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On the boundary values of functions of several complex variables, I.

By

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1. Introduction. Let $f(\zeta)$ be a function regular for $|\zeta| < 1$. Fatou's classical result asserts that if $f(\zeta)$ is bounded there, then for almost every point $e^{i\theta}$ on $|\zeta|=1$ the limit of $f(\zeta)$ exists as ζ approaches $e^{i\theta}$ along any non-tangential path. A more general result of Nevanlinna and Ostrowski asserts that the conclusion of Fatou's theorem holds if the boundedness of the function f is replaced by the condition

$$(1) \quad \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta = O(1).$$

In a sense, this is the best possible result since, if $\omega(u)$ is any positive and increasing function tending to $+\infty$ with u but less rapidly than $\log u$, then there is a function $f(\zeta)$ regular for $|\zeta| < 1$, satisfying the condition $\int_0^{2\pi} \omega(|f(e^{i\theta})|) d\theta = O(1)$, and deprived even of radial limit for almost every θ^1 .

The main problem of this paper is that of boundary values of regular functions of several complex variables. Let us begin by the simplest case of two variables, and let $f(z, \zeta)$ be regular in the bicylinder $|z| < 1$, $|\zeta| < 1$. Thus

$$f(z, \zeta) = \sum_{m,n} c_{mn} z^m \zeta^n.$$

The first question that naturally occurs is whether an analogue of Fatou's theorem holds here. The answer is affirmative: if $f(z, \zeta)$ is bounded for $|z| < 1$, $|\zeta| < 1$, then

$$\lim_{z \rightarrow e^{i\alpha}, \zeta \rightarrow e^{i\theta}} f(z, \zeta)$$

¹ See Paley and Zygmund [1].

exists for almost every point (x, θ) of the square $0 \leq x \leq 2\pi$, $0 \leq \theta \leq 2\pi$, provided that z tends to e^{ix} along any non-tangential path and ζ tends to $e^{i\theta}$ along any non-tangential path. (In what follows we shall say for brevity that the point (z, ζ) tends to $(e^{ix}, e^{i\theta})$ along any non-tangential path. Similarly for any number of complex variables²⁾. More difficult is the problem of an extension to the case of two variables of the Nevanlinna-Ostrowski theorem. We shall show that the conclusion of the preceding result holds if the boundedness of $f(z, \zeta)$ for $|z| < 1$, $|\zeta| < 1$ is replaced by the condition

$$(2) \quad \int_0^{2\pi} \int_0^{2\pi} \log^+ |f(re^{ix}, \varrho e^{i\theta})| \log^+ |f(re^{ix}, \varrho e^{i\theta})| dx d\theta < M < +\infty.$$

More generally, we shall prove the following theorem.

Theorem 1. Let $f(z_1, z_2, \dots, z_k)$ be regular in the k -cylinder

$$(I_k) \quad |z_1| < 1, \quad |z_2| < 1, \quad \dots, \quad |z_k| < 1 \quad (k \geq 2)$$

and let the integral

$$(3) \quad \int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f(r_1 e^{ix_1}, \dots, r_k e^{ix_k})| \{\log^+ |f(r_1 e^{ix_1}, \dots, r_k e^{ix_k})|\}^{k-1} dx_1 \dots dx_k$$

be bounded for all $r_i < 1$. Then, for almost every point (x_1, x_2, \dots, x_k) of the k -dimensional interval

$$(Q_k) \quad 0 \leq x_1 \leq 2\pi, \quad 0 \leq x_2 \leq 2\pi, \quad \dots, \quad 0 \leq x_k \leq 2\pi$$

the limit $f(e^{ix_1}, e^{ix_2}, \dots, e^{ix_k})$ of $f(z_1, z_2, \dots, z_k)$ exists as (z_1, z_2, \dots, z_k) approaches $(e^{ix_1}, e^{ix_2}, \dots, e^{ix_k})$ along any non-tangential path³⁾.

²⁾ See Zygmund [2], where the result is stated explicitly for the radial approach only (as a result about Abel summability of double Fourier series), though the proof is essentially the same for non-tangential paths. More precisely,

among other results, it is shown there that if $\sum_{m,n} |c_{mn}|^2$ converges and if we set

$$u(x, \theta; r, \varrho) = \sum_{m,n} c_{mn} r^{|m|} \varrho^{|n|} e^{i(mx + n\theta)}$$

then at almost every point x, θ the expression u tends to a finite limit as $r, \varrho \rightarrow 1$.

If $f(z, \zeta) = \sum_{m,n} c_{mn} z^m \zeta^n$ is bounded for $|z| < 1$, $|\zeta| < 1$, then clearly $\sum |c_{mn}|^2 < \infty$.

³⁾ By a limit we always mean a finite limit.

At the first sight, the condition of boundedness of the integral (3) may appear rather strange. To grasp its significance, one has to recall a result from the theory of Lebesgue integration, namely that if $g(x_1, x_2, \dots, x_k)$ is measurable and $|g|(\log^+ |g|)^{k-1}$ is integrable, then the integral of g is strongly differentiable at almost every point⁴⁾. The resemblance between these two theorems is by no means coincidental since their common source is the Hardy-Littlewood Max Theorem⁵⁾.

One may ask about the behaviour of f if the iterated logarithm in (3) is omitted. To a fuller discussion of this problem we return in the second part of this paper. Here we limit ourselves to a result which will be needed in the proof of Theorem 1. Here and hereafter, unless otherwise stated, by a limit we shall always mean a limit along every non-tangential path. For the sake of clearness we begin by stating the result for $k=2$.

Theorem 2'. Let $f(z, \zeta)$, regular for $|z| < 1$, $|\zeta| < 1$, satisfy the condition

$$(4) \quad \int_0^{2\pi} \int_0^{2\pi} \log^+ |f(re^{ix}, \varrho e^{i\theta})| dx d\theta < M < \infty.$$

Then, for almost every θ in $(0, 2\pi)$ the limit

$$(5) \quad f_\theta(z) = \lim_{\zeta \rightarrow e^{i\theta}} f(z, \zeta)$$

exists uniformly in every circle $|z| \leq r < 1$, and so is a regular function in $|z| < 1$. Moreover, for almost every θ the function $f_\theta(z)$ satisfies the condition

$$(6) \quad \int_0^{2\pi} \log^+ |f_\theta(re^{ix})| dx < M < +\infty.$$

Hence, in particular, for every such θ the limit

$$f_{\theta x} = \lim_{z \rightarrow e^{ix}} f_\theta(z)$$

exists almost everywhere in x .

The point of Theorem 2' is that under condition (4) the function $f(z, \zeta)$ has an iterated limit almost everywhere. The extension of Theorem 2' to general $k \geq 2$ is as follows.

⁴⁾ See Jessen, Marcinkiewicz and Zygmund [1], or Saks [1], p. 147.

⁵⁾ See Hardy and Littlewood [1], or Zygmund [1], p. 241 sqq. The latter book will henceforth be quoted TS.

Theorem 2. Let $f(z_1, z_2, \dots, z_k)$, regular in Γ_k , satisfy the condition

$$(7) \quad \int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f(r_1 e^{ix_1}, \dots, r_k e^{ix_k})| dx_1 \dots dx_k < M < \infty \quad (r_1, \dots, r_k < 1).$$

Then for almost every x_1 the function

$$f_{x_1}(z_2, \dots, z_k) = \lim_{z_1 \rightarrow e^{ix_1}} f(z_1, z_2, \dots, z_k)$$

exists uniformly in every $(k-1)$ -cylinder $|z_2| \leq r_2 < 1, \dots, |z_k| \leq r_k < 1$, and so is a regular function of z_2, \dots, z_k for $|z_2| < 1, \dots, |z_k| < 1$. Moreover, for almost every x_1 ,

$$(7a) \quad \int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f_{x_1}(r_2 e^{ix_2}, \dots, r_k e^{ix_k})| dx_2 \dots dx_k < M_{x_1} < +\infty.$$

It follows from Theorem 2 that

$$f_{x_1 x_2}(z_3, \dots, z_k) = \lim_{z_2 \rightarrow e^{ix_2}} f_{x_1}(z_2, \dots, z_k)$$

exists almost everywhere in x_1, x_2 and is a regular function of z_3, \dots, z_k . This argument can be repeated and, in particular, we see that the function $f_{x_1 x_2 \dots x_k}$ exists almost everywhere in Q_k .

Let α be any positive number. A function $f(z_1, \dots, z_k)$ regular in Γ_k will be said to belong to the class H^α , if

$$(8) \quad \int_0^{2\pi} \dots \int_0^{2\pi} |f(r_1 e^{ix_1}, \dots, r_k e^{ix_k})|^\alpha dx_1 \dots dx_k < M^\alpha < +\infty \quad (r_1, \dots, r_k < 1).$$

Obviously, if f satisfies (8), then the integral (3) is bounded so that the limit $f(e^{ix_1}, \dots, e^{ix_k})$ exists almost everywhere.

Theorem 3. If $f(z_1, \dots, z_k)$ is of the class H^α , then

$$(10) \quad \lim_{r_1, \dots, r_k \rightarrow 1} \int_0^{2\pi} \dots \int_0^{2\pi} |f(r_1 e^{ix_1}, \dots, r_k e^{ix_k}) - f(e^{ix_1}, \dots, e^{ix_k})|^\alpha dx_1 \dots dx_k = 0.$$

This theorem is a corollary of another one which we shall state in a moment.

If A_1, A_2, \dots, A_k are sets of points situated respectively in the planes of the variables z_1, z_2, \dots, z_k , then by $A_1 \times A_2 \times \dots \times A_k$ we shall mean the set of the systems (z_1, z_2, \dots, z_k) such that $z_1 \in A_1, z_2 \in A_2, \dots, z_k \in A_k$.

Let $0 < \eta < R$, $0 \leq \theta \leq 2\pi$. In what follows, by $\Delta(\theta, \eta, R)$ we shall mean the open domain limited by the tangents from the point $Re^{i\theta}$ to the circle $|\zeta| = \eta$, and by the longer arc of that circle between the points of contact. Instead of $\Delta(\theta, \eta, 1)$ we shall often write $\Delta(\theta, \eta)$.

Theorem 4. Suppose that $f(z_1, \dots, z_k)$ is regular in Γ_k and satisfies condition (8). Let

$$F^\eta(x_1, \dots, x_k) = \sup |f(z_1, \dots, z_k)|$$

for $(z_1, \dots, z_k) \in \Delta(x_1, \eta) \times \dots \times \Delta(x_k, \eta)$.

Then

$$(11) \quad \int_0^{2\pi} \dots \int_0^{2\pi} \{F^\eta(x_1, \dots, x_k)\}^\alpha dx_1 \dots dx_k \leq (A_{\eta, \alpha} M)^\alpha$$

with $A_{\eta, \alpha}$ finite and depending on η, α only.

That this theorem implies Theorem 3 is immediate since, on account of (11), and of Theorem 1, the integrand in (10) tends to 0 for almost every (x_1, \dots, x_k) and is majorized by an integrable function.

Before we pass to the proofs of the above (and some other) theorems, a few remarks may be useful.

(i) In the case of regular functions $f(\zeta)$ of a single variable, both the theorem of Fatou and its Nevanlinna-Ostrowski generalization (the latter in a suitable form) can be immediately extended to, say, simply connected domains. It is enough to map conformally the domain considered onto the unit circle, and to „transplant” the function from the former into the latter. By considering subdomains of the given domain we can also considerably relax the condition of simple connectedness. For obvious reasons such an argument is applicable to functions of more than one complex variable in the case of some special domains only. For more general domains in the complex space of k dimensions, if we wanted to obtain extensions of the theorems stated above, we should have to give special proofs, imitating to a certain degree the proofs of these theorems⁶⁾. This we shall not do here and we concentrate

⁶⁾ For the behavior of analytic functions in such domains see various papers of Bergman, in particular Bergman [1]. See also Bergman and Marcinkiewicz [1], and Bers [1]. The latter paper contains an extensive list of Bergman's papers in this field.

on the domains Γ_k which are the most important ones for the study of power series and of trigonometric series of several variables.

(ii) In the proof of Theorem 1 we shall use the fact that $\log^+ |f(z_1, \dots, z_k)|$ is a subharmonic function of each z_j , the other z 's being fixed. Considerable part of the argument can be immediately extended to multiply subharmonic functions, that is functions of several complex variables, subharmonic in each of these variables. However, here again we shall not go beyond what is strictly needed for the proofs of the theorems enounced, leaving generalizations aside.

(iii) The case of regular functions $f(\zeta)$, $|\zeta| < 1$, of a single complex variable is exceptional insofar as we have here at our disposal a decomposition of f into a non-vanishing regular function and the Blaschke product formed with the zeros of f . If we had a similar result for functions of several complex variables, the proofs in that case would merely be repetitions of proofs for one variable. However, no such decomposition exists for functions of several complex variables, and it is this fact which makes the proof more difficult there (see also Bergman [2], and the remarks in Bochner [1]). Roughly speaking, the proof of Theorem 1 will proceed as follows. First of all, using the fact that $\log^+ |f|$ is multiply subharmonic we prove Theorem 1 in a weaker form, where the convergence of f along nontangential paths is replaced by the boundedness of f along such paths. At this stage, the usefulness of the mere subharmonic character of $\log^+ |f|$ will have been exhausted because for subharmonic functions the most we could hope for would be the existence of limits along radii or slightly more general curves, but not inside angles. (A good illustration here is provided by a result of Littlewood [1]. He proved that if $\omega(\zeta)$ is subharmonic in $|\zeta| < 1$, and

if $\int_0^{2\pi} \omega^+(e^{i\theta}) d\theta = 0(1)$ for $\varrho < 1$, then the radial limit of $\omega(\zeta)$ exists

almost everywhere. Since $\omega(\zeta)$ can be equal to $-\infty$ in a set dense in $|\zeta| < 1$, his result obviously cannot be extended to nontangential paths. As one easily sees, the situation is not improved even if one considers bounded subharmonic functions $\omega(\zeta)$). The passage from the boundedness of f to the existence of a limit will already take into account some specific properties of regular functions.

(iv) Theorems 3 and 4 are known for $\alpha > 1$, and in this case the result is valid for multiply harmonic functions, that is continuous functions of several complex variables, harmonic in each variable separately. (See e.g. Sokół-Sokołowski [1], Bergman and Marcinkiewicz [1], Bochner [1]). For $\alpha = 1$, Bochner [1] proved that $f(z_1, \dots, z_k)$ converges to a limit in the L -metric (he did not prove the pointwise convergence). The remaining parts of Theorems 3 and 4 seem to be new.

(v) The plan of the paper is as follows. In Section 2 we give certain lemmas needed for the proofs of our theorems. The proofs of the theorems are completed in Section 3, which also contains some additional remarks about the theorems. Finally, Section 4 contains applications of Theorems 3 and 4 to conjugate functions in the case of several variables. For details we refer the reader to that section.

2. Auxiliary lemmas. *Lemma 1.* Let $\varphi_1(\zeta), \varphi_2(\zeta), \dots$ be functions regular and uniformly bounded for $|\zeta| < 1$. Suppose that the sequence of boundary values $\varphi_1(e^{i\theta}), \varphi_2(e^{i\theta}), \dots$ converges for every θ belonging to a set E of positive measure¹⁾. Then for almost every $\theta_0 \in E$ $\{\varphi_n(\zeta)\}$ converges uniformly in $\Delta(\theta_0, \eta)$, no matter what is $\eta < 1$ ²⁾.

Let us first consider an arbitrary function $\varphi(\zeta)$ regular and satisfying $|\varphi(\zeta)| < 1$ for $|\zeta| < 1$. Let us suppose that $|\varphi(e^{i\theta})| < \varepsilon$ for $\theta \in E$. The function $\psi(\zeta) = \log |\varphi(\zeta)|$ is a negative subharmonic function in $|\zeta| < 1$. Let $P(\varrho, \theta)$ be the Poisson kernel,

$$P(\varrho, \theta) = \frac{1}{2} \frac{1 - \varrho^2}{1 - 2\varrho \cos \theta + \varrho^2}.$$

¹⁾ Each φ_n obviously has boundary values $\varphi_n(e^{i\theta})$ almost everywhere. We recall that boundary values are defined as limits along nontangential paths.

²⁾ As I learnt recently, Lemma 1 is not entirely new. Cotlar and Vignaux [1] had shown that if the assumptions of Lemma 1 are satisfied, and if in addition the functions φ_n are all continuous on an arc of the unit circle containing E , then the sequence $\{\varphi_n(\zeta)\}$ converges uniformly over almost every radius terminating in E ; more precisely, over every radius terminating at a point of density of E .

In Lemma 1, the uniform boundedness of the functions $\varphi_n(\zeta)$ can be replaced by the condition that the subharmonic functions $\log^+ |\varphi_n(\zeta)|$ have a common harmonic majorant $h(\varrho, \theta)$ which is the Poisson integral of an L -integrable function (or, what is the same thing, that the integrals $\int h(\varrho, \theta) d\theta$ are uniformly absolutely continuous). The proof undergoes but little change.

One easily sees that for $\varrho < 1$ one has ^{a)}

$$(12) \quad \begin{aligned} \psi(\varrho e^{i\theta}) &\leq \frac{1}{\pi} \int_0^{2\pi} \psi(e^{iu}) P(\varrho, \theta - u) du \\ &\leq \frac{1}{\pi} \int_E \psi(e^{iu}) P(\varrho, \theta - u) du \leq -(\log 1/\varepsilon) \cdot \frac{1}{\pi} \int_E P(\varrho, \theta - u) du. \end{aligned}$$

The last integral here is the Poisson integral of the characteristic function $h(\theta)$ of the set E . In virtue of the classical result of Fatou, at every point θ_0 at which the integral $H(\theta)$ of $h(\theta)$ has a derivative, the Poisson integral of h tends to $H'(\theta_0)$ as $\varrho e^{i\theta}$ approaches $e^{i\theta_0}$ within any $\Delta(\theta_0, \eta)$. In particular, if θ_0 is a point of density of E , then $H'(\theta_0) = 1$, and the Poisson integral tends to limit 1 there. Since this integral represents a positive function inside the unit circle, one immediately sees that there is a positive constant $K = K(\theta_0, E, \eta)$ such that

$$\frac{1}{\pi} \int_E P(\varrho, \theta - u) du \geq K \quad \text{for } \zeta = \varrho e^{i\theta} \in \Delta(\theta_0, \eta).$$

This and (12) imply that

$$\log |\varphi(\zeta)| \leq -K \log(1/\varepsilon) \quad \text{for } \zeta \in \Delta(\theta_0, \eta).$$

The point of this inequality is that if $|\varphi(\zeta)| < 1$ for $|\zeta| < 1$, and if $|\varphi(e^{i\theta})|$ is uniformly small on E , then $|\varphi(\zeta)|$ is small in $\Delta(\theta_0, \eta)$.

It is now easy to complete the proof of Lemma 1. We may suppose that $|\varphi_n(\zeta)| < \frac{1}{2}$ for $|\zeta| < 1$ and for all n . Hence the functions $\Phi_{mn}(\zeta) = \varphi_m(\zeta) - \varphi_n(\zeta)$ are absolutely less than 1 in $|\zeta| < 1$. Without loss of generality, using Egoroff's theorem, we may assume that $\{\varphi_n(e^{i\theta})\}$ converges uniformly on E . Then $|\Phi_{mn}(e^{i\theta})|$ is uniformly small on E for m and n sufficiently large and so, by what has just been proved, $|\Phi_{mn}(\zeta)|$ is uniformly small in $\Delta(\theta_0, \eta)$ for such m, n , if θ_0 is a point of density of E . This completes the proof of Lemma 1.

^{a)} The first inequality that follows is obtained from the inequality

$$\psi(\varrho e^{i\theta}) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi(R e^{iu}) \frac{R^2 - \varrho^2}{R^2 - 2R\varrho \cos(\theta - u) + \varrho^2} du \quad (\varrho < R < 1)$$

which is obvious if φ has no zeros on $|\zeta| = R$. It is enough to make here $R \rightarrow 1$, and to use the very well known Fatou lemma.

The fact that under the conditions of Lemma 1 the sequence $\{\varphi_n(\zeta)\}$ converges uniformly in every circle $|\zeta| \leq r < 1$ is, of course, well known. For our purpose however, it is important to have uniform convergence in the remaining part of $\Delta(\theta_0, \eta)$.

Remark. As the above proof shows, the Lemma holds if the subscript n tends to a limit, infinite or finite, in a continuous way. In our applications, n will be a complex variable, or even a set of complex variables, tending to a finite limit.

Let D be a domain in the z -plane limited by a simple closed curve C . A subdomain D' of D limited by a simple closed curve C' will be called of type Δ , if

- a) C' has exactly one point z_0 in common with C , and otherwise is situated totally in the interior of C ;
- b) the curve C has a tangent t at z_0 ;
- c) the curve C' is non-tangential to C at z_0 .

Condition c) means that in the neighborhood of z_0 the curve C' is contained in an angle with vertex at z_0 , situated on one side of t , with neither side of the angle on t .

Lemma 2. Let $f_1(z), f_2(z), \dots$ be a sequence of functions regular and uniformly bounded in a domain D limited by a simple closed and rectifiable curve C . Suppose that the boundary values of the $f_n(z)$ exist at every point of a set E situated on C and of positive linear measure, and that these boundary values converge to a limit, as $n \rightarrow \infty$, at every point of E . Then almost every point z_0 of E has the following property: the sequence $\{f_n(z)\}$ converges uniformly inside every subdomain of D having z_0 on its boundary and of type Δ .

This lemma can be easily deduced from Lemma 1. For let $z = \mu(\zeta)$ be a function regular in the circle $|\zeta| < 1$ and mapping that circle conformally onto D . The functions $\varphi_n(\zeta) = f_n[\mu(\zeta)]$ are regular and uniformly bounded in $|\zeta| < 1$. Since the mapping can be extended in a continuous and one-one way to the closed circle $|\zeta| \leq 1$, and is conformal with respect to non-tangential paths at almost every point of $|\zeta| = 1$, and since the sets of linear measure zero on either boundary are mapped into sets of measure zero, one sees immediately that there is a set E' of positive measure situated on $|\zeta| = 1$ and such that at almost every point of it the boundary values of the functions $\varphi_n(\zeta)$ exist and converge to a limit as $n \rightarrow \infty$. The set E' is merely the image of E through the mapping $z = \mu(\zeta)$. If we omit from E' a subset of measure zero, the φ_n will possess the required

properties at every point of E' . Hence, by Lemma 1, for almost every point $e^{i\theta} \in E'$ the sequence $\{f_n(\zeta)\}$ converges uniformly in every $\Delta(\theta_0, \eta)$. If the mapping is conformal at $\zeta_0 = e^{i\theta_0}$, this implies the uniform convergence of $\{f_n(z)\}$ in every subdomain D' of D of type Δ and having $z_0 = \mu(e^{i\theta_0})$ on its boundary. Obviously almost every point $z_0 \in E$ has the required property, and the lemma follows.

Also this lemma holds if n is a continuous variable (or a set of variables) tending to a limit.

Lemma 3. Let $U_\lambda(\zeta)$ be a family of subharmonic functions in $|\zeta| < 1$ depending on a real-valued parameter λ and non-decreasing in λ for every fixed ζ , $|\zeta| < 1$. If

$$(13) \quad \int_0^{2\pi} U_\lambda(\rho e^{i\theta}) d\theta < M < +\infty \quad (0 \leq \rho < 1)$$

with M independent of λ and ρ , then there is a harmonic function $H(\zeta)$ in $|\zeta| < 1$ majorizing all the $U_\lambda(\zeta)$ there.

This is a classical result for harmonic functions. The passage to subharmonic functions is immediate.

Lemma 4. Under the assumptions of Theorem 1, for almost every point $(x_1^0, x_2^0, \dots, x_k^0)$, the function $f(z_1, z_2, \dots, z_k)$ is bounded in

$$(14) \quad \Delta(x_1^0, \eta) \times \Delta(x_2^0, \eta) \times \dots \times \Delta(x_k^0, \eta), \quad \text{whatever } \eta < 1^{11}.$$

Let η and λ denote two positive numbers less than 1. If $u(z)$ is a non-negative harmonic function in $|z| < \lambda$, and continuous for $|z| \leq \lambda$ we set

$$U^{\eta, \lambda}(x) = \text{Max } u(z) \quad \text{for } z \in \Delta(x, \lambda, \eta).$$

We shall need two inequalities for $U^{\eta, \lambda}(x)$ ¹⁰. The first is

$$(15) \quad \left\{ \int_0^{2\pi} [U^{\eta, \lambda}(x)]^\delta dx \right\}^{1/\delta} \leq B_{\eta, \delta} \int_0^{2\pi} u(\lambda e^{i\theta}) dx,$$

¹⁰ Both inequalities easily follow from the principal results of Hardy and Littlewood [1], though neither is stated there explicitly. A proof of (15) will be found in *TS*, p. 248.

¹¹ This lemma, whose refinement to Theorem 1 represents the main result of the present paper, was presented at the meeting of the Mathematical Club of the University of Michigan March, 1942.

where δ is any positive number less than 1. The second inequality is

$$(16) \quad \int_0^{2\pi} \chi_{\alpha-1} \{U^{\eta, \lambda}(x)\} dx \leq A_{\eta, \alpha} \int_0^{2\pi} \chi_\alpha \{u(\lambda e^{i\theta})\} dx + A_{\eta, \alpha},$$

where $\alpha \geq 1$ and

$$\chi_\alpha(t) = t \log^\alpha(t+1), \quad (t \geq 0).$$

Here and in what follows by letters A, B, \dots with various subscripts we shall denote constants, not necessarily always the same, depending only on the parameters exhibited in the subscripts.

It is seen immediately that the inequalities (15) and (16) remain valid if $u(z)$ is a function non-negative, continuous and subharmonic in $|z| < 1$. The condition of continuity can be dropped here, but the point is without importance for us, and the passage from subharmonic functions to their harmonic majorants is particularly simple if the former are continuous. For it is enough to apply the inequalities (15) and (16) to the function harmonic in $|z| < \lambda$, continuous in $|z| \leq \lambda$ and coinciding with the given subharmonic function u on $|z| = \lambda$, and then to use the fact that this harmonic function majorizes u in $|z| < \lambda$.

Let us write

$$(17) \quad \log^+ |f(z_1, \dots, z_k)| = u(z_1, \dots, z_k).$$

Taking, as before, $0 < \eta < 1$, $0 < \lambda < 1$, we set

$$\begin{aligned} U_{x_1}^{\eta, \lambda}(z_2, \dots, z_k) &= \text{Max } u(z_1, \dots, z_k) & \text{for } z_1 \in \Delta(x_1, \eta, \lambda), \\ U_{x_1 x_2}^{\eta, \lambda}(z_3, \dots, z_k) &= \text{Max } U_{x_1}^{\eta, \lambda}(z_2, \dots, z_k) & \text{for } z_2 \in \Delta(x_2, \eta, \lambda), \\ &\dots \dots \dots \\ U_{x_1 \dots x_k}^{\eta, \lambda} &= \text{Max } U_{x_1 \dots x_{k-1}}^{\eta, \lambda}(z_k) & \text{for } z_k \in \Delta(x_k, \eta, \lambda). \end{aligned}$$

Each of the expressions on the left is a continuous and subharmonic function of each of the variables z_j in parenthesis. If no confusion arises, we shall omit the superscripts η, λ in notation.

By (15),

$$(18) \quad \left\{ \int_0^{2\pi} (U_{x_1 \dots x_k})^\delta dx_k \right\}^{1/\delta} \leq B_{\eta, \delta} \int_0^{2\pi} U_{x_1 \dots x_{k-1}}(\lambda e^{i\theta_k}) dx_k.$$

If we integrate both sides with respect to x_1, \dots, x_{k-1} and apply Hölder's inequality to the left side, we get

$$(18a) \quad \left\{ \int_0^{2\pi} \dots \int_0^{2\pi} (U_{x_1 \dots x_k})^\delta dx_1 \dots dx_{k-1} \right\}^{1/\delta} \leq B_{\eta, \delta} \int_0^{2\pi} \dots \int_0^{2\pi} U_{x_1 \dots x_{k-1}}(\lambda e^{i\theta_k}) dx_1 \dots dx_{k-1}.$$

Similarly, by (16),

$$\int_0^{2\pi} U_{x_1 \dots x_{k-1}}(\lambda e^{ix_k}) dx_{k-1} \leq A_{\eta,1} \int_0^{2\pi} \chi_1 \{U_{x_1 \dots x_{k-2}}(\lambda e^{ix_{k-1}}, \lambda e^{ix_k})\} dx_{k-1} + A_{\eta,1},$$

and integration with respect to x_1, \dots, x_{k-2}, w_k gives

$$\begin{aligned} \int_0^{2\pi} \dots \int_0^{2\pi} U_{x_1 \dots x_{k-1}}(\lambda e^{ix_k}) dx_1 \dots dx_k &\leq \\ &\leq A_{\eta,1} \int_0^{2\pi} \dots \int_0^{2\pi} \chi_1 \{U_{x_1 \dots x_{k-2}}(\lambda e^{ix_{k-1}}, \lambda e^{ix_k})\} dx_1 \dots dx_k + A_{\eta,1}. \end{aligned}$$

Proceeding in the same way and constantly using (16) we get

$$\begin{aligned} \int_0^{2\pi} \chi_{k-2} \{U_{x_1}(\lambda e^{ix_2}, \dots, \lambda e^{ix_k})\} dx_1 &\leq \\ &\leq A_{\eta,k-1} \int_0^{2\pi} \chi_{k-1} \{u(\lambda e^{ix_1}, \dots, \lambda e^{ix_k})\} dx_1 + A_{\eta,k-1}, \end{aligned}$$

and so

$$\begin{aligned} (19) \quad \int_0^{2\pi} \dots \int_0^{2\pi} \chi_{k-2} \{U_{x_1}(\lambda e^{ix_2}, \dots, \lambda e^{ix_k})\} dx_1 \dots dx_k &\leq \\ &\leq A_{\eta,k-1} \int_0^{2\pi} \dots \int_0^{2\pi} \chi_{k-1} \{u(\lambda e^{ix_1}, \dots, \lambda e^{ix_k})\} dx_1 \dots dx_k + A_{\eta,k-1}. \end{aligned}$$

Using this chain of inequalities from (18a) through (19) we find

$$\begin{aligned} (20) \quad \left\{ \int_0^{2\pi} \dots \int_0^{2\pi} (U_{x_1 \dots x_k}^{\lambda \eta, \lambda})^\delta dx_1 \dots dx_k \right\}^{1/\delta} &\leq \\ &\leq C_{\eta, \delta, k} \int_0^{2\pi} \dots \int_0^{2\pi} \chi_{k-1} \{u(\lambda e^{ix_1}, \dots, \lambda e^{ix_k})\} dx_1 \dots dx_k + C_{\eta, \delta, k}. \end{aligned}$$

Let us now observe that the boundedness of the integral (3) is equivalent to an inequality

$$(21) \quad \int_0^{2\pi} \dots \int_0^{2\pi} \chi_{k-1} \{u(r_1 e^{ix_1}, \dots, r_k e^{ix_k})\} dx_1 \dots dx_k < M < +\infty \quad (r_1, \dots, r_k < 1).$$

Hence the left side of (20) never exceeds $C_{\eta, \delta, k} M + C_{\eta, \delta, k}$. The integrand on the left is a non-decreasing function of λ . Hence, making λ tend to 1 and setting

$$U_{x_1 \dots x_k}^\eta = \lim_{\lambda \rightarrow 1} U_{x_1 \dots x_k}^{\lambda \eta, \lambda}$$

we get

$$\left\{ \int_0^{2\pi} \dots \int_0^{2\pi} (U_{x_1 \dots x_k}^\eta)^\delta dx_1 \dots dx_k \right\}^{1/\delta} \leq C_{\eta, \delta, k} M + C_{\eta, \delta, k}.$$

Hence, in particular,

$$(22) \quad U_{x_1 \dots x_k}^\eta < +\infty$$

for almost every point x_1, \dots, x_k . Taking for η a sequence of values tending to 1, we obtain Lemma 4 since, as immediately seen, $U_{x_1 \dots x_k}$ is the upper bound of $u = \log^+ |f|$ in the domain (14).

Incidentally, we also proved that under the assumptions of Theorem 1 the function $U_{x_1 \dots x_k}$ belongs to L^δ for every $\delta < 1$.

3. Proofs of Theorems 1, 2 and 4. We begin with the proof of Theorem 2. Let $u(z)$ be subharmonic, non-negative and, say, continuous for $|z| < 1$. Let $0 < r_0 < r < 1$. Then

$$(23) \quad 2\pi u(z) \leq \frac{r+r_0}{r-r_0} \int_0^{2\pi} u(re^{ix}) dx \quad \text{for } |z| \leq r_0.$$

For u harmonic, this is a classical inequality. The extension to subharmonic functions is immediate.

For fixed values of z_2, \dots, z_k , each absolutely less than 1, the function $\log^+ |f(z_1, \dots, z_k)|$ is subharmonic in z_1 . Hence

$$I_r(z_1) = \int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f(z_1, r e^{ix_2}, \dots, r e^{ix_k})| dx_2 \dots dx_k$$

is a function of z_1 subharmonic and continuous in $|z_1| < 1$, for every value of the parameter r , $0 \leq r < 1$. It is a non-decreasing function of r . To prove the latter, it is enough to show that if we replace in the integral the numbers $r e^{ix_2}, \dots, r e^{ix_k}$ by $r_2 e^{ix_2}, \dots, r_k e^{ix_k}$ the resulting expression is a non-decreasing function of r_2, \dots, r_k . This is seen, for

example in the case r_2 , from the fact that $\int_0^{2\pi} \log^+ |f(z_1, r_2 e^{ix_2}, \dots, r_k e^{ix_k})| dx_2$

is a non-decreasing function of r_2 for every system of z_1, x_3, \dots, x_k and so remains a non-decreasing function of r_2 after integration with respect to x_3, \dots, x_k over the interval $(0, 2\pi)$.

$$\int_0^{2\pi} I_r(r_1 e^{ix_1}) dx_1 < M < +\infty,$$

$$(24) \quad I_T(z_1) = \int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f(z_1, re^{ix_2}, \dots, re^{ix_k})| dx_2 \dots dx_k < M(x_1, \eta)$$

[illegible]

$$\log^+ |f(z_1, z_2, \dots, z_k)| \leq \left(\frac{1}{2\pi}\right)^{k-1} \left(\frac{1+r_0}{1-r_0}\right)^{k-1} M(x_1^0, \eta),$$

$$(25) \quad |z_2| < 1, \dots, |z_k| < 1$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \left\{ \int_0^{2\pi} \log^+ |f(r_1 e^{ix_1}, r_2 e^{ix_2}, \dots, r_k e^{ix_k})| dx_1 \right\} dx_2 \dots dx_k < M.$$

$$\int_0^{2\pi} \log^+ |f(r_1 e^{ix_1}, r_2^0 e^{ix_2^0}, \dots, r_k^0 e^{ix_k^0})| dx_1 = o(1) \quad \text{as } r_1 \rightarrow 1.$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f(z_1, r_2 e^{ix_2}, \dots, r_k e^{ix_k})| dx_2 \dots dx_k < M(x_1, \eta), \quad \text{for } z_1 \in \Delta(x_1, \eta)$$

and for every set of numbers r_2, \dots, r_k less than 1. If x_1 also has the property that $f_x(z_2, \dots, z_k)$ exists in the sense just established, the

last inequality gives

$$\int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f_{x_1}(r_2 e^{ix_2}, \dots, r_k e^{ix_k})| dx_2 \dots dx_k \leq M(x_1, \eta) \quad (r_2, \dots, r_k < 1).$$

This completes the proof of Theorem 2.

We now pass to the proof of Theorem 1. The proof will be by induction from $k-1$ to k . For $k=1$, Theorem 1 reduces to the Nevanlinna-Ostrowski theorem.

Let $\chi(t)$ be a non-negative, increasing and convex function of $t \geq 0$, and suppose that $f(z_1, \dots, z_k)$ is a function regular in the k -cylinder I_k and satisfying the condition

$$(26) \quad \int_0^{2\pi} \dots \int_0^{2\pi} \chi \{ \log^+ |f(r_1 e^{ix_1}, \dots, r_k e^{ix_k})| \} dx_1 \dots dx_k = O(1)$$

for $r_1, \dots, r_k < 1$. Since $\chi(t) > kt$ for large t , where k is a positive constant, condition (26) implies an inequality (7). In particular, for almost every x_1 the function

$$(27) \quad f_{x_1}(z_2, \dots, z_k) = \lim_{z_1 \rightarrow e^{ix_1}} f(z_1, \dots, z_k)$$

exists and is regular for $|z_2| < 1, \dots, |z_k| < 1$. Moreover, for almost every x_1 we shall have not only the inequality (7a) but the stronger inequality

$$(28) \quad \int_0^{2\pi} \dots \int_0^{2\pi} \chi \{ \log^+ |f_{x_1}(r_2 e^{ix_2}, \dots, r_k e^{ix_k})| \} dx_2 \dots dx_k < M_{x_1} < \infty.$$

(In particular, if $f(z_1, \dots, z_k) \in H^\alpha$, so does $f_{x_1}(z_2, \dots, z_k)$ for almost every x_1).

The proof of this is merely a repetition of the proof of Theorem 2 if we replace in it $\log^+ |f|$ by $\chi(\log^+ |f|)$ (the proof is even slightly shorter because the existence of (27) has already been established). For the only property of $\log^+ |f|$ we used was the subharmonic character with respect to each of the variables z_1, \dots, z_k , a property which is obviously shared by $\chi(\log^+ |f|)$.

Let us now take for $\chi(t)$ the function $t \log^{k-1}(t+1)$. Under the assumptions of Theorem 1 condition (26) is then satisfied, and so for almost every x_1 we have (28), which is equivalent to

$$\int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f_{x_1}(r_2 e^{ix_2}, \dots, r_k e^{ix_k})| dx_2 \dots dx_k < M_{x_1} < \infty$$

(with M_{x_1} not necessarily the same as before). In particular, we have

$$\int_0^{2\pi} \dots \int_0^{2\pi} \log^+ |f_{x_1}(r_2 e^{ix_2}, \dots, r_k e^{ix_k})| dx_2 \dots dx_k < M_{x_1} < \infty$$

for almost every x_1 . Hence, if we assume the validity of Theorem 1 with k replaced by $k-1$, we get that for almost every x_1 the non-tangential limit

$$(29) \quad f_{x_1}(e^{ix_2}, \dots, e^{ix_k}) = \lim_{z_2 \rightarrow e^{ix_2}, \dots, z_k \rightarrow e^{ix_k}} f_{x_1}(z_2, \dots, z_k)$$

exists almost everywhere in (x_2, \dots, x_k) .

Let us now consider any point (x_2^0, \dots, x_k^0) with the following properties:

(i) For almost every x_1 the function $f(z_1, \dots, z_k)$ is bounded in

$$(30) \quad \Delta(x_1, \eta) \times \Delta(x_2^0, \eta) \times \dots \times \Delta(x_k^0, \eta)$$

whatever $\eta < 1$;

(ii) $f_{x_1}(e^{ix_2^0}, \dots, e^{ix_k^0})$ (see (29)) exists for almost every x_1 .

Almost every point (x_2^0, \dots, x_k^0) satisfies condition (i), and almost every such point satisfies condition (ii), and so also conditions (i) and (ii) simultaneously. Let us fix such a point (x_2^0, \dots, x_k^0) .

Let us also fix an η , $0 < \eta < 1$. On account of condition (i),

a) there is a perfect set P of measure arbitrarily close to 2π and such that $f(z_1, \dots, z_k)$ is uniformly bounded in (30) for $x_1 \in P$.

We may also assume that

b) the function $f_{x_1}(z_2, \dots, z_k)$ exists for every $x_1 \in P$ and is regular for $|z_2| < 1, \dots, |z_k| < 1$;

c) $f_{x_1}(e^{ix_2^0}, \dots, e^{ix_k^0})$ exists for $x_1 \in P$.

Let us now set

$$D = \sum_{x_1 \in P} \Delta(x_1, \eta).$$

Thus D is an open domain of a (familiar) starlike shape limited by a simple closed and rectifiable curve C . The latter has the points e^{ix_1} , $x_1 \in P$, and only such points, in common with the circumference $|z_1| = 1$. The function $f(z_1, \dots, z_k)$ is bounded for $z_1 \in D$ and $(z_2, \dots, z_k) \in \Delta(x_2^0, \eta) \times \dots \times \Delta(x_k^0, \eta)$. Let us consider here (z_2, \dots, z_k) as a parameter tending to $(e^{ix_2^0}, \dots, e^{ix_k^0})$. Thus we have a family of functions regular and uniformly bounded for $z_1 \in D$. On account

of property b), each function has a limit $f_{x_1}(z_2, \dots, z_k)$ at every point e^{ix_1} with $x_1 \in P$, and on account of c) these boundary values converge on P for $(z_2, \dots, z_k) \rightarrow (e^{ix_2^0}, \dots, e^{ix_k^0})$. On account of Lemma 2, for almost every $x_1 \in P$ the function $f(z_1, z_2, \dots, z_k)$ tends to a limit as (z_2, \dots, z_k) tends to $(e^{ix_2^0}, \dots, e^{ix_k^0})$ and z_1 approaches e^{ix_1} from within of a subdomain of D of type Δ . For almost every $x_1 \in P$ the tangents to $|z_1|=1$ and to C coincide (they certainly coincide, though this is not important for us, at every point of density of P). Thus for almost every point of P , and no matter what is $\eta' < 1$, the function $f(z_1, \dots, z_k)$ tends to a limit as z_1 tends to e^{ix_1} within $\Delta(x_1, \eta')$ and (z_2, \dots, z_k) tends to $(e^{ix_2^0}, \dots, e^{ix_k^0})$ within $\Delta(x_2^0, \eta) \times \dots \times \Delta(x_k^0, \eta)$. This property holds for almost every x_1 since the measure of P is arbitrarily close to 2π . Taking now η arbitrarily close to 1 we conclude that for almost every (x_2^0, \dots, x_k^0) we can find a set of measure 2π of values x_1^0 such that no matter what is $\eta < 1$, the function $f(z_1, \dots, z_k)$ tends to a limit as (z_1, \dots, z_k) tends to $(e^{ix_1^0}, e^{ix_2^0}, \dots, e^{ix_k^0})$ remaining within $\Delta(x_1^0, \eta) \times \Delta(x_2^0, \eta) \times \dots \times \Delta(x_k^0, \eta)$.

Thus we proved the existence of the limit $f(e^{ix_1^0}, \dots, e^{ix_k^0})$ for almost every point (x_1^0, \dots, x_k^0) , under the assumption that Theorem 1 is true with k replaced by $k-1$. Since the assumption is valid for $k=2$, Theorem 1 is established completely.

We now pass to the proof of Theorem 4. We restrict ourselves to the case $k=2$ which is entirely typical.

Let us consider any function $u(z)$ subharmonic and, say, continuous in $|z| < 1$. Using the same notation as in the proof of Lemma 4, we have the inequality

$$\int_0^{2\pi} \{U^{p, \lambda}(x)\}^p dx \leq A_\eta \left(\frac{p}{p-1}\right)^p \int_0^{2\pi} u^p(\lambda e^{ix}, \lambda e^{i\theta}) d\theta \quad (p > 1),$$

due to Hardy and Littlewood (a proof can also be found in *TS*, p. 248).

Let now $u(z, \zeta)$ be non-negative, continuous and subharmonic in z and ζ separately, for $|z| < 1$, $|\zeta| < 1$. Let $0 < \eta < 1$, $0 < \lambda < 1$. Let

$$(31) \quad U^{p, \lambda}(x, \theta) = \text{Max } u(z, \zeta) \quad \text{for } z \in \Delta(x, \lambda\eta, \lambda), \quad \zeta \in \Delta(\theta, \lambda\eta, \lambda).$$

We shall prove that

$$(32) \quad \int_0^{2\pi} \int_0^{2\pi} \{U^{p, \lambda}(x, \theta)\}^p dx d\theta \leq \left\{A_\eta \left(\frac{p}{p-1}\right)^p\right\}^2 \int_0^{2\pi} \int_0^{2\pi} u^p(\lambda e^{ix}, \lambda e^{i\theta}) dx d\theta \quad (p > 1).$$

For, given any ζ , $|\zeta| < 1$, let us set

$$(33) \quad V^{p, \lambda, \lambda}(x, \zeta) = \text{Max } u(z, \zeta) \quad \text{for } z \in \Delta(x, \eta\lambda, \lambda).$$

Then

$$U^{p, \lambda, \lambda}(x, \theta) = \text{Max } V^{p, \lambda, \lambda}(x, \zeta) \quad \text{for } \zeta \in \Delta(\theta, \eta\lambda, \lambda).$$

Since, for η, λ, x fixed, $V^{p, \lambda, \lambda}(x, \zeta)$ is subharmonic and continuous in $|\zeta| < 1$,

$$\int_0^{2\pi} \{U^{p, \lambda, \lambda}(x, \theta)\}^p d\theta \leq A_\eta \left(\frac{p}{p-1}\right)^p \int_0^{2\pi} \{V^{p, \lambda, \lambda}(x, \lambda e^{i\theta})\}^p d\theta,$$

or, integrating both sides with respect to x and inverting the order of integration on the right,

$$(34) \quad \int_0^{2\pi} \int_0^{2\pi} \{U^{p, \lambda, \lambda}(x, \theta)\}^p dx d\theta \leq A_\eta \left(\frac{p}{p-1}\right)^p \int_0^{2\pi} d\theta \int_0^{2\pi} \{V^{p, \lambda, \lambda}(x, \lambda e^{i\theta})\}^p dx.$$

For any fixed ζ , $|\zeta| < 1$, the function $u(z, \zeta)$ is subharmonic in z , and so

$$\int_0^{2\pi} \{V^{p, \lambda, \lambda}(x, \lambda e^{i\theta})\}^p dx \leq A_\eta \left(\frac{p}{p-1}\right)^p \int_0^{2\pi} u^p(\lambda e^{ix}, \lambda e^{i\theta}) dx.$$

Substituting this inequality into the right side of (34) we get (32).

For $p \geq 2$ the expression $\{p/(p-1)\}^p$ does not exceed 4. Hence, if we take for p in (32) the value $2k$, multiply both sides of the inequality by $a^{2k}/(2k)!$ and sum over $k=1, 2, 3, \dots$ we obtain

$$\int_0^{2\pi} \int_0^{2\pi} \cosh \{a U^{2, \lambda, \lambda}(x, \theta)\} dx d\theta \leq 16 A_\eta^2 \int_0^{2\pi} \int_0^{2\pi} \cosh \{a u(\lambda e^{ix}, \lambda e^{i\theta})\} dx d\theta,$$

assuming, as we may, that $16A_\eta^2 \geq 1$. Or, on account of the inequality $\frac{1}{2}e^t \leq \cosh t \leq e^t$, valid for $t \geq 0$,

$$(35) \quad \int_0^{2\pi} \int_0^{2\pi} \exp \{a U^{\eta, \lambda, \lambda}(x, \theta)\} dx d\theta \leq 32 A_\eta^2 \int_0^{2\pi} \int_0^{2\pi} \exp \{a u(\lambda e^{ix}, \lambda e^{i\theta})\} dx d\theta.$$

It is now easy to complete the proof of Theorem 4. For let $f(z, \zeta)$ be a function regular for $|z| < 1$, $|\zeta| < 1$, and satisfying

$$(36) \quad \int_0^{2\pi} \int_0^{2\pi} |f(re^{ix}, \rho e^{i\theta})|^\alpha dx d\theta \leq M^\alpha \quad (r, \rho < 1, \alpha > 0).$$

Let us now set

$$\begin{aligned} u(z, \zeta) &= \log^+ |f(z, \zeta)| \\ F^{\eta, \lambda, \lambda}(x, \theta) &= \max |f(z, \zeta)| \quad \text{for } (z, \zeta) \in \Delta(x, \lambda\eta, \lambda) \times \Delta(\theta, \lambda\eta, \lambda) \\ F^\eta(x, \theta) &= \lim_{\lambda \rightarrow 1} F^{\eta, \lambda, \lambda}(x, \theta). \end{aligned}$$

Then (see (31)),

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \{F^{\eta, \lambda, \lambda}(x, \theta)\}^\alpha dx d\theta &\leq \int_0^{2\pi} \int_0^{2\pi} \exp \{a U^{\eta, \lambda, \lambda}(x, \theta)\} dx d\theta \\ &\leq 32 A_\eta^2 \int_0^{2\pi} \int_0^{2\pi} \exp \{a u(\lambda e^{ix}, \lambda e^{i\theta})\} dx d\theta \\ &\leq 32 A_\eta^2 \int_0^{2\pi} \int_0^{2\pi} \{|f(\lambda e^{ix}, \lambda e^{i\theta})|^\alpha + 1\} dx d\theta. \end{aligned}$$

Let us compare the extreme terms here. If we apply the resulting inequality to the function Kf , where K is a positive constant, and then make K tend to $+\infty$, we get

$$\int_0^{2\pi} \int_0^{2\pi} \{F^{\eta, \lambda, \lambda}(x, \theta)\}^\alpha dx d\theta \leq 32 A_\eta^2 \int_0^{2\pi} \int_0^{2\pi} |f(\lambda e^{ix}, \lambda e^{i\theta})|^\alpha dx d\theta \leq 32 A_\eta^2 M^\alpha,$$

and, making $\lambda \rightarrow 1$,

$$\int_0^{2\pi} \int_0^{2\pi} \{F^\eta(x, \theta)\}^\alpha dx d\theta \leq 32 A_\eta^2 M^\alpha,$$

which completes the proof of Theorem 4.

Theorem 5. Under the conditions of Theorem 1, at almost every point (x_1, x_2, \dots, x_k) the non-tangential limit

$$f(e^{ix_1}, e^{ix_2}, \dots, e^{ix_k}) = \lim f(z_1, z_2, \dots, z_k)$$

is equal to the iterated limit $f_{x_1 x_2 \dots x_k}$.

The proof is almost immediate and it is enough to consider the case $k=2$. Since, by hypothesis, $f(e^{ix}, e^{ix_2})$ exists almost everywhere, given any $\varepsilon > 0$ we can find a set E of points (x_1, x_2) situated in the square $Q(0 \leq x_1 \leq 2\pi, 0 \leq x_2 \leq 2\pi)$ such that the measure of $Q - E$ is arbitrarily small and that

$$(37) \quad \begin{aligned} &|f(r_1 e^{ix_1}, r_2 e^{ix_2}) - f(e^{ix_1}, e^{ix_2})| < \varepsilon \\ &\text{for } (x_1, x_2) \in E, \quad 1 - r_1 < \delta, \quad 1 - r_2 < \delta, \quad \delta = \delta(\varepsilon) \end{aligned}$$

(we use here the existence of the radial limit only). We know that $f_{x_1}(x_2)$ exists for almost every x_1 . Hence, making r_1 tend to 1 in (37), we see that

$$|f_{x_1}(r_2 e^{ix_2}) - f(e^{ix_1}, e^{ix_2})| \leq \varepsilon \quad (1 - r_2 < \delta)$$

for almost every x_1 such that $(x_1, x_2) \in E$ or, in other words, for almost every (x_1, x_2) in E . Making now r_2 tend to 1 we see that

$$|f_{x_1 x_2} - f(e^{ix_1}, e^{ix_2})| \leq \varepsilon$$

for almost every point in E . Since ε and the measure of $Q - E$ are arbitrarily small, Theorem 5 is established.

It follows from Theorem 5 that under the assumptions of Theorem 5 (in particular for the functions of the class H^*) the iterated limit of the function f is almost always independent of the order in which the variables z_1, z_2, \dots, z_k tend successively to limits. It is likely that the result remains valid for the functions f satisfying the conditions of Theorem 2.

Added in proof. 15. IX. 1949. It can be shown that under the assumptions of Theorem 2 the function f tends to a limit as the point (z_1, \dots, z_k) approaches almost every point $(e^{i\theta_1}, \dots, e^{i\theta_k})$ non-tangentially and *regularly*. „Regularly” means that all the ratios $(1 - r_j)/(1 - r_k)$ remain bounded. A proof will be given in a paper by A. P. Calderón and myself.

4. Applications to conjugate series of several variables. First we recall a few familiar facts from the theory of trigonometric series ¹²⁾

$$(38) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

of a single variable (we restrict ourselves to series with real coefficients). This series is the real part of the power series

$$(39) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n - ib_n)z^n$$

if we set here $z = e^{ix}$. The imaginary part of (39), under the same assumption, is

$$(40) \quad \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)$$

and is called the *conjugate* series of (39). If we denote (39) by S , its conjugate will be denoted by \tilde{S} . If S is the Fourier series of a (Lebesgue) integrable function f , then almost everywhere \tilde{S} is summable Abel (or even (C,1)) to sum

$$(41) \quad \tilde{f}(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \frac{1}{2} \cot \frac{t}{2} dt = -\frac{1}{\pi} \lim_{\epsilon \rightarrow +0} \int_{-\pi}^{(\epsilon)\pi}$$

where $\int_{-\pi}^{(\epsilon)\pi}$ denotes the integral extended over the part of $(-\pi, \pi)$ situated outside the interval $(-\epsilon, \epsilon)$. The result includes the assertion that $\tilde{f}(x)$ exists almost everywhere for every integrable f .

If we write S in the form

$$(42) \quad \sum_{n=-\infty}^{+\infty} c_n e^{inx}, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 0; \quad c_{-n} = \bar{c}_n,$$

then \tilde{S} can be written

$$(43) \quad \sum_{n=-\infty}^{+\infty} (-i \operatorname{sign} n) c_n e^{inx}.$$

¹²⁾ All the results needed here will be found in *TS*, Ch. VII. For the inequality (45), see *TS*, p. 165. Ex. 7.

The very well known result of M. Riesz asserts that if f belongs to L^p , $p > 1$, so does \tilde{f} . In what follows we shall also need the following two inequalities

$$(44) \quad \left(\int_{-\pi}^{\pi} |\tilde{f}|^{\delta} dx \right)^{1/\delta} \leq A_{\delta} \int_{-\pi}^{\pi} |f| dx \quad (0 < \delta < 1),$$

$$(45) \quad \int_{-\pi}^{\pi} |\tilde{f}| (\log^{+} |\tilde{f}|)^{\alpha-1} dx \leq A_{\alpha} \int_{-\pi}^{\pi} |f| (\log^{+} |f|)^{\alpha} dx + A_{\alpha} \quad (\alpha \geq 1),$$

the latter especially in the case $\alpha = 1$.

Let us now consider a double trigonometric series

$$(46) \quad \sum_{m,n=-\infty}^{+\infty} c_{mn} e^{i(mx+ny)} \quad c_{-m,-n} = \bar{c}_{m,n}.$$

It can be written in the equivalent form

$$(47) \quad \sum_{m,n=0}^{\infty} (a_{mn} \cos mx \cos ny + \beta_{mn} \sin mx \cos ny + \gamma_{mn} \cos mx \sin ny + \delta_{mn} \sin mx \sin ny) \lambda_{mn},$$

where the multipliers λ_{mn} are introduced for the sake convenience only and are defined by the equations

$$\lambda_{mn} = 1, \quad \lambda_{m0} = \lambda_{0n} = \frac{1}{2}, \quad \lambda_{00} = \frac{1}{4} \quad (m, n \neq 0).$$

The coefficients $\alpha, \beta, \gamma, \delta$ are then easily expressible in terms of the c_{mn} . For example, $a_{mn} = c_{m,n} + c_{m,-n} + c_{-m,n} + c_{-m,-n}$, etc. If $c_{mn} = 0$ whenever $mn = 0$, we shall say that (46), or (47), is of *restricted* type.

Let us now consider a power series

$$(48) \quad \sum_{m,n=0}^{\infty} (A_{mn} - iB_{mn}) z^m \zeta^n$$

of two variables. The real part of it for $z = e^{ix}$, $\zeta = e^{iy}$ is

$$(49) \quad \sum_{m,n=0}^{\infty} A_{mn} \cos (mx + ny) + B_{mn} \sin (mx + ny),$$

which shows that only special series (47) are real parts of power series of two variables. Writing (49) in the form (46), we see that a necessary and sufficient condition for (46) to be the real part of a power

series is that $c_{mn}=0$ for $mn<0$. If (49) is the real part of (48), the imaginary part is

$$(50) \quad \sum_{m,n=0}^{\infty} A_{mn} \sin(mx+ny) - B_{mn} \cos(mx+ny).$$

Thus the notion of conjugate series does not appear naturally in the case of series of two variables. We may however introduce it formally. For using the definition (43) we may consider alongside (46) the three series

$$(51) \quad \sum c_{mn}(-i \operatorname{sign} m) e^{i(mx+ny)}, \quad \sum c_{mn}(-i \operatorname{sign} n) e^{i(mx+ny)}, \\ \sum c_{mn}(-\operatorname{sign} mn) e^{i(mx+ny)}$$

which will be the conjugates of S with respect to x , to y , and to x, y . If we denote (46) by S , these series will be written \tilde{S}^x , \tilde{S}^y and \tilde{S}^{xy} respectively. The series $\tilde{S}^{xy} = \tilde{S}^{yx}$ is always of restricted type. If S is of restricted type and of the form (49), then (50) is both \tilde{S}^x and \tilde{S}^y . The series (50) will be called the *conjugate* of (49). If the constant term of the power series (48) is real, an assumption which we can always make, the constant term of (50) is zero.

It is important to observe that, for the most general series S in (48), $S - \tilde{S}^{xy}$ is always the real part of a power series. If, in addition, S is of restricted type then the conjugate of $S - \tilde{S}^{xy}$ is $\tilde{S}^x + \tilde{S}^y$. For

$$S - \tilde{S}^{xy} = \sum c_{mn}(1 + \operatorname{sign} mn) e^{i(mx+ny)}$$

and the coefficients on the right are zero if $mn<0$. The second part of the italicized statement follows immediately from the formula

$$(1 + \operatorname{sign} mn) \operatorname{sign} m = \operatorname{sign} m + \operatorname{sign} n \quad (m, n \neq 0).$$

The importance of this lies in the fact that though (unlike in the case of series of one variable) the complex methods cannot be applied to the general series S , they are applicable to $S - \tilde{S}^{xy}$. If we can obtain some information about the behavior of the latter series, and if we know the behavior of S , we get information about \tilde{S}^{xy} .

As an application of this idea we are going to prove the following result, in which f is of period 2π with respect to x and y , and Q denotes the square $|x| \leq \pi$, $|y| \leq \pi$.

Theorem 6'. If $f(x, y) \log^+ |f(x, y)| \in L$, then

$$(52) \quad \tilde{f}(x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v) \frac{1}{2} \cot \frac{1}{2} u \frac{1}{2} \cot \frac{1}{2} v du dv = \\ = \frac{1}{\pi^2} \lim_{\varepsilon, \eta \rightarrow +0} \int_{-\pi}^{(\varepsilon)\pi} \int_{-\pi}^{(\eta)\pi} f(x+u, y+v) \frac{1}{2} \cot \frac{1}{2} u \frac{1}{2} \cot \frac{1}{2} v du dv$$

exists almost everywhere. Moreover

$$\left(\int_Q |\tilde{f}|^p dx dy \right)^{1/p} \leq A_\delta \int_Q |f| \log^+ |f| dx dy + A_\delta.$$

The existence of \tilde{f} for $f \in L^p$, $p > 1$, was proved by Sokół-Sokołowski [1]. He even showed that \tilde{f} exists if $f (\log^+ |f|)^3$ is integrable, and he conjectured that this result holds if the exponent 3 is replaced by 1.

In proving Theorem 6' we shall be concise and we shall omit certain details which can be found in the paper just quoted. Let us denote the $(C, 1, 1)$ means of the series (51) respectively by $\tilde{\sigma}_{mn}^x(x, y)$, $\tilde{\sigma}_{mn}^y(x, y)$, $\tilde{\sigma}_{mn}^{xy}(x, y)$, and the Abel means by $\tilde{F}_{re}^x(x, y)$, $\tilde{F}_{re}^y(x, y)$, $\tilde{F}_{re}^{xy}(x, y)$. Sokołowski proves (l. c., Theorem 4) that, under the assumptions of Theorem 6',

$$(53) \quad \tilde{\sigma}_{mn}^{xy}(x, y) - \frac{1}{\pi^2} \int_{-\pi}^{(\varepsilon)\pi} \int_{-\pi}^{(\eta)\pi} f(x+u, y+v) \frac{1}{2} \cot \frac{1}{2} u \frac{1}{2} \cot \frac{1}{2} v du dv \rightarrow 0 \\ \left(\varepsilon, \eta \rightarrow 0, m = \left[\frac{1}{\varepsilon} \right], n = \left[\frac{1}{\eta} \right] \right)$$

almost everywhere, and exactly the same proof (for Fejér's and Poisson's kernels satisfy similar inequalities) shows that

$$(54) \quad \tilde{F}_{re}^{xy}(x, y) - \frac{1}{\pi^2} \int_{-\pi}^{(\varepsilon)\pi} \int_{-\pi}^{(\eta)\pi} f(x+u, y+v) \frac{1}{2} \cot \frac{1}{2} u \frac{1}{2} \cot \frac{1}{2} v du dv \rightarrow 0 \\ (\varepsilon = 1-r \rightarrow 0, \eta = 1-\rho \rightarrow 0).$$

Let us assume first that S , defined by (46), is of restricted type. Since $f \log^+ |f|$ is integrable, both \tilde{S}^x and \tilde{S}^y are Fourier series of integrable functions $\tilde{f}^x(x, y)$ and $\tilde{f}^y(x, y)$, and so $V = \tilde{S}^x + \tilde{S}^y$ is the Fourier series of $h = \tilde{f}^x + \tilde{f}^y \in L$. Moreover V is the imaginary part,

on the unit circle, of a (restricted) power series $F(z, \zeta)$ defined by (48). Hence the real part $U = S - \tilde{S}^{xy} = \sum u_{mn} \epsilon^{i(mx+ny)}$ of that power series is of the class L^δ , for every $0 < \delta < 1$. More precisely, if we set $u_{re}(x, y) = \sum u_{mn} \epsilon^{i[mx+ny]}$, then

$$(55) \quad \left(\int_Q |u_{re}(x, y)|^\delta dx dy \right)^{1/\delta} \leq A_\delta \int_Q |h(x, y)| dx dy \\ \leq A_\delta \int_Q |f| \log^+ |f| dx dy + A_\delta.$$

It follows that $F(z, \zeta) \in H^\delta$ for every $\delta < 1$, and so, on account of Theorem 1, has a non-tangential limit almost everywhere. In particular, $S - \tilde{S}^{xy}$ is Abel summable at almost every point. Since the same holds for S (see Jessen, Marcinkiewicz and Zygmund [1]), \tilde{S}^{xy} is Abel summable almost everywhere. On account of (54), this completes the proof of the first part of Theorem 6', under the assumptions that S is of restricted type.

To dispose of that assumption we consider the function

$$(56) \quad g(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y) dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y) dy - \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) dx dy,$$

which has the same "marginal" coefficients c_{m0}, c_{0n} as f , the remaining coefficients being all equal to zero. Thus the Fourier series of $f_1 = f - g$ is of restricted type, and since $g \log^+ |g| \in L$ the function $f_1 \log^+ |f_1|$ is integrable, so that $\tilde{f}^{xy}(x, y)$ exists almost everywhere. On the other hand, $\tilde{g}^{xy}(x, y)$ exists everywhere and vanishes identically. Hence \tilde{f}^{xy} exists almost everywhere.

The second part of Theorem 6' follows immediately from (55), and the extension to "non-restricted" series is again achieved by considering the function (56).

The purpose of the foregoing argument was primarily to show how complex methods can be applied to double trigonometric series, and we are not interested in exploiting the field of conjugate series here. We may however state without proof some additional result which can easily be proved.

First of all, from Theorem 6' and from (53) follows that \tilde{S}^{xy} is almost everywhere summable by the method $(C, 1, 1)$ (or even summable (C, α, β) , $\alpha > 0$, $\beta > 0$, as can be shown similarly). We have already stated the fact that \tilde{S}^{xy} is almost everywhere summable

Abel. That $\tilde{F}_{re}^{xy}(x, y)$ has non-tangential limits almost everywhere can be proved similarly. An application of (45) shows that

$$\int_Q |\tilde{f}^{xy}| \{ \log(2 + |\tilde{f}^{xy}|) \}^{a-2} dx dy \leq A_a \int_Q |f| |\log^+ |f||^a dx dy + A_a \quad (a > 1),$$

an inequality which for $a=1$ degenerates into the second part of Theorem 6'. What is more, both inequalities hold if instead of \tilde{f}^{xy} we write there

$$\tilde{f}^*(x, y) = \sup_{\epsilon, \eta} \left| \tau^{-2} \int_{-\pi}^{(\epsilon)\pi} \int_{-\pi}^{(\eta)\pi} f(x+u, y+v) \frac{1}{2} \cot \frac{1}{2} u \frac{1}{2} \cot \frac{1}{2} v du dv \right| \quad (\epsilon < \pi, \eta < \pi)$$

$$\text{or } \tilde{\sigma}^*(x, y) = \sup_{m, n} |\tilde{\sigma}_{m, n}^{xy}(x, y)|, \text{ or } \tilde{P}^*(x, y) = \sup_{re} |\tilde{L}_{re}^{xy}(x, y)|.$$

We end by a few remarks concerning trigonometric series S of k variables

$$(57) \quad \sum_{n_1, \dots, n_k = -\infty}^{+\infty} c_{n_1, \dots, n_k} \epsilon^{i(n_1 x_1 + \dots + n_k x_k)}.$$

The notions of restricted series and of conjugate series are defined as before. A necessary and sufficient condition for S to be the real part of a power series $\sum c_{n_1, \dots, n_k} z_1^{n_1} \dots z_k^{n_k}$ is that all the coefficients c_{n_1, \dots, n_k} whose indices distinct from 0 are not of the same sign vanish.

Theorem 6. Let $f(x_1, \dots, x_k)$ be of period 2π with respect to each variable and let $f \log^+ |f|^{k-1} \in L$. Then

$$\tilde{f}(x_1, \dots, x_k) = \lim_{\epsilon_1, \dots, \epsilon_k \rightarrow 0} \frac{(-1)^k}{\pi^k} \int_{-\pi}^{(\epsilon_1)\pi} \dots \int_{-\pi}^{(\epsilon_k)\pi} f(x_1 + u_1, \dots, x_k + u_k) \frac{1}{2} \cot \frac{1}{2} u_1 \dots \frac{1}{2} \cot \frac{1}{2} u_k du_1 \dots du_k$$

exists almost everywhere, and

$$\left(\int_Q |\tilde{f}|^\delta dx_1 \dots dx_k \right)^{1/\delta} \leq A_{\delta, k} \int_Q |f| (\log^+ |f|)^{k-1} dx_1 \dots dx_k + A_{\delta, k}.$$

If we use (44) and (45) the proof of Theorem 6 follows the same line as that of Theorem 6, provided we show that a proper linear combination of S and of its conjugates is always the real part of a power series. We confine our attention to this problem only.

Let S be an arbitrary series (57). Let $\sum S^{x_1}$ denote the sum of the conjugates of S with respect to one variable, $\sum S^{x_1 x_2}$ — the sum of the conjugates with respect to two distinct variables, etc. We shall show that

$$(58) \quad S - \sum S^{x_1 x_2} + \sum S^{x_1 x_2 x_3 x_4} - \dots$$

is always the real part of a power series of k variables, and that if S is of restricted type, the imaginary part of that power series will be

$$(59) \quad \sum S^{x_1} - \sum S^{x_1 x_2 x_3} + \dots$$

For let n_1, n_2, \dots, n_k be any set of indices all distinct from zero, and let $\eta_j = \text{sign } n_j$. We have to show that if the n_j are not of the same sign then

$$(60) \quad 1 + \sum \eta_1 \eta_2 + \sum \eta_1 \eta_2 \eta_3 \eta_4 + \dots$$

is zero. For let us assume for the sake of definiteness that k is even (the proof for k odd is similar) and let us denote the sum (60) by P_k . Let us denote a similar sum formed with $\eta_1, \dots, \eta_{k-2}$ by P_{k-2} and let the sum

$$\sum \eta_1 + \sum \eta_1 \eta_2 \eta_3 + \dots$$

be N_{k-2} . Then one easily verifies the relation $P_k = P_{k-2}(1 + \eta_{k-1} \eta_k) + N_{k-2}(\eta_{k-1} + \eta_k)$, from which immediately follows that $P_k = 0$ if for example $\eta_{k-1} \eta_k < 0$.

If S is of restricted type, so is (58), and one easily verifies that the conjugate of (58) with respect, say, to x_1 is (59).

The other remarks we made about double trigonometric series are immediately extensible to the series (57).

Buenos Aires, R. Argentina 1. VI. 1948.

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