On the third symmetric potency of the circumference.

By

Karol Borsuk (Warszawa).

1. ***Symmetric products.*** Let X be an arbitrary metric space. Denote by 2^X the *hyperspace* of X, that is the metric space whose points are all closed bounded and not empty sets $E \subset X$ and which is metrized by the formula [1])

$$\varrho(E_1, E_2) = \mathrm{Max}\,[\sup_{\xi \in E_1} \varrho(\xi, E_2), \sup_{\xi \in E_2} \varrho(\xi, E_1)].$$

Let $E_1, E_2, \ldots, E_n$ be arbitrary subsets (different or not) of the space X. We shall denote

$$E_1 \circ E_2 \circ \ldots \circ E_n$$

and call *symmetric product* [2]) of the sets $E_1, E_2, \ldots, E_n$ the subset of 2^X consisting of all sets of the form $\{\xi_1, \xi_2, \ldots, \xi_n\}$, where $\xi_\nu \in E_\nu$ for $\nu = 1, 2, \ldots, n$.

We see at once that if h is a homeomorphism mapping the space X onto an other metric space $h(X)$, then the symmetric product $E_1 \circ E_2 \circ \ldots \circ E_n$ is homeomorphic with the symmetric product $h(E_1) \circ h(E_2) \circ \ldots \circ h(E_n)$. In fact, it suffices to assign to each set $\{\xi_1, \xi_2, \ldots, \xi_n\} \in E_1 \circ E_2 \circ \ldots \circ E_n$ the set

$$\{h(\xi_1), h(\xi_2), \ldots, h(\xi_n)\} \in h(E_1) \circ h(E_2) \circ \ldots \circ h(E_n)$$

in order to obtain a homeomorphism mapping $E_1 \circ E_2 \circ \ldots \circ E_n$ onto $h(E_1) \circ h(E_2) \circ \ldots \circ h(E_n)$.

It is clear that the symmetric product satisfies the commutative law, i. e., if $j_1, j_2, \ldots, j_n$ is a permutation of the indices $1, 2, \ldots, n$, then

$$E_1 \circ E_2 \circ \ldots \circ E_n = E_{j_1} \circ E_{j_2} \circ \ldots \circ E_{j_n}.$$

[1]) Introduced by D. Pompéju, Ann. de Toulouse (2) **7** (1905).

[2]) In the case $E_1 = E_2 = \ldots = E_n = E$ this notion is identical to the n-th symmetric product of E introduced by S. Ulam and myself in the paper *On symmetric products of topological spaces*, Bull. Amer. Math. Soc. **37** (1931), p. 875-882.

If $0 = n_0 < n_1 < n_2 < \ldots < n_k = n$ and if for $n_{j-1} < \nu \leqslant n_j$ and $j = 1, 2, \ldots, k$ we have $E_\nu = X_j$, then we shall denote the symmetric product $E_1 \circ E_2 \circ \ldots \circ E_n$ also by the symbol

$$X_1^{(n_1)} \circ X_2^{(n_2 - n_1)} \circ \ldots \circ X^{(n_k - n_{k-1})}.$$

In particular if $E_\nu = X$ for $\nu = 1, 2, \ldots, n$, then the symmetric product $E_1 \circ E_2 \circ \ldots \circ E_n = X^{(n)}$ will be called the *n-th symmetric potency* of the set X.

If all sets $E_1, E_2, \ldots, E_n$ are disjoint, then the symmetric product $E_1 \circ E_2 \circ \ldots \circ E_n$ is homeomorphic to the cartesian product $E_1 \times E_2 \times \ldots \times E_n$. If one (at least) of the sets $E_1, E_2, \ldots, E_n$ is empty, then the symmetric product $E_1 \circ E_2 \circ \ldots \circ E_n$ is also empty.

The investigation of the topological structure of the symmetric product is in general very difficult. In particular the relation between the homological properties of factors and of their symmetric product is quite unknown.

The purpose of this paper is to investigate the topological structure of the third symmetric potency of the circumference.

2. ***Decomposition of $S^{(3)}$ into some closed sets.*** Let S denote the unit circumference defined as the set of all complex numbers z with $|z| = 1$. The real part of z will be denoted by Rz, the imaginary part — by Iz. The points $z \in S$ with $Iz \geqslant 0$ constitute a hemicircumference, which we will denote by H_1. The points $z \in S$ with $Iz \leqslant 0$ constitute the complementary hemicircumference, denoted H_1. Thus

(1) $$H_1 + H_{-1} = S \quad \text{and} \quad H_1 \cdot H_{-1} = \{1, -1\}.$$

For every element $\{z_1, z_2, z_3\}$ of the third symmetric potency $S^{(3)}$ of S two at least of the points z_1, z_2, z_3 belong to one of the hemicircumferences H_1 and H_{-1}. Consequently putting

(2) $$A_1 = H_1^{(2)} \circ S, \quad A_{-1} = H_{-1}^{(2)} \circ S$$

we have

(3) $$S^{(3)} = A_1 + A_{-1}.$$

Putting

(4) $$T = A_1 \cdot A_{-1}$$

we see at once that every point of T may be represented in the form $\{z_1, z_2, z_3\}$ where either $z_1 = z_2 \in H_1$ and $z_3 \in H_{-1}$ or $z_1 \in H_1$, $z_2 \in H_{-1}$ and $z_3 \in H_1 \cdot H_{-1}$.

Furthermore let us denote by L_1 the simple arc composed of all $z \in S$ such that $Rz \geqslant 0$ and $Iz \leqslant 0$ and by L_{-1} the simple arc composed of all $z \in S$ such that $Rz \leqslant 0$ and $Iz \leqslant 0$ (see figure 1). It is easy to see that denoting by T_1 the subset of T consisting of all points $\{z_1, z_2, z_3\} \in T$ such that $\{z_1, z_2, z_3\} \cdot L_1 \neq 0$ and by T_{-1} the subset of T consisting of all points $\{z_1, z_2, z_3\} \in T$ such that $\{z_1, z_2, z_3\} \cdot L_{-1} \neq 0$, we have

(5) $$T = T_1 + T_{-1}.$$

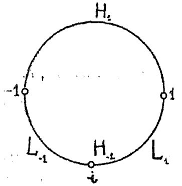

Fig. 1.

Moreover we have

(6) $$L_1 + L_{-1} = H_{-1} \quad \text{and} \quad L_1 \cdot L_{-1} = \{-i\}.$$

From (1), (2) and (6) we infer

(7) $$A_1 = H_1^{(3)} + H_1^{(2)} \circ L_1 + H_1^{(2)} \circ L_{-1}.$$

3. ***Structure of the set*** $\boldsymbol{H_1^{(3)}}$. It is known [3]) that the set $H_1^{(3)}$ is a 3-cell, its boundary is the set B_1 consisting of all elements $\{z_1, z_2, z_3\} \in H_1^{(3)}$ such that at least one of the numbers z_1, z_2, z_3 is equal to 1 or to -1. Consequently

(8) $$B_1 = C_1 + C_{-1}$$

where

(9) $$C_1 = H_1^{(2)} \circ \{1\}, \quad C_{-1} = H_1^{(2)} \circ \{-1\}.$$

Furthermore

(10) $$H_1^{(3)} \cdot (H_1^{(2)} \circ L_1) = C_1, \quad H_1^{(3)} \cdot (H_1^{(2)} \circ L_{-1}) = C_{-1}.$$

The common part of the sets C_1 and C_{-1} consists of all points of the form $\{1, -1, z\}$ with $z \in H_1$. This set may be obtained from the hemicircumference H_1 by matching its endpoints 1 and -1. Hence it is a simple closed curve; denote it J_0. Thus

(11) $$J_0 = C_1 \cdot C_{-1} = (H_1^{(2)} \circ L_1) \cdot (H_1^{(2)} \circ L_{-1}).$$

[3]) l. c. p. 880.

Since the simple closed curve J_0 is the common part of the closed sets C_1 and C_{-1} constituting a decomposition of the surface B_1 of the 3-cell $H_1^{(3)}$, we conclude that C_1 and C_{-1} are 2-cells and the point $\{1, 1, 1\} \in C_1 - J_0$ is an inner point of C_1 and the point $\{-1, -1, -1\} \in C_{-1} - J_0$ an inner point of C_{-1}.

Consider in the euclidean 3-dimensional space the 3-cell D (see figure 2) which is the sum of all segments joining the points $a_1 = (1, 0, 1)$ and $a_{-1} = (1, 0, -1)$ with all points $(x, y, 0)$ of the circle Q_0 determined by the inequality

$$x^2 + y^2 \leqslant 1.$$

The circumference S_0 of the circle Q_0 cuts the boundary P of the cell D into two 2-cells: the cell P_1 containing the point a_1 and the cell P_{-1} containing the point a_{-1}.

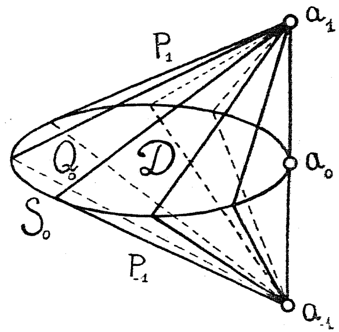

Fig. 2.

It is clear that there exists a homeomorphism h_0 mapping $H_1^{(3)}$ onto D in such a way that the simple closed curve J_0 is mapped into the circumference S_0 and the 2-cells C_1 and C_{-1} respectively into the cells P_1 and P_{-1}. Moreover we can assume that the point $\{1, -1, 1\} = \{1, -1, -1\}$ of J_0 is mapped by h_0 onto the point $a_0 = (1, 0, 0)$ of S_0, the simple arc $H_1 \circ \{1\} \circ \{1\} \subset C_1$ onto the segment joining a_0 with a_1 and the simple arc $H_1 \circ \{-1\} \circ \{-1\} \subset C_{-1}$ onto the segment joining a_0 with a_{-1}.

4. ***Structure of the set*** $\boldsymbol{H_1^{(2)} \circ L_1}$. The set $H_1^{(2)} \circ L_1$ may be obtained from the cartesian product $H_1^{(2)} \times L_1$ by matching every point of the form $(\{z, 1\}, 1)$, where $z \in H_1$, with the point of the form $(\{z, z\}, 1)$. Every point of the cartesian product $H_1^{(2)} \times L_1$ may be written (in the unique manner) in the form

$$\xi = (\{z_1, z_2\}, z_3),$$

where $z_1, z_2 \in H_1$, $z_3 \in L_1$ and $-1 \leqslant Rz_1 \leqslant Rz_2 \leqslant 1$, $0 \leqslant Rz_3 \leqslant 1$. If we assign to ξ the point

$$h_1(\xi) = (Rz_1, Rz_2, Rz_3),$$

we obtain a homeomorphism mapping $H_1^{(2)}\times L_1$ onto the prism Π (see figure 3) its lower base is the triangle Δ_0 with the vertices

$$b_{-1}=(-1,-1,0),\quad b_0=(-1,1,0),\quad b_1=(1,1,0)$$

and the upper base is the triangle Δ_1 with the vertices

$$c_{-1}=(-1,-1,1),\quad c_0=(-1,1,1),\quad c_1=(1,1,1).$$

The homeomorphism h_1 maps the points of the form

$$(\{z,1\},1),\quad \text{where}\quad z\in H_1$$

onto the points $(Rz,1,1)$ lying on the edge $\overline{c_0c_1}$ of the upper base and the points of the form

$$(\{z,z\},1),\quad \text{where}\quad z\in H_1$$

onto the points $(Rz,Rz,1)$ lying on the edge $\overline{c_{-1}c_1}$ of the upper base. Moreover the upper base Δ_1 corresponds to the 2-cell $H_1^{(2)}\times\{1\}$ and the lower base Δ_0 to the 2-cell $H_1^{(2)}\times\{-i\}$. It follows that the homeomorphism h_1 induces the homeomorphism h_2 mapping the set $H_1^{(2)}\circ L_1$ onto the set Π^* obtained from the prism Π by matching the edge $\Gamma'=\overline{c_0c_1}$ with the edge $\Gamma''=\overline{c_{-1}c_1}$.

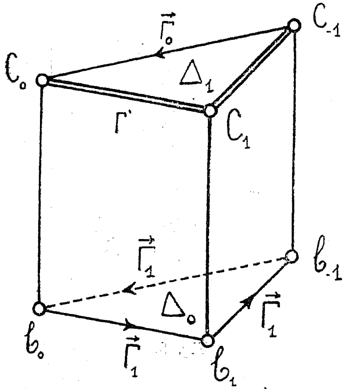

Fig. 3.

Accordingly to the homeomorphism h_2, the 2-cell Δ_1^*, obtained from the upper base Δ_1 by matching the edges Γ' and Γ'', corresponds to the 2-cell C_1 given by the formula (9) and the simple arc, obtained by matching the edges Γ' and Γ'', corresponds to the simple arc $H_1\circ\{1\}\circ\{1\}$.

Let Λ denote the sum of three squares constituting the sides of the prism Π. Evidently the set Λ^*, obtained from Λ by matching the edges Γ' and Γ'', is homeomorphic to Λ. It is clear that the 1-dimensional circuit $\vec{\Gamma}_0'$ of Λ^*, corresponding to the oriented segment $\overrightarrow{c_{-1}c_0}$, is homologous on Λ^* to the 1-dimensional circuit

$$\vec{\Gamma}_1=\overrightarrow{b_0b_1}+\overrightarrow{b_1b_{-1}}+\overrightarrow{b_{-1}b_0}.$$

According to the homeomorphism h_2, the circuit $\vec{\Gamma}_0$ corresponds to the 1-dimensional circuit $\vec{J}_0$, i. e. to the simple closed curve J_0 (defined by the formula (11)) oriented by the increase of Rz. Similarly the circuit $\vec{\Gamma}_1$ corresponds to the 1-dimensional circuit $\vec{J}_1$, i. e. to the simple closed curve J_1 consisting of all points of the form $\{z_1,z_2,-1\}\in H_1^{(2)}\circ\{1\}$, where either $z_1=z_2$ or $\{1,-1\}\cdot\{z_1,z_2\}=0$, oriented by the following order of its three points:

$$\{-1,1,-1\},\quad \{1,1,-1\},\quad \{-1,-1,-i\}.$$

Finally let us observe that according to the homeomorphism h_2, the set Λ^* corresponds to the subset of $H_1^{(2)}\circ L_1$ consisting of all points which may be represented in the form $\{z_1,z_2,z_3\}$, where either $z_1=z_2\in H_1$ and $z_3\in L_1$ or $z_1\in H_1$, $z_2\in L_1$ and $z_3\in L_1\cdot L_{-1}$. This means that Λ^* corresponds to the set T_1 defined in the Nr **2**. Consequently

(12) $\vec{J}_0$ is homologous to $\vec{J}_1$ on T_1.

If we consider, instead of the set $H_1^{(2)}\circ L_1$ the set $H_1^{(2)}\circ L_{-1}$, we infer in quite analogous manner that

(13) $\vec{J}_0$ is homologous to $\vec{J}_1$ on T_{-1}.

5. ***The structure of the set $H_1^{(3)}+H_1^{(2)}\circ L_1$.*** Applying the formula (10) and the properties of the homeomorphism h_0 stated in the Nr **3** we see that the set $H_1^{(3)}+H_1^{(2)}\circ L_1$ may be considered as the set obtained from the sets D and Π^* by matching the 2-cell P_1 of D with the 2-cell Δ_1^* of Π^*. This matching may be realized in the 3-dimensional euclidean space in the following manner:

Let K_1 denote the cylinder constituted of all points (x,y,z) such that

$$x^2+y^2\leqslant 1,\quad 0\leqslant t\leqslant 2.$$

The cylindric part of the surface of K_1, that is the set consisting of all points (x,y,t) such that

$$x^2+y^2=1,\quad 0\leqslant t\leqslant 2,$$

will be denoted by N_1. The lower base of K_1 is the circle Q_0. The upper base will be denoted by Q_1 and its circumference by S_1.

The set Π^* is evidently homeomorphic to the closure $\overline{K_1-D}$ of the set K_1-D (see figure 4) in such a manner that the lower base Δ_0 of Π^* corresponds to the circle Q_1 and the upper base Δ_1^* — to the common part of $\overline{K_1-D}$ and D, that is to the 2-cell P_1. It follows that there exists a homeomorphism h_3 mapping the set $H_1^{(3)}+H_1^{(2)}\circ L_1$ onto the set $F_1=D+K_1$ and satisfying the conditions:

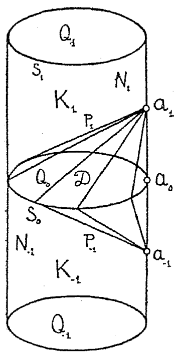

Fig. 4.

(14) $h_3(\xi)=h_0(\xi)$ for every $\xi\in H_1^{(3)}$,

(15) $h_3(H_1^{(2)}\circ\{-i\})=Q_1$,

(16) $h_3(J_0)=S_0$,

(17) $h_3(J_1)=S_1$,

(18) $h_3(T_1)=N_1$.

By (16) and (17) h_3 maps the circuits $\overrightarrow{J_0}$ and $\overrightarrow{J_1}$ onto the circuits $h_3(\overrightarrow{J_1})$ and $h_3(\overrightarrow{J_0})$ obtained by suitable orientation of the circumferences S_0 and S_1. From (11) and (17) we get

(19) $h_3(\overrightarrow{J_0})$ is homologous on N_1 to $h_3(\overrightarrow{J_1})$.

6. ***Structure of the set*** A_1. Let K_{-1} denote the cylinder (see figure 4) constituted of all points (x,y,t) such that

$$x^2+y^2\leqslant 1,\quad -2\leqslant t\leqslant 0.$$

The circle Q_0 is the upper base of K_{-1}. Denote by Q_{-1} the circle which is the lower base of K_{-1} and its circumference by S_{-1}. The cylindric part of the surface of K_{-1}, that is the set consisting of all points (x,y,t) such that

$$x^2+y^2=1 \quad\text{and}\quad -2\leqslant t\leqslant 0$$

will be denoted by N_{-1}.

Repeating the reasoning of Nr **5**, we infer that there exists a homeomorphism h_{-3} mapping the set $H_1^{(3)}+H_1^{(2)}\circ L_{-1}$ onto the set $F_{-1}=D+K_{-1}$ and satisfying to the conditions:

(20) $$h_{-3}(\xi)=h_0(\xi)\quad\text{for every}\quad \xi\in H_1^{(3)},$$

(21) $$h_{-3}(H_1^{(2)}\circ\{-i\})=Q_{-1},$$

(22) $$h_{-3}(J_0)=S_0,$$

(23) $$h_{-3}(J_1)=S_{-1},$$

(24) $$h_{-3}(T_{-1})=N_{-1}.$$

From (13) and (24) we get

(25) $h_{-3}(\overrightarrow{J_0})$ is homologous on N_{-1} to $h_{-3}(\overrightarrow{J_1})$.

It follows by (19) that

(26) $h_{-3}(\overrightarrow{J_1})$ is homologous on N_1+N_{-1} to $h_3(\overrightarrow{J_1})$.

Now let us observe that the common part of the sets $H_1^{(3)}+H_1^{(2)}\circ L_{-1}$ and $H_1^{(3)}+H_1^{(2)}\circ L_1$ consists not only of the set $H_1^{(3)}$, but also of the set $H_1^{(2)}\circ\{-i\}$. It follows, by (14), (15), (20) and (21 that both homeomorphisms h_3 and h_{-3} may be considered as one homeomorphism g_1 mapping the set

$$A_1=H_1^{(3)}+H_1^{(2)}\circ L_{-1}+H_1^{(2)}\circ L_1$$

onto the set obtained from the cylinder $K=K_1+K_{-1}$ by matching for every $\xi\in H_1^{(2)}\circ\{-i\}$, the point $h_3(\xi)\in Q_1$ with the point $h_{-3}(\xi)\in Q_{-1}$.

From (25) it follows that this matching identifies the circumferences of the lower and the upper bases of K, with consistent (on $N=N_1+N_{-1}$) orientation. Consequently the set K^* obtained in this manner from K is an anchor ring.

7. ***Structure of the set*** $S^{(3)}$. We have shown that there exists a homeomorphism g_1 mapping the set A_1 onto an anchor ring K^*. The toral surface of the anchor ring K^*, obtained from N by matching the circumferences S_1 and S_{-1} corresponds by (18), (24) and (4) to the set $T=A_1\cdot A_{-1}$. From (16) and (22) we conclude that g_1 maps the simple closed curve J_0 onto the circumference S_0 of the circle $Q_0\subset K$.

Denote by $\bar{z}$, for every complex number z, the complex number $Rz-Iz$. Put

$$\alpha(\{z_1,z_2,z_3\})=\{\bar{z}_1,\bar{z}_2,\bar{z}_3\}\quad\text{for every}\quad \{z_1,z_2,z_3\}\in S^{(3)}.$$

We obtain thus a homeomorphism mapping $S^{(3)}$ onto itself in such a manner that $\alpha(A_{-1})=A_1$. Evidently the homeomorphism α maps the simple closed curve J_0 onto itself. If we put

$$g_{-1}(\xi)=g_1\alpha(\xi)\quad\text{for every}\quad \xi\in A_{-1},$$

we obtain a homeomorphism mapping A_{-1} onto K^* and satisfying to the condition

$$g_{-1}(J_0)=g_1(J_0)=S_0.$$

With these facts in mind, let us consider a manifold M made up of two copies K'^* and K''^* of the anchor ring K^* by matching their toral surfaces in such a manner that for every $\xi \in T$ the points $g_1(\xi)$ and $g_{-1}(\xi)$ are identified. It is clear that the mapping g defined as g_1 in A_1 and as g_{-1} in A_{-1} constitutes a homeomorphism mapping the set $S^{(3)}=A_1+A_{-1}$ onto M. But M is a 3-dimensional manifold obtained from two anchor rings K'^* and K''^* by matching their toral surfaces in such a manner that the circumference S_0' of the generating circle of K'^* is matched with the circumference S_0'' of the generating circle of K''^*. It is known[4]) that this condition determines completely the structure of the manifold M. Namely M is an oriented manifold with the Heegaard diagram[5]) consisting of the anchor ring and the system of two circumferences-boundaries of two generating circles. This manifold is homeomorphic to the cartesian product of the circumference and the 2-dimensional sphere[4]).

Thus we may state the following

Theorem. *The third symmetric potency of the circumference is homeomorphic to the cartesian product of the circumference and the 2-dimensional sphere.*

[4]) H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig 1934, p. 220.

[5]) P. Heegaard, *Sur l'Analysis Situs*, Bull. Soc. Math. France **44** (1916), p. 161.

A theorem on the structure of homomorphisms.

By

Roman Sikorski (Warszawa).

This paper is a supplement to my paper [1]. Terminology and notation are in this paper the same as in [1].

Let $\boldsymbol{A}$ be a σ-complete Boolean algebra and let $\mathcal{S}$ and $\mathfrak{s}(A)$ denote respectively the set of all prime ideals of $\boldsymbol{A}$ and the set of all prime ideals of $\boldsymbol{A}$ which do not contain an element $A \in \boldsymbol{A}$. As Stone has proved[1]), $S=\mathfrak{s}(A)$ is an isomorphism of $\boldsymbol{A}$ on a field of sets $\boldsymbol{S}=\mathfrak{s}(\boldsymbol{A})$.

Let $\boldsymbol{N}^0$ denote the class of all sets

$$\mathfrak{s}(A)-\sum_{n=1}^{\infty}\mathfrak{s}(A_n)$$

where $A, A_n \in \boldsymbol{A}$, and $A=\sum_{n=1}^{\infty}A_n$.

The class $\boldsymbol{N}$ of all subsets of sets $\sum_{n=1}^{\infty}N_n$ where $N_n \in \boldsymbol{N}^0$ is a σ-ideal of subsets of $\mathcal{S}$.

Let $\boldsymbol{Z}$ denote the class of all sets $Z \subset \mathcal{S}$ which can be represented in the form

$$Z=S-N_1+N_2$$

where $S \in \mathcal{S}$ and $N_1, N_2 \in \boldsymbol{N}$. Obviously $\boldsymbol{N} \subset \boldsymbol{Z}$ and $\boldsymbol{S} \subset \boldsymbol{Z}$.

We have then[2]):

(i) *$\boldsymbol{Z}$ is a σ-field of subsets of $\mathcal{S}$.*

(ii) *The mapping $\bar{\mathfrak{s}}$ defined by the formula*

$$\bar{\mathfrak{s}}(A)=[\mathfrak{s}(A)] \quad \textit{for} \quad A \in \boldsymbol{A}$$

is an isomorphism of $\boldsymbol{A}$ on the σ-quotient algebra $\boldsymbol{Z}/\boldsymbol{N}$.

[1]) Stone [1], p. 106. In general, the field $\boldsymbol{S}$ is not a σ-field.

[2]) The proof of lemmas (i) and (ii) is similar to the proof of theorems 5.2 and 5.1 respectively in my paper [2] (every set $N \in \boldsymbol{N}$ is of first category in Stone's space $\mathcal{S}$). See also Loomis [1], p. 757.