative subclass of $\boldsymbol{A}$ is called a *field* in $\boldsymbol{A}$. A σ-additive field in a Boolean σ-algebra $\boldsymbol{A}$ is called *σ-field* in $\boldsymbol{A}$.

We use the symbols $+$, $\cdot$, $\subset$, etc. in both meanings: in the sense of Boolean algebra for elements of $\boldsymbol{A}$ and in the sense of Theory of sets for subclasses of $\boldsymbol{A}$. For each $E\in\boldsymbol{A}$ we denote by E' the complement of E.

A class $\boldsymbol{I}\subset\boldsymbol{A}$ is called *ideal in* $\boldsymbol{A}$ if it is hereditary (i. e. if the relations: $A\in\boldsymbol{A}$, $B\in\boldsymbol{I}$ and $A\subset B$ imply $A\in\boldsymbol{I}$) and additive. An ideal $\boldsymbol{I}$ is *proper* if $\boldsymbol{A}\neq\boldsymbol{I}$. A proper ideal is *prime* if, for each $E\in\boldsymbol{A}$, we have $E\in\boldsymbol{I}$ or $E'\in\boldsymbol{I}$.

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[1]) Comp. O. Nikodym, *Sur les fonctions d'ensemble*. Comptes Rendus du I Congrès des Math. des Pays Slaves, Varsovie 1929 (1930), pp. 304-313, esp. pp. 310-312.

[2]) Our metod has been recently used by Sikorski in his study of homomorphisms in Boolean algebras. See R. Sikorski, *A theorem on extension of homomorphisms*, Ann. Soc. Pol. Math. **21** (1948), pp. 332-335.

[3]) Presented to the Polish Mathematical Society, Warsaw Section, on September 21, 1945.

[4]) The possibility of extension in this case is well known. See e. g. A. Horn and A. Tarski, *Measures in Boolean algebras*, Trans. Amer. Math. Soc. **64** (1948), pp. 467-497, esp. pp. 476-477 and, for σ-measures, Nikodym, l. c.

[5]) Presented to the Polish Mathematical Society, Wrocław Section, on March 25, 1949.

We call *measure* in a field $\mathbf{M}$ in $\mathbf{A}$ each real additive function $\mu(E)$ defined for $E \in \mathbf{M}$, such that $0 \leqslant \mu(E) \leqslant +\infty$. If $\mathbf{M}$ is a σ-field in a σ-algebra, then each σ-additive measure in $\mathbf{M}$ is called *σ-measure*. A measure which assumes precisely two values is called *two-valued*.

If μ is a measure in $\mathbf{M}$ and ν a measure in $\mathbf{N}$ and if moreover $\mathbf{M} \subset \mathbf{N}$ and $\mu(M)=\nu(M)$ for $M \in \mathbf{M}$ then ν is called an *extension of μ to $\mathbf{N}$*. An extension which is a σ-measure is called *σ-extension*.

1. Extension of fields. $\mathbf{M}$ being a class of elements of $\mathbf{A}$ and Z an element of $\mathbf{A}$, we denote by $[\mathbf{M}, Z]$ the class of all elements

$$MZ+NZ' \quad \text{where} \quad M, N \in \mathbf{M}.$$

It is easy to see that

(i) $[\mathbf{M}\, Z]=[\mathbf{M}, Z]$.

(ii) If $\mathbf{M}$ is a field in $\mathbf{A}$, then $[\mathbf{M}, Z]$ is the smallest field in $\mathbf{A}$ which includes $\mathbf{M}$ and (Z).

(iii) If $E_j \in [\mathbf{M}\, Z]$ (for $j=1,2$) and $E_1E_2=0$, then there is M_j, $N \in \mathbf{M}$ $(j=1,2)$ such that

$$E_j=M_jZ+N_jZ' \qquad (j=1,2)$$
$$M_1M_2=0=N_1N_2.$$

2. Interior and exterior measure. μ being a measure in a field $\mathbf{M}$ in $\mathbf{A}$, we define in the whole algebra $\mathbf{A}$ the *interior* measure μ_i [and the *exterior* measure μ_e] by putting for each $E \in \mathbf{A}$

$$\mu_i(E)=\sup \mu(M) \quad [\text{and} \quad \mu_e(E)=\inf \mu(M)]$$

where M runs over the class of all $M \in \mathbf{M}$ such that $M \subset E$ [or $M \supset E$ respectively].

It is known that, for each $A, B \in \mathbf{A}$:

(i) If $AB=0$ and $A+B \in \mathbf{M}$ then

$$\mu_i(A)+\mu_e(B)=\mu(A+B).$$

(ii) If $A \subset M$ and $B \subset N$, where $M, N \in \mathbf{M}$ and $MN=0$, then

$$\mu_i(A+B)=\mu_i(A)+\mu_i(B) \quad \mu_e(A+B)=\mu_e(A)+\mu_e(B).$$

3. Problem of extensions of measure. The following lemma is obvious:

(i) Let $\mathbf{M} \subset \mathbf{N}$ be two fields in $\mathbf{A}$, μ a measure in $\mathbf{M}$, and ν an extension of μ to $\mathbf{N}$. Then

(*) $$\mu_i(N) \leqslant \nu(N) \leqslant \mu_e(N) \quad \text{for} \quad N \in \mathbf{N}.$$

Thus the following problem arises [6]:

Let: μ be a measure in a field $\mathbf{M}$ in $\mathbf{A}$, $Z \in \mathbf{A}$ and ξ a number such that $\mu_i(Z) \leqslant \xi \leqslant \mu_e(Z)$: Does there exist an extension ν of μ to $[\mathbf{M}, Z]$ such that $\nu(Z)=\xi$?

4. Canonical extensions.

Lemma. *If μ is a measure in a field $\mathbf{M}$ in $\mathbf{A}$, $Z \in \mathbf{A}$ and for each $E \in \mathbf{A}$*

$$\nu_*(E)=\mu_i(EZ) \quad \nu^*(E)=\mu_e(EZ)$$

then the functions ν_* and ν^* are measures in $[\mathbf{M}, Z]$.

Proof. Let

$$E_j \in [\mathbf{M}, Z] \quad (\text{for } j=1,2) \quad \text{and} \quad E_1E_2=0,$$

then, by 1 (iii),

$$E_j=M_jZ+N_jZ', \quad M_j, N_j \in \mathbf{M}, \quad M_1M_2=0=N_1N_2.$$

Consequently $E_jZ=M_jZ$, whence

$$\nu_*(E_1+E_2)=\mu_i[(E_1+E_2)Z]=\mu_i(M_1Z+M_2Z)$$

and, by 2 (ii),

$$\mu_i(M_1Z+M_2Z)=\mu_i(M_1Z)+\mu_i(M_2Z)=\nu_*(E_1)+\nu_*(E_2),$$

whence finally

$$\nu_*(E_1+E_2)=\nu_*(E_1)+\nu_*(E_2).$$

The proof for ν^* is quite analogous.

Theorem 1. *Let μ be a measure in a field $\mathbf{M}$ in $\mathbf{A}$, and $Z \in \mathbf{A}$. We put for each $E \in [\mathbf{M}, Z]$:*

(1) $$\underline{\mu}(E)=\mu_i(EZ)+\mu_e(EZ')$$

(2) $$\overline{\mu}(E)=\mu_e(EZ)+\mu_i(EZ').$$

Then $\underline{\mu}$ and $\overline{\mu}$ are extensions of μ to $[\mathbf{M}, Z]$ and

(1′) $\underline{\mu}(Z)=\mu_i(Z)$ (2′) $\overline{\mu}(Z)=\mu_e(Z)$.

Proof. By lemma and 1 (i), $\underline{\mu}$ and $\overline{\mu}$ are measures in $[\mathbf{M}, Z]$. Let $M \in \mathbf{M}$. Consequently, by 2 (i),

$$\underline{\mu}(M)=\mu_i(MZ)+\mu_e(MZ')=\mu(M).$$

and analogically, $\overline{\mu}(M)=\mu(M)$. Thus $\overline{\mu}$ and $\underline{\mu}$ are extensions of μ. By putting $E=Z$ in (1) and (2), we obtain (1′) and (2′). Theorem 1 is thus proved.

[6] Cf. e. g. Nikodym, op. cit., p. 312.

By transfinite induction we obtain from Theorem 1 and the well-ordering principle:

(i) *For each measure μ in a field $\boldsymbol{M}$ in $\boldsymbol{A}$ there is an extension ν to $\boldsymbol{A}$ such that the closures of sets of all values of μ and ν are equal*[7]).

and therefore, in particular[8]):

(ii) *For each two-valued finite measure μ in a field $\boldsymbol{M}$ in $\boldsymbol{A}$ there is a two-valued finite extension ν to $\boldsymbol{A}$,*

or, equivalently,

(iii) *For each proper ideal $\boldsymbol{I}$ in $\boldsymbol{A}$ there is a prime ideal $\boldsymbol{P}$ in $\boldsymbol{A}$ which includes $\boldsymbol{I}$.*

With the help of Theorem 1 we answer by positive (and effectively) the general problem of this paper under the hypothesis $\mu(Z)<+\infty$. Namely we obtain directly from Theorem 1 the

Theorem 2. *Assume the hypotheses of Theorem 1, and moreover $\mu_e(Z)<+\infty$. Let*

(3) $$\mu_i(Z)\leqslant\xi\leqslant\mu_e(Z)$$

or, in other terms,

(4) $$\xi=(1-\vartheta)\mu_i(Z)+\vartheta\mu_e(Z) \qquad \textit{where } 0\leqslant\vartheta\leqslant1.$$

Then, the function

(5) $$m(E)=(1-\vartheta)\underline{\mu}(E)+\vartheta\overline{\mu}(E)$$

is an extension of μ to $[\boldsymbol{M},Z]$ such that $m(Z)=\xi$.

The proof is trivial. The hypothesis $\mu_e(Z)<+\infty$ is used for the logical equivalence of (3) and (4).

We call *canonical* each extension defined by formula (5) and, in particular, one defined by (1) or (2).

5. Case of infinite measure. We pass now to the case $\mu_e(Z)=+\infty$. We shall prove with the help of 4 (iii) (and therefore *non effectively*) the following

Theorem 3. *Let μ be a measure in a field $\boldsymbol{M}$ in $\boldsymbol{A}$, and $Z\in\boldsymbol{A}$. We assume $\mu_e(Z)=+\infty$. Then, for each $\xi\geqslant\mu_i(Z)$ there is an extension ν of μ to $[\boldsymbol{M}, Z]$ such that $\nu(Z)=\xi$.*

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[7]) The existence of an extension of μ to the whole algebra is well-known (see e. g. Horn-Tarski, op. cit., p. 477, Theorem 1.21). The condition concerning the closures is due to R. Sikorski.

[8]) See e. g. Horn-Tarski, l. c.

Proof. In view of Theorem 1 we may assume without loss of generality that $\xi<+\infty$. Denote by $\boldsymbol{I}$ the class of all elements of $\boldsymbol{M}$ of the form

(6) $$M+N, \quad \text{where} \quad M,N\in\boldsymbol{M},\ \mu(M)<+\infty,\ NZ=0.$$

The class $\boldsymbol{I}$ is obviously an ideal in $\boldsymbol{M}$ and we shall prove that $\boldsymbol{I}$ is a proper ideal. In fact, if $E\in\boldsymbol{I}$, then E is of the form (6), whence

$$E=M+N\subset M+Z',\quad EZ\subset MZ,\quad \mu_e(EZ)\leqslant\mu_e(M)<+\infty.$$

Since $\mu_e(Z)=+\infty$, we have $EZ\neq Z$ and therefore the unit element of $\boldsymbol{A}$ does not belong to $\boldsymbol{I}$. By 4 (iii) (for $\boldsymbol{A}=\boldsymbol{M}$), there is a prime ideal $\boldsymbol{P}$ in $\boldsymbol{M}$, containing $\boldsymbol{I}$.

Now, we put $m(E)=0$ for $E\in\boldsymbol{P}$ and $m(E)=1$ for $E\in\boldsymbol{M}-\boldsymbol{P}$. The function m is therefore a two-valued measure in $\boldsymbol{M}$, such that

(7) $$m(N)=0 \quad \text{whenever} \quad \mu(N)<+\infty \quad \text{or} \quad NZ=0.$$

If $M\in\boldsymbol{M}$ and $M\supset Z$, then $M'Z=0$, whence, by (7), $m(M')=0$ and consequently $m(M)=1$. Hence

(8) $$m_e(Z)=1.$$

By Theorem 1 there is an extension $\overline{m}$ of m to $[\boldsymbol{M},Z]$ such that $\overline{m}(Z)=m_e(Z)$ and an extension $\underline{\mu}$ of μ to $[\boldsymbol{M},Z]$ such that $\underline{\mu}(Z)=\mu_i(Z)$. We put for $E\in[\boldsymbol{M},Z]$

(9) $$\nu(E)=\underline{\mu}(E)+\eta\overline{m}(E), \quad \text{where} \quad \eta=\xi-\mu_i(Z).$$

The function ν is clearly a measure in $[\boldsymbol{M},Z]$ and we have by (8) and (9)

$$\nu(Z)=\underline{\mu}(Z)+\eta\overline{m}(Z)=\mu_i(Z)+\eta\cdot m_e(Z)=\xi.$$

The measure ν is an extension of μ to $[\boldsymbol{M},Z]$. In fact, if $M\in\boldsymbol{M}$, then

$$\nu(M)=\underline{\mu}(M)+\eta\overline{m}(M)=\mu(M)+\eta m(M),$$

whence

if $\mu(M)<+\infty$, then $\nu(M)=\mu(M)$ by (7);

if $\mu(M)=+\infty$, then a fortiori $\nu(M)=+\infty$.

This completes the proof.

6. Problem of effectivity. In the preceding paragraph we have proved Theorem 3 non effectively. Now, we shall prove that Theorem 3 implies effectively a well known theorem, all known proofs of which are also non effective.

Denote by E^* for each set E of numbers the set of all numbers x, such that $|x| \epsilon E$, and, for each class Q of sets of numbers, by Q^* the class of all sets E^*, where $E \epsilon Q$.

Of course, *for each field* $\boldsymbol{M}$ *of subsets of a set* I *of positive numbers*

(i) $\boldsymbol{M}^*$ *is a field of subsets of* I^*.

(ii) *The field* $[\boldsymbol{M}^*, \boldsymbol{I}]$ *contains* $\boldsymbol{M}$.

(iii) *If* $E_1, E_2 \epsilon \boldsymbol{M}^*$ *and* $E_1 \subset \boldsymbol{I} \subset E_2$, *then* $E_1 = 0$ *and* $E_2 = \boldsymbol{I}^*$.

Now we shall prove that

(iv) *Theorem 3 implies effectively the following proposition:*

(*) *There is a finite measure in the field* $\boldsymbol{M}$ *of all sets of positive integers which vanishes for all finite sets.*

For if, let us put $\mu(E) = 0$ for each finite $E \epsilon \boldsymbol{M}^*$ and $\mu(E) = +\infty$ for each infinite $E \epsilon \boldsymbol{M}^*$. Therefore the function μ is a measure in $\boldsymbol{M}^*$. Denoting by I the set of all positive integers, we have by (iii)

$$\mu_i(I) = 0, \quad \mu_e(I) = +\infty.$$

Thus, by Theorem 3 for $\xi = 1$ and by (ii), we obtain a finite measure ν in $\boldsymbol{M}$ vanishing for all finite sets.

This completes the proof.

Notice finally that Theorem (*) follows easily from 4 (ii). Proposition 4 (ii) implies also the following theorem stronger than (*):

($\overset{*}{*}$) *There is a finite two-valued measure in the field of all sets of positive integers which vanishes for all finite sets* [9].

W. Sierpiński has proved that Theorem ($\overset{*}{*}$) implies effectively the existence of a set non measurable in the sense of Lebesgue [10]. The analogous problem for Theorem (*) remains open.

[9]) Theorem of Ulam and Tarski. See e. g. A. Tarski, *Une contribution à la théorie de la mesure*, Fund. Math. **15** (1930), pp. 42-50.

[10]) W. Sierpiński, *Fonctions additives non complètement additives et fonctions non mesurables*, Fund. Math. **30** (1938), pp. 96-99.

7. Problem of unicity. The following easy examples A and B prove that

(i) *The canonical extension is not unique.*

A. Let us put $I = (a,b,c,d)$ and denote by $\boldsymbol{M}$ the field which consists of four sets: 0, I, (a,b) and (c,d). Let $Z = (a,c)$. Consequently $[\boldsymbol{M}, Z]$ is the field of all subsets of I. Denote for $E \epsilon \boldsymbol{M}$ by $\mu(E)$ the number of points of E. For each non negative number $\varepsilon \leqslant 1$ we denote by $\mu_\varepsilon(E)$ the measure in $[\boldsymbol{M}, Z]$ defined by the equalities:

$$\mu_\varepsilon((a)) = \mu_\varepsilon((d)) = 1 - \varepsilon, \quad \mu_\varepsilon((b)) = \mu_\varepsilon((c)) = 1 + \varepsilon.$$

Thus we obtain a family of different extensions μ_ε of μ such that $\mu_\varepsilon(Z) = 2$.

B. For the same $\boldsymbol{M}$ and Z let us put

$$\nu((a,b)) = 2, \quad \nu((c,d)) = +\infty$$

and, for $0 \leqslant \varepsilon \leqslant 1$:

$$\nu_\varepsilon((a)) = 1 - \varepsilon, \quad \nu_\varepsilon((d)) = +\infty, \quad \nu_\varepsilon((b)) = \nu_\varepsilon((c)) = 1 + \varepsilon.$$

Then we obtain a family of different extensions ν_ε of ν such that $\nu_\varepsilon(Z) = 2$.

It may be noticed however, that

(ii) *In case*

$$\mu_i(Z) < +\infty \tag{10}$$

the canonical extension $\underline{\mu}$ *defined by* (1) *is the unique extension satisfying* (1′) [11])

Let ν be an extension of μ on $[\boldsymbol{M}, Z]$ such that

$$\nu(Z) = \mu_i(Z). \tag{11}$$

For each $M \epsilon \boldsymbol{M}$ we have

$$\nu(Z) = \nu(MZ) + \nu(M'Z)$$
$$\mu_i(Z) = \mu_i(MZ) + \mu_i(M'Z) \qquad \text{(by 2 (ii))}$$

and

$$\mu_i(MZ) \leqslant \nu(MZ), \quad \mu_i(M'Z) \leqslant \nu(M'Z) \qquad \text{(by 3 (i))}$$

whence, in view of (10) and (11),

$$\nu(MZ) = \mu_i(MZ). \tag{12}$$

For each $N \epsilon \boldsymbol{M}$ we have $\nu(N) = \mu(N)$ and

$$\nu(N) = \nu(NZ) + \nu(NZ')$$
$$\mu(N) = \mu_i(NZ) + \mu_e(NZ') \qquad \text{(by 2 (i))}.$$

Hence, on account of (12) (for $M = N$) and (10), if $\mu(N) = +\infty$, then

$$\nu(NZ') = +\infty = \mu_e(NZ')$$

and if $\mu(N) < +\infty$, then

$$\nu(NZ') = \nu(N) - \nu(NZ) = \mu(N) - \mu_i(NZ) = \mu_e(NZ').$$

[11]) Propositions (ii) and (iv) are due to Professor V. Jarník.

Thus

(13) $$\nu(NZ') = \mu_e(NZ').$$

The formulas (12) and (13) imply the identity $\nu = \underline{\mu}$.

Analogously

(iii) *In case*

(14) $$\mu_e(Z) < +\infty$$

the canonical extension $\bar{\mu}$ *by* (2) *is the unique extension satisfying* (2').

The following simple example shows that

(iv) *The condition* (10) *in* (ii) *cannot be omitted*[11].

Let us put $I = (a,b,c)$, and denote by $\boldsymbol{M}$ the field which consists of four sets: I, 0, (a,b) and (c). Let $Z = (b,c)$; then $[\boldsymbol{M}\ Z]$ is the field of all subsets of I.

Denote by μ the measure in $\boldsymbol{M}$ defined by the equalities:

$$\mu((a,b)) = 2, \quad \mu((c)) = +\infty.$$

Obviously $\mu_i(Z) = +\infty$ and there are different extensions ν of μ to $[\boldsymbol{M}, Z]$ such that $\nu(Z) = +\infty$.

Notice finally that

(v) *The condition* (14) *in* (iii) *cannot be replaced by* (10).

Denote by I the set of all positive integers and $\boldsymbol{M}$ the class of all subsets of I. Denote finally by $\mu(M)$, for each $M \in \boldsymbol{M}^*$ (where the asterisk is used in the sense of n⁰ 6) the number of points of M ($+\infty$ in case M infinite).

By 6 (iii) we have $\mu_i(I) = 0$ and $\mu_e(I) = +\infty$. It is easy to see that there are different extensions ν' and ν'' of μ to $[\boldsymbol{M}^*, I]$ such that $\nu'(I) = +\infty = \nu''(I)$: e. g. $\nu'(E)$ defined as the number of points belonging to E and $\nu''(E)$ defined as $2\nu'(EI)$.

8. σ-extensions. Let us suppose that $\boldsymbol{A}$ is a *Boolean σ-algebra*. Then

(i) If $\boldsymbol{M}$ is a σ-field in $\boldsymbol{A}$, then $[\boldsymbol{M}, Z]$ is the smallest σ-field in $\boldsymbol{A}$ which includes $\boldsymbol{M}$ and (Z).

For σ-fields and σ-measures propositions 1 (iii) and 2 (ii) hold for infinite sequences of elements of $\boldsymbol{A}$. Consequently, it is easy to prove that our effective method of extensions may be used for σ-measures:

Theorem 4. *If μ is a σ-measure in a σ-field $\boldsymbol{M}$ in $\boldsymbol{A}$, then the canonical extensions of μ are σ-extensions.*

On the contrary, our non-effective Theorem 3 fails for σ-extensions:

Theorem 5. *There is a σ-field of sets $\boldsymbol{A}$, a σ-field $\boldsymbol{M} \subset \boldsymbol{A}$, a σ-measure μ in $\boldsymbol{M}$ and a set $Z \in \boldsymbol{A}$, such that $\mu_i(Z) = 0$, $\mu_e(Z) = +\infty$ and that there is no σ-extension ν of μ to $[\boldsymbol{M}, Z]$ with $0 < \nu(Z) < +\infty$.*

Proof. Let us denote by Z the interval $0 < t \leqslant 1$, by $\boldsymbol{B}$ the class of Borel subset of Z and by $\boldsymbol{K}$ the ideal in $\boldsymbol{B}$ which consists of all Borel sets of the first category contained in Z.

By applying the notation of n⁰ 6, we put $\boldsymbol{M} = \boldsymbol{B}^*$ and

(15) $$\mu(E) = \begin{cases} 0 & \text{for } E \in \boldsymbol{K}^* \\ +\infty & \text{for } E \in \boldsymbol{M} - \boldsymbol{K}^*. \end{cases}$$

The function μ is, of course, a σ-measure in $\boldsymbol{M}$.

Suppose ν is a σ-extension of μ to $[\boldsymbol{M}\ Z]$. By 6 (ii) the σ-field $[\boldsymbol{M}\ Z]$ contains $\boldsymbol{B}$ and, consequently, ν is a σ-measure in $\boldsymbol{B}$. In view of (15), we have $\nu(E) = 0$ for each $E \in \boldsymbol{K}$. By a known theorem[12]) we have $\nu(Z) = 0$ or $\nu(Z) = +\infty$, q. e. d.

9. Table of results. E denote an effective proof, E' — a non effective proof of existence, U — the unicity of extension, U' — the existence of different extensions for some measure. N denote that the theorem fails for at least one measure.

Extensions.

If μ is a measure in a field $\boldsymbol{M}$ and	then there exists an extension ν of μ to $[\boldsymbol{M}.Z]$ such that		
	(a) $\nu(Z) = \mu_i(Z)$	(b) $\nu(Z) = \xi$ where $\mu_i(Z) < \xi < \mu_e(Z)$	(c) $\nu(Z) = \mu_e(Z)$
(A) $\mu_e(Z) < +\infty$	E — Th. 1 U — 7 (ii)	E — Th. 2 U' — 7 (i) A	E — Th. 1 U — 7 (iii)
(B) $\mu_i(Z) < \mu_e(Z) = +\infty$	E — Th. 1 U — 7 (ii)	E' — Th. 3 U' — 7 (i) B	E — Th. 1 U' — 7 (v)
(C) $\mu_i(Z) = +\infty$	E — Th. 1 U' — 7 (iv)	N — Contradiction between (C) and (b)	E — Th. 1 U' — 7 (iv)

[12]) E. Szpilrajn-Marczewski, *Remarques sur les fonctions complètement additives d'ensemble et sur les ensembles jouissant de la propriété de Baire*, Fund. Math. **22** (1934), pp. 303-311, esp. p. 305, Corollaire.

σ-extensions.

If μ is a σ-measure in a σ-field M and	then there exists a σ-extension ν of μ to $[M, Z]$ such that		
	(a) $\nu(Z)=\mu_i(Z)$	(b) $\nu(Z)=\xi$ where $\mu_i(Z)<\xi<\mu_e(Z)$	(c) $\nu(Z)=\mu_e(Z)$
(A) $\mu_e(Z)<+\infty$	E — Th. 4 U — 7 (ii)	E — Th. 4 U′— 7 (i)	E — Th. 4 U — 7 (iii)
(B) $\mu_i(Z)<\mu_e(Z)=+\infty$	E — Th. 4 U — 7 (ii)	N — Th. 5	E — Th. 4 U′— 7 (v)
(C) $\mu_i(Z)=+\infty$	E — Th. 4 U′— 7 (iv)	N — Contradiction between (C) and (b)	E — Th. 4 U′— 7 (iv)

Quelques généralisations des théorèmes sur les coupures du plan[1]).

Par

Casimir Kuratowski (Warszawa).

Soit sur le plan des nombres complexes, augmenté du point à l'infini, $A_0, \dots, A_{n-1}$ un système de n ($\geqslant 3$) ensembles arbitraires. Posons

(1) $\mathfrak{X}=A_0+\dots+A_{n-1}$, (2) $B_k=A_{k+1}+\dots+A_{k+n-1}$,

(3) $C_k=A_{k+1}+\dots+A_{k+n-2}$, (4) $P=C_0\cdot\ldots\cdot C_{n-1}$,

les indices étant réduits mod. n (dans les formules (2) et (3)).

La seule hypothèse faite sur les ensembles $A_0, \dots, A_{n-1}$ est que

(i) *les ensembles $C_0, \dots, C_{n-1}$ sont connexes.*

Nous nous proposons d'établir les deux théorèmes suivants[2]):

Théorème 1. *Soient p et q deux points situés en dehors de $\mathfrak{X}$. Sous les hypothèses que:*

(ii) *aucun des ensembles B_k ne coupe le plan entre p et q*[3]),

(iii) $P\neq 0$,

— *l'ensemble $\mathfrak{X}$ ne coupe pas le plan entre p et q.*

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[1]) Communication présentée au Congrès des mathématiciens polonais et tchéquoslovaques à Prague, le 30. VIII. 1949.

[2]) Pour $n=3$, les théorèmes 1 et 2 ont été établis par S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), p. 78 et 79. Les théorèmes de Eilenberg généralisent le théorème „*sur trois continus*" (voir ma note des Monatsh. Math.-Phys. **36** (1929), p. 77), ainsi que certains théorèmes de E. Čech, publiés dans sa note *Trois théorèmes sur l'homologie*, Publ. Univ. Mas. **19** (1931), p. 20.

[3]) Un ensemble E *coupe* le plan entre les points p et q lorsqu'il n'existe aucun continu unissant ces points en dehors de E.