The lattice of all open sets of a Hausdorff space satisfies conditions (1),...,(5) of the preceding theorem if and only if the space is locally compact and totally-disconnected. Therefore (by virtue of Stone's theorem on the topological representation of Boolean rings)[9] theorem 2 gives also the characterization of the lattice of all open sets of a locally compact totally-disconnected space. The compact case is obtained by adding condition (6) or (6′).

[9]) See M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., vol. **41** (1937), pp. 375-481.

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An undecidable arithmetical statement.

By

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The purpose of this paper is to give an alternative proof of the existence of formally undecidable sentences. Instead of the arithmetization of syntax and the diagonal process which were used by Gödel in his famous paper of 1931[1]), I shall make use of some simple set-theoretic lemmas and of the Skolem-Löwenheim theorem.

My result is in some respect stronger than that of Gödel. The sentence constructed by his method ceases to be undecidable if one enlarges the underlying logic by a new rule of proof, in the simplest case by the rule of infinite induction[2]). The undecidability of the sentence to be constructed here is, on the contrary, independent of whether we accept the absolute notion of integers or the relative (axiomatic) one[2a]).

On the other hand the proof of undecidability to be given below is unlike that of Gödel non-finitary. It rests on the axioms of the Zermelo-Fraenkel set-theory including the axiom of choice and an additional axiom ensuring the existence of at least one inaccessible aleph[3]). Finally the method of Gödel gives undecidable sentences expressed in terms of the arithmetic of natural numbers whereas we shall obtain here a sentence from the arithmetic of reals.

[1]) Gödel [4]. Numbers in brackets refer to the bibliography on p. 163.

[2]) Tarski [12].

[2a]) Other such sentences have been constructed by Rosser [10] and Tarski [15]. My method is different from theirs.

[3]) Tarski [14]. Using Tarski's terminology we would have to say that $\aleph_\xi$ is weakly inaccessible.

1. Axioms of set-theory. V. Neumann [4]), Bernays [5]), and Gödel [6]) have shown how to build up an axiomatic system of set theory, presupposing only the restricted functional calculus (with identity) as logical basis, in such a way that it contains only a finite number of axioms. A slight modification of the systems of Bernays and Gödel allows us to reduce the number of primitive notions to one, viz. a binary relation ϵ [7]).

We shall state here explicitly the axioms of the resulting system (S).

Elements of the field of ϵ are called classes and elements of the domain of ϵ are called sets. Thus every set is a class but not conversely. The lower case Latin letters will be used as variables for sets and the upper case Latin letters as variables for classes.

The axioms fall into several groups [8]):

Group A.

$A1.$ $\prod_u(u \epsilon X \equiv u \epsilon Y) \rightarrow X = Y$,

$A2.$ $\sum_z \prod_u [u \epsilon z \equiv (u = x + u = y)]$.

The set z whose existence is stated in $A2$ is called the non-ordered pair of x and y and denoted by $\{x,y\}$. The ordered pair of x and y is defined as

$$\langle x,y \rangle = \{\{x,y\}, \{x,x\}\}.$$

Group B.

$B1.$ $\sum_A \prod_{xy}(\langle x,y\rangle \epsilon A \equiv x \epsilon y)$,

$B2.$ $\sum_C \prod_u [u \epsilon C \equiv (u \epsilon A)\cdot(u \epsilon B)]$,

$B3.$ $\sum_B \prod_u [u \epsilon B \equiv (u \epsilon A)']$,

$B4.$ $\sum_B \prod_x (x \epsilon B \equiv \sum_y \langle y,x\rangle \epsilon A)$,

$B5.$ $\sum_B \prod_{xy}(\langle y,x\rangle \epsilon B \equiv x \epsilon A)$,

$B6.$ $\sum_B \prod_{xy}(\langle x,y\rangle \epsilon B \equiv \langle y,x\rangle \epsilon A)$,

$B7.$ $\sum_B \prod_{xyz}(\langle x,\langle y,z\rangle\rangle \epsilon B \equiv \langle y,\langle z,x\rangle\rangle \epsilon A)$,

$B8.$ $\sum_B \prod_{xyz}(\langle x,\langle y,z\rangle\rangle \epsilon B \equiv \langle x,\langle z,y\rangle\rangle \epsilon A)$.

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[4]) v. Neumann [8].

[5]) Bernays [1].

[6]) Gödel [5].

[7]) The possibility of this reduction has been realised by A. Tarski. See a remark made on p. 208 of [7].

[8]) The logical symbols to be used in this paper are as follows: $\rightarrow$ (if, then), $+$ (or), $\cdot$ (and), $\equiv$ (if and only if), $'$ (not), Π (for every), Σ (there is). The axioms should have been preceded by general quantifiers so as to bind all the variables.

From these axioms it follows that there is a uniquely determined class 0 (called the null-class) such that $\prod_x(x \epsilon 0)'$.

We shall use the following abbreviations:

$X \subset Y$ for $\prod_u(u \epsilon X \rightarrow u \epsilon Y)$,

$J(X)$ for $\prod_{uvw}(\langle v,u\rangle \epsilon X \cdot \langle w,u\rangle \epsilon X \rightarrow v = w)$.

Group C.

$C1.$ $\sum_x((x \neq 0)\cdot \prod_y\{(y \epsilon x) \rightarrow \sum_z[(z \epsilon x)\cdot(z \neq y)\cdot(y \subset z)]\})$,

$C2.$ $\sum_y \prod_{uv}[(u \epsilon v)\cdot(v \epsilon x) \rightarrow (u \epsilon y)]$,

$C3.$ $\sum_y \prod_u [(u \subset x) \rightarrow (u \epsilon y)]$,

$C4.$ $J(A) \rightarrow \sum_y \prod_u\{(u \epsilon y) \equiv \sum_v[(v \epsilon x)\cdot(\langle u,v\rangle \epsilon A)]\}$.

Group D.

$D1.$ $A \neq 0 \rightarrow \sum_u\{(u \epsilon A)\cdot \prod_v[(v \epsilon u) \rightarrow (v \epsilon A)']\}$.

Group E.

$E1.$ $\sum_A(J(A)\cdot \prod_x\{(x \neq 0) \rightarrow \sum_y[(\langle y,x\rangle \epsilon A)\cdot(y \epsilon x)]\})$.

2. Preliminary lemmas. We shall assume as known the derivation of set-theoretical theorems from the above axioms and we shall make free use of them stating them in the current notations and terminology [9]). This applies in particular to theorems concerning transfinite ordinals and to definitions by transfinite induction [10]).

We define by transfinite induction the sets t_ξ as follows:

$$t_0 = 0, \quad t_{\xi+1} = \hat{E}_x[x \subset t_\xi], \quad t_\lambda = \sum_{\eta<\lambda} t_\eta$$

(λ — limit number).

It is easy to see that $t_\eta \subset t_\xi$ for $\eta < \xi$.

If $x \epsilon t_{\xi+1} - t_\xi$, then x is said to be of the *type* ξ. It follows from the axiom $D1$ that for every set x there is an ordinal ξ such that x is of the type ξ.

Indeed, the axioms of group B entail the existence of the class A of those sets which are of no type. If A were non void, there

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[9]) It will therefore not always be possible to use small and capital letters in the manner explained on p. 144.

[10]) For the treatment of ordinals on the basis of axioms $A1$—$E1$ cf. [1], [5], [8], and [9].

would exist by $D1$ an element u such that $u \in A$ but no element of u is in A. Hence every element v of u would be of a type, say $\xi(v)$, and all the ordinals $\xi(v)$ would form a set (according to the axiom $C4$). Now there exists for every set of ordinals $\xi(v)$ an ordinal ξ surpassing all of them; hence we would obtain $u \subset t_\xi$ i. e. $u \in t_{\xi+1}$, which proves that u is of a type not greater than $\xi+1$ and hence u non $\in A$. This contradiction shows that $A=0$.

The sets

$$0,\quad \{0\},\quad \{0,\{0\}\},\quad \{0,\{0\},\{0,\{0\}\}\},\ldots$$

will be identified with the integers

$$1,\quad 2,\quad 3,\quad 4,\ldots$$

and their set will be denoted by N. It is known how to define the arithmetical operations, e. g.

$$x+y,\quad x\cdot y,\quad x^y,\quad 2^{x-1}(2y-1),\ldots$$

on elements of N.

Every class of ordered pairs is called (binary) *relation*. If R is a relation, then xRy means the same as $\langle x,y\rangle \in R$. The classes

$$E_x \textstyle\sum_y xRy,\quad E_y \sum_x xRy,\quad E_x \sum_y (xRy+yRx)$$

are called *domain*, *converse domain*, and *field* of R and denoted by $D_+(R)$, $D_-(R)$, and $C(R)$.

If $B \subset C(R)$, $x \in B$ and no $y \in B$ satisfies the condition yRx, then x is called a *minimum* of R in B. If R has at least one minimum in every non void class $B \subset C(R)$, then R is called *well-founded* [11]).

R is *internal* if for every two elements x_1, x_2 of its field the following equivalence holds:

$$\textstyle\prod_y (yRx_1 \equiv yRx_2) \equiv (x_1 = x_2).$$

An internal relation R has at most one minimum in $C(R)$.

For any set x we denote by ϵ_x the ϵ-relation limited to x, i. e. such that

$$u\,\epsilon_x\,v \equiv (u \in x)\cdot(u \in v)\cdot(v \in x).$$

[11]) Zermelo [17].

Theorem 1. *ϵ_x is an internal relation if and only if x satisfies the following condition $E(x)$:*

$$\textstyle\prod_{uv}((u \in x)\cdot(u \neq v)\cdot(v \in x) \rightarrow \{[(u-v)+(v-u)]\cdot x \neq 0\}).$$

Proof is obvious.

Theorem 2. *ϵ_x is well-founded for every non-void set x.*

Proof. Evidently there is an ordinal ξ such that $x \subset t_\xi$. Suppose that c is a non-void subset of the field of ϵ_x. Since $c \subset x$, there are ordinals $\eta \leqslant \xi$ such that $c\cdot t_\eta \neq 0$. If ζ is the smallest ordinal of this kind, then every element of $c\cdot t_\zeta$ is a minimum of ϵ_x in c. The existence of these minima shows that ϵ_x is well-founded, q. e. d.

A set s is called *complete* [12]) if every element of s is a part of s: $\prod_y (y \in s \rightarrow y \subset s)$.

We assume as known the notion of *isomorphism* of two relations.

Theorem 3. *For every well-founded and internal relation R whose field is a set there is a set s such that $E(s)$, R is isomorphic with ϵ_s, and the field of ϵ_s is complete.*

Proof. R being internal and well-founded, there is exactly one minimum z_0 of R in $C(R)$. Put

$$\{z_0\} = m_0,\quad f(z_0) = 0$$

and suppose that sets m_α are already defined for $\alpha < \xi$ and that a function f is defined on the sum $\sum_{\alpha<\xi} m_\alpha$.

Let m_ξ be the set of all $x \in C(R) - \sum_{\alpha<\xi} m_\alpha$ for which the following condition is satisfied

$$\textstyle\prod_y [yRx \rightarrow y \in \sum_{\alpha<\xi} m_\alpha]$$

and let $f(x)$ be defined on m_ξ by the equation

(1) $$f(x) = E_{f(y)}[yRx].$$

This definition of $f(x)$ is correct since yRx implies (for $x \in m_\xi$) that $y \in \sum_{\alpha<\xi} m_\alpha$ and therefore $f(y)$ is defined according to the inductive assumption.

[12]) Gödel [5], p. 23.

It is evident that $m_\xi \cdot m_\eta = 0$ for $\xi \neq \eta$ and that $m_\xi \subset C(R)$ for every ξ. Since $C(R)$ is a set, there must be a (smallest) ordinal ζ such that $m_\zeta = 0$.

We shall show that the difference $D = C(R) - \sum_{\alpha<\zeta} m_\alpha$ is void. Otherwise there would be a minimum z of R in D and z would satisfy the condition

$$yRz \rightarrow y \text{ non } \epsilon\, D \rightarrow y \,\epsilon \sum_{\alpha<\zeta} m_\alpha,$$

i. e. z would be an element of m_ζ contrary to the definition of ζ. Hence $D=0$ and consequently

$$C(R) = \sum_{\alpha<\zeta} m_\alpha .$$

We now show by induction on ξ that f is a one-one mapping of the sum $\sum_{\alpha<\xi} m_\alpha$ on a subset of t_ξ and satisfies the equivalence

(2) $$xRy \equiv f(x) \,\epsilon\, f(y)$$

for every pair x,y of the elements of $\sum_{\alpha<\xi} m_\alpha$.

This is evident for $\xi=1$; it is also easy to see, that if β is a limit number and the theorem holds for $\xi<\beta$, it holds also for $\xi=\beta$.

It remains to consider the case $\beta=\gamma+1$ under the assumption that the proposition holds for $\xi=\gamma$.

If $x \,\epsilon\, m_\gamma$, then by (1) $f(x)$ is a set of elements each of which is of a type $<\gamma$. Hence $f(x)$ is at most of the type γ and therefore $f(x) \,\epsilon\, t_{\gamma+1} = t_\beta$. The function f maps thus the sum $\sum_{\alpha<\beta} m_\alpha$ on a subset of t_β.

In order to prove that this is a one-one mapping let us suppose that x and y are two different elements of the sum $\sum_{\alpha<\beta} m_\alpha$. Since R is internal, there is an element u such that either $(uRx)\cdot(uRy)'$ or $(uRx)'\cdot(uRy)$. It will be sufficient to consider the first case. From uRx it follows that $x \neq z_0$ and hence by (1) $f(u) \,\epsilon\, f(x)$. Furthermore $u \,\epsilon \sum_{\alpha<\gamma} m_\alpha$ since x is an element of at most m_γ. If $f(u)$ were an element of $f(y)$, there would be an element $y_1 \,\epsilon \sum_{\alpha<\gamma} m_\alpha$ such that $f(u)=f(y_1)$ and $y_1 R y$. Since f is a one-one mapping on $\sum_{\alpha<\gamma} m_\alpha$, it would follow $u=y_1$, and consequently uRy what contradicts the assumption. Hence $f(u)$ non $\epsilon\, f(y)$ which shows that $f(x) \neq f(y)$. Hence

$$x \neq y \rightarrow f(x) \neq f(y),$$

i. e. f is a one-one mapping.

If x and y are elements of $\sum_{\alpha<\beta} m_\alpha$ and xRy, then $f(x) \,\epsilon\, f(y)$ according to (1). If $f(x) \,\epsilon\, f(y)$, then by (1) there is a z such that zRy and $f(x)=f(z)$. f being a one-one mapping, we obtain $x=z$ and xRy. The equivalence (2) is thus proved.

Putting $\xi=\zeta$ we infer that f maps $C(R)$ on a set s and satisfies the equivalence (2) for every pair x,y of the elements of $C(R)$. f being a one-one mapping, it follows that the relations R and ϵ_s are isomorphic. Since R is internal, ϵ_s is internal too and theorem 1 gives $E(s)$.

In order to prove that $C(\epsilon_s)$ is complete let us suppose that $y \,\epsilon\, C(\epsilon_s)$. It follows that $y \,\epsilon\, s$ and hence $f^{-1}(y) \,\epsilon\, C(R)$ and consequently there is an $\alpha<\zeta$ such that $f^{-1}(y) \,\epsilon\, m_\alpha$. If $\alpha=0$, then $f^{-1}(y)=z_0$, $y=0$ and $y \subset C(\epsilon_s)$. If $\alpha \neq 0$, then $y = ff^{-1}(y)$ is by (1) identical with the set of all $f(z)$ for which $zRf^{-1}(y)$. Since $f(z)\,\epsilon_s\, y$, it follows that all these $f(z)$ belong to the field of ϵ_s and therefore $y \subset C(\epsilon_s)$, which completes the proof of theorem 3.

For every set $A \subset N$ we put

$$R_A = E_{\langle m,n\rangle}[2^{m-1}(2n-1) \,\epsilon\, A].$$

In this way a one-one correspondence is established between subsets of N and binary relations whose fields are contained in N.

The following theorem is evident:

Theorem 4. *For every relation R with at most denumerable field there is a set $A \subset N$ such that the relations R and R_A are isomorphic.*

3. Set-theoretical and arithmetical formulae. A *set-theoretical formula* is an expression built up from elementary expressions of the form

$$a=b, \quad a \,\epsilon\, b$$

with the help of the logical connectives and quantifiers $\prod_a$ and $\sum_a$. The letters a, b may be replaced by any other letters.

The formulae included in a given formula are called its constituents.

If one wishes to make general statements about formulae one has to distinguish between the object- and syntax languages and recur to intuitively clear but sometimes clumsy semantical notions. For the readers benefit we may dispense with these complications since we shall never make general statements about formulae: we shall limit ourselves to the consideration of a finite number of formulae. Symbols like $\Phi, \Psi, \ldots$ or $\Phi(\epsilon, a, \ldots, m)$, $\Psi(\epsilon, a, \ldots, m)$ are not names of formulae but abbreviations of them and specifically of those of them in which variables $a,\ldots,m$ are free. We postpone till § 5 the enumeration of formulae to be used in our proof, we remark only that their class contains with every formula its constituents.

Saying that all formulae have a property we wish to say that all formulae of the considered finite class have this property. It will be easily seen that we eliminate in this way all the meta-mathematical notions and that all lemmas to be proved below belong entirely to the object-language.

Suppose that the letter x does not occur in Φ. Replace in this formula the unrestricted quantifiers $\prod_h$, $\sum_h$ (where h is any letter) by the restricted ones

$$\prod_h[h \in x \rightarrow \ldots], \quad \sum_h[(h \in x)\ldots]$$

(which we will sometimes write, more conveniently, as $\prod_{h \in x}$ and $\sum_{h \in x}$).

The resulting formula will be abbreviated as Φ_x and called the *formula Φ relativized to x*.

The symbol $\Phi(R, a, \ldots, m)$ will be used as an abbreviation of the formula resulting from $\Phi(\epsilon, a, \ldots, m)$ by substituting in it the letter R for the letter ϵ.

Theorem 5. $(\Phi_x)_y \equiv \Phi_{x \cdot y}$.

Proof. The theorem is evident for quantifier-free formulas. Since its validity for the formulae Φ and Ψ entails its validity for the formulae Φ' and $\Phi + \Psi$ it will be sufficient to prove that if it is true for a formula Φ, it is true also for the formula $\sum_h \Phi$. Let Ψ be an abbreviation of the latter formula. Then

$$\Psi_x \equiv \sum_h[(h \in x)\cdot\Phi], \quad (\Psi_x)_y \equiv \sum_{h \in y}[(h \in x)\cdot(\Phi_x)_y]$$

and therefore

$$(\Psi_x)_y \equiv \sum_h[(h \in x)\cdot(h \in y)\cdot(\Phi_x)_y] \equiv \sum_{h \in x \cdot y}(\Phi_x)_y.$$

On the other hand $\Psi_{x \cdot y} \equiv \sum_{h \in x \cdot y}\Phi_{x \cdot y}$ and since $\Phi_{x \cdot y} \equiv (\Phi_x)_y$ by the inductive assumption, we obtain $(\Psi_x)_y \equiv \Psi_{x \cdot y}$ which proves the theorem.

Theorem 6. *If $s = C(\epsilon_x)$ and $a \in s, \ldots, m \in s$, then*

$$\Phi_s(\epsilon, a, \ldots, m) \equiv \Phi_s(\epsilon_x, a, \ldots, m). \tag{3}$$

Proof. If Φ is an elementary formula $a = b$, then (3) reduces to the tautological equivalence $a = b \equiv a = b$.

If Φ is an elementary formula $a \in b$, then (3) is equivalent to

$$a \in b \equiv [(a \in x)\cdot(a \in b)\cdot(b \in x)]$$

which is true since $a \in s$ and $b \in s$ by the hypothesis and s is evidently a subset of x.

From (2) it follows

$$\Phi'_s(\epsilon, a, \ldots, m) \equiv \Phi'_s(\epsilon_x, a, \ldots, m)$$

which proves that the validity of (3) for a formula Φ implies its validity for the formula Φ'.

If besides (2) the following equivalence holds

$$\Psi_s(\epsilon, a, \ldots, h, n, \ldots, q) \equiv \Psi_s(\epsilon_x, a, \ldots, h, n, \ldots, q),$$

then

$$\Phi_s(\epsilon, a, \ldots, m)\cdot\Psi_s(\epsilon, a, \ldots, h, n, \ldots, q) \equiv \Phi_s(\epsilon_x, a, \ldots, m)\cdot\Psi_s(\epsilon, a, \ldots, h, n, \ldots, q),$$

which shows that if the theorem is true for two formulae, it is true for their conjunction.

Suppose now that (3) holds for a formula Φ and let $\Psi(\epsilon, b, \ldots, m)$ be the abbreviation of $\sum_a \Phi(\epsilon, a, b, \ldots, m)$. Let $b, \ldots, m$ be elements of s. If $\Psi_s(\epsilon, b, \ldots, m)$, then there is an $a \in s$ such that $\Phi_s(\epsilon, a, \ldots, m)$ which yields according to (3) $\Phi_s(\epsilon_x, a, \ldots, m)$ and hence $\sum_{a \in s}\Phi(\epsilon_x, a, \ldots, m)$. Thus

$$\Psi_s(\epsilon, b, \ldots, m) \rightarrow \Psi_s(\epsilon_x, b, \ldots, m).$$

If $\Psi_s(\epsilon_x, b, \ldots, m)$, then there is an $a \in s$ such that $\Phi_s(\epsilon_x, a, \ldots, m)$ and therefore by (3) $\Phi_s(\epsilon, a, \ldots, m)$ which gives $\sum_{a \in s}\Phi_s(\epsilon, a, \ldots, m)$. Hence

$$\Psi_s(\epsilon_x, b, \ldots, m) \rightarrow \Psi_s(\epsilon, b, \ldots, m).$$

Theorem 6 is thus proved for the formula Ψ, q. e. d.

Theorem 7[13]. *If the relations R_1 and R_2 are isomorphic, $s_1 = C(R_1)$, $s_2 = C(R_2)$, $a_1, \ldots, m_1 \in C(R_1)$ and $a_2, \ldots, m_2$ correspond to $a_1, \ldots, m_1$ in the isomorphism between R_1 and R_2, then*

$$\Phi_{s_1}(R_1, a_1, \ldots, m_1) \equiv \Phi_{s_2}(R_2, a_2, \ldots, m_2).$$

[13]) Theorem 7 is a special case of a theorem proved by A. Tarski and A. Lindenbaum [16].

Proof. The theorem is evident if Φ has the form $a=b$ or $a \epsilon b$. We show similarly as in the proof of theorem 6 that if the theorem holds for two formulae, it holds also for their negations and their conjunction.

It remains to consider the formula $\sum_a \Phi(\epsilon, a, \ldots, m)$ which we shall abbreviate as $\Psi(\epsilon, b, \ldots, m)$.

If $\Psi_{s_1}(R_1, b_1, \ldots, m_1)$, then there is an $a_1 \epsilon s_1$ such that $\Phi_{s_1}(R_1, a_1, b_1, \ldots, m_1)$. If a_2 is the element of s_2 corresponding to a_1 in the isomorphism between R_1 and R_2, then the inductive assumption gives $\Phi_{s_2}(R_2, a_2, b_2, \ldots, m_2)$ from which we infer that

$$\sum_{a_2 \epsilon s_2} \Phi_{s_2}(R_2, a_2, b_2, \ldots, m_2).$$

Hence

$$\Psi_{s_1}(R_1, b_1, \ldots, m_1) \rightarrow \Psi_{s_2}(R_2, b_2, \ldots, m_2).$$

The converse implication is proved in the same manner.

Besides the set-theoretical formulae we shall consider the *arithmetical formulae*. In order to define them we first explain what is meant by a *term*: the symbols 1,2,3,... and the small Latin letters in italics are terms; if m and n are terms, then $m+n$, $m-n$, $m \cdot n$, m^n are terms.

The simplest arithmetical formulae are equalities $m=n$ and expressions $m \epsilon A$ where m and n are terms and A any upper case Latin letter.

Other arithmetical formulae are built up from the simplest ones with the help of the logical connectives *and*, *or*, *not*, etc. and the quantifiers of the following four forms

$$(*) \qquad \begin{array}{ll} \prod_p[p \epsilon N \rightarrow \ldots], & \sum_p[(p \epsilon N) \cdot \ldots] \\ \prod_A[A \subset N \rightarrow \ldots], & \sum_A[(A \subset N) \cdot \ldots]. \end{array}$$

These quantifiers will be written afterwards as

$$\prod_{p \epsilon N}, \quad \sum_{p \epsilon N}, \quad \prod_{A \subset N}, \quad \sum_{A \subset N}.$$

The letters p and A may be replaced here by any other letters.

An arithmetical formula is called *elementary* if it does not contain quantifiers of the form (*).

The following theorem is an immediate consequence of the admitted definitions and of the equivalence $pR_A q \equiv 2^{p-1}(2q-1) \epsilon A$:

Theorem 8. *If $\Phi(\epsilon, a, \ldots, m)$ is a set-theoretical formula and $A \subset N$, then $\Phi_{C(R_A)}(R_A, a, \ldots, m)$ as well as $\Phi_N(R_A, a, \ldots, m)$ are equivalent to elementary arithmetical formulae.*

It is evident that every proposition concerning only natural or real numbers is expressible as an arithmetical formula. Furthermore those arithmetical formulae which correspond to usual axioms of real number arithmetics are derivable from the axioms $A1-E1$ of the system (S). It follows that the whole classical mathematics may be expressed and proved in the system (S).

4. Reduction of certain set-theoretical formulae to the arithmetical formulae.

The theorem to be proved in this section is the main result of the whole paper. It is a simple corollary to the well-known theorem of Skolem-Löwenheim which for our purpose may be stated in the following form:

Theorem 9. *If $a \epsilon x, \ldots, m \epsilon x$ and $\Phi_x(\epsilon, a, \ldots, m)$, then there is an at most denumerable subset y of x such that*

(4)	$a \epsilon y, \ldots, m \epsilon y,$	(5)	$E(y),$
(6)	$C(\epsilon_y) = y,$	(7)	$\Phi_y(\epsilon, a, \ldots, m).$

Proof[14]). We may evidently assume that Φ has the normal form

$$\prod_{x_1} \cdots \prod_{x_m} \sum_{y_1} \cdots \sum_{y_n} \prod_{z_1} \cdots \prod_{z_p} \sum_{t_1} \cdots \sum_{t_q} \cdots \Psi,$$

where

$$\Psi = \Psi(\epsilon, x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p, t_1, \ldots, t_q, \ldots)$$

is quantifier-free.

Owing to the axiom of choice the condition $\Phi_x(\epsilon, a, \ldots, m)$ is equivalent to the existence of a set of functions

$$f_i(x_1, \ldots, x_m), \quad g_j(x_1, \ldots, x_m, z_1, \ldots, z_p), \ldots$$

(where $i=1,2,\ldots,n$, $j=1,2,\ldots,q,\ldots$) with the following properties:

1° *they are defined for all the values of their arguments running through x;*

2° *if $x_1, \ldots, x_m, z_1, \ldots, z_p, \ldots$ are elements of x, then*

$$(8) \quad \Psi(\epsilon, x_1, \ldots, x_m, f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m), z_1, \ldots, z_p, g_1(x_1, \ldots, x_m, z_1, \ldots, z_p), \ldots, g_q(x_1, \ldots, x_m, z_1, \ldots, z_p), \ldots).$$

[14]) For this proof compare Skolem [11].

Let $h(z,t)$ be a function defined for $z,t \epsilon x$ and such that $h(z,t) \epsilon (z-t)+(t-z)$ for $z \neq t$. The existence of such a function follows again from the axiom of choice.

Now define y as a smallest set such that

$$a,\dots,m \epsilon y,$$

if $x_1,\dots,x_m,z_1,\dots,z_p,\dots \epsilon y$, *then* $f_i(x_1,\dots,x_m), g_j(x_1,\dots,x_m,z_1,\dots,z_p),\dots \epsilon y$ (*for* $i=1,2,\dots,n,\ j=1,2,\dots,q,\dots$),

if $z,t \epsilon y$ *and* $z \neq t$, *then* $h(z,t) \epsilon y$.

It is evident that y is an at most denumerable set and satisfies the condition (4). If $u,v \epsilon y$ and $u \neq v$, then $h(u,v) \epsilon [(u-v)+(v-u)]\cdot y$, whence $[(u-v)+(v-u)]\cdot y \neq 0$ which proves that y satisfies the condition (5). Since $0 \epsilon C(\epsilon_y)$ and $h(z,0) \epsilon z$ for $z \neq 0$ we see easily that $C(\epsilon_y)=y$ and hence y satisfies the condition (6). Finally y satisfies the condition (7) since (8) holds for every $x_1,\dots,x_m,z_1,\dots,z_p,\dots \epsilon y$ and the values of the functions $f_i, g_j, \dots$ occurring in (8) belong to y.

Theorem 9 is thus proved.

It will be convenient to use the abbreviation $[x]$ for $C(\epsilon_x)$. It is evident that

$$(y \epsilon [x]) \equiv (y \epsilon x)\cdot \textstyle\sum_t\{(t \epsilon x)\cdot[(y \epsilon t)+(t \epsilon y)]\}$$

from which it follows that

$$(9) \quad \Phi_{[x]}(\epsilon) \equiv \textstyle\sum_s\{\Phi_s(\epsilon)\cdot \prod_y[(y \epsilon s) \equiv (y \epsilon x)\cdot \sum_t\{(t \epsilon x)\cdot[(y \epsilon t)+(t \epsilon y)]\}]\}$$

for any formula $\Phi(\epsilon)$. This equivalence will be used in section 5.

We shall now prove the following theorem:

Theorem 10. *For every set-theoretical formula* $\Phi(\epsilon)$ *there is an elementary arithmetical formula* $\mathcal{G}(A,B)$ *with two free variables such that*

$$\textstyle\sum_x\{\Phi_{[x]}(\epsilon)\cdot \prod_y[(y \epsilon [x]) \rightarrow (y \subset [x])]\} \equiv \sum_{A \subset N} \prod_{B \subset N} \mathcal{G}(A,B).$$

Proof. By theorem 8 there is an elementary arithmetical formula $\mathcal{H}(A)$ such that

$$\Phi_{C(R_A)}(R_A) \equiv \mathcal{H}(A).$$

Let $\mathcal{K}(A)$ be the elementary arithmetical formula equivalent to

$$\textstyle\prod_{m,n \epsilon C(R_A)}[m \neq n \rightarrow \sum_{r \epsilon C(R_A)}(rR_A m \equiv rR_A n)'].$$

$\mathcal{K}(A)$ says of course that R_A is internal.

Let $\mathcal{L}(A,B)$ be the elementary arithmetical formula equivalent to

$$\textstyle\sum_{m \epsilon N}(m \epsilon B)\cdot \prod_{p \epsilon N}\{(p \epsilon B) \rightarrow \sum_{n \epsilon N}[(nR_A p)+(pR_A n)]\} \rightarrow$$
$$\textstyle\rightarrow \sum_{m \epsilon N}\{(m \epsilon B)\cdot \prod_{n \epsilon N}[(n \epsilon B) \rightarrow (nR_A m)']\}.$$

$\mathcal{L}(A,B)$ says that if B is a non void subset of $C(R_A)$, then R_A has a minimum in B.

Now take as $\mathcal{G}(A,B)$ the conjunction

$$\mathcal{H}(A)\cdot\mathcal{K}(A)\cdot\mathcal{L}(A,B).$$

We shall show that this formula has the desired property.

Suppose first that there is a set x such that $\Phi_{[x]}(\epsilon)$ and $[x]$ is complete.

By theorem 9 there is an at most denumerable set y such that

$$E(y), \quad C(\epsilon_y)=y, \quad \text{and } \Phi_y(\epsilon).$$

The second and third conditions give in virtue of theorem 6 $\Phi_y(\epsilon_y)$. Since the field of ϵ_y is at most denumerable, there is by theorem 4 a set $A \subset N$ such that the relations ϵ_y and R_A are isomorphic. Applying theorem 7 we obtain from $\Phi_y(\epsilon_y)$

$$\Phi_{C(R_A)}(R_A),$$

which proves that $\mathcal{H}(A)$.

We show further that $\mathcal{K}(A)$. For this purpose we assume that $m \neq n$ and $m,n \epsilon C(R_A)$. Let f be a function which establishes an isomorphism between R_A and ϵ_y. Thus $f(m)$ and $f(n)$ are different elements of y. Since $E(y)$, there is a $z \epsilon y$ such that $[z \epsilon f(m) \equiv z \epsilon f(n)]'$. Putting $r=f^{-1}(z)$ we get $r \epsilon C(R_A)$ and $(rR_A m \equiv rR_A n)'$ which proves that $\mathcal{K}(A)$.

We shall finally show that if $B \subset N$, then $\mathcal{L}(A,B)$. This is evident if $B=0$. Assume now that $B \neq 0$ and that B is a non void subset of the field of R_A. Since R_A is isomorphic with ϵ_y and ϵ_y is well-founded (by theorem 2), R_A is well-founded too. If m is a minimum of R_A in B, then $m \epsilon B$ and

$$\textstyle\prod_{n \epsilon N}[(n \epsilon B) \rightarrow (nR_A m)'].$$

This proves that $\mathcal{L}(A,B)$.

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We have thus proved, that if $\Phi_{[x]}(\epsilon)$, then there is a set $A \subset N$ such that $\mathcal{H}(A)$, $\mathcal{K}(A)$, and (for every $B \subset N$) $\mathcal{L}(A,B)$. Hence

$$\sum_x \Phi_{[x]}(\epsilon) \to \sum_{A\subset N} \prod_{B\subset N} \mathcal{G}(A,B).$$

Suppose now that there is a set $A \subset N$ such that

$$\mathcal{G}(A,B)$$

for every set $B \subset N$. It follows that

$$\mathcal{H}(A), \quad \mathcal{K}(A), \quad \text{and} \quad \mathcal{L}(A,B)$$

for every $B \subset N$. From $\mathcal{K}(A)$ we see immediately that R_A is internal and from $\prod_{B\subset N}\mathcal{L}(A,B)$ that R_A is well founded. By theorem 3 there is a set x such that ϵ_x is isomorphic with R_A and the field $[x]$ of ϵ_x is complete. From $\mathcal{H}(A)$ we obtain $\Phi_{C(R_A)}(R_A)$ and applying theorem 7 we obtain $\Phi_{[x]}(\epsilon_x)$. This gives in virtue of theorem 6 $\Phi_{[x]}(\epsilon)$, whence, $[x]$ being complete

$$\sum_{A\subset N} \prod_{B\subset N} \mathcal{G}(A,B) \to \sum_x \{\Phi_{[x]}(\epsilon) \cdot \prod_y [(y \in [x]) \to (y \subset [x])]\}.$$

The proof of theorem 10 is thus complete.

5. Construction of an undecidable sentence. A limit ordinal ξ is called *inaccessible* if it satisfies the following condition:

if $x \subset t_\xi$ and x is not of the same power as t_ξ, then $x \in t_\xi$.

We shall add to the axioms of the system (S) the following axiom

*F*1. *There is at least one inaccessible ordinal $\xi > \omega$*

and shall call (S_1) the resulting system of axioms.

In this section we shall prove in (S_1) theorems about the system (S).

Let $\Phi(\epsilon)$ be the conjunction of all the axioms of the system (S) preceded by universal quantifiers so as to render all the variables apparent.

We shall apply the theorems established in sections 3 and 4 taking all constituents of $\Phi(\epsilon)$ as well as their normal forms as elements of the finite class of formulae which were till now left unspecified.

Theorem 11. *If ξ is an inaccessible ordinal, then* $\Phi_{t_{\xi+1}}(\epsilon)$.

We omit the proof of this theorem since it is very easy and essentially known [15]).

Let $\Delta(\epsilon)$ be the following formula

$$(10) \quad \sum_x \sum_s \{\Phi_s(\epsilon) \cdot \prod_y [(y \in s) \equiv (y \in x) \cdot \sum_t \{(t \in x) \cdot [(y \in t) + (t \in y)]\}] \cdot \prod_y \prod_t [(y \in s) \cdot (t \in y) \to (t \in s)]\}.$$

According to (9) (see p. 154) this formula could have been written as

$$(10^*) \quad \sum_x \{\Phi_{[x]}(\epsilon) \cdot \prod_y [(y \in [x]) \to (y \subset [x])]\},$$

we prefer however the more complicated expression (10) since it is important for what follows to have the formula $\Delta(\epsilon)$ written without abbreviations such as „$[x]$" and „$\subset$".

Theorem 12. *There is a set a such that* $\Phi_{[a]}(\epsilon) \cdot \Delta_{[a]}(\epsilon)$.

Proof. Let ξ be the first inaccessible ordinal greater than ω and put $a = t_{\xi+1}$. Since $[a] = a$, we obtain from theorem 11

$$(11) \quad \Phi_{[a]}(\epsilon).$$

It follows from theorem 9 that there is an at most denumerable subset y of a such that

$$(12) \quad E(y), \qquad (13) \quad [y] = C(\epsilon_y) = y, \qquad \text{and} \qquad (14) \quad \Phi_y(\epsilon).$$

The relation ϵ_y is well-founded and internal (see (12) and theorems 1 and 2) and therefore (see theorem 3) there is a set s such that

$$(15) \quad \epsilon_y \textit{ and } \epsilon_s \textit{ are isomorphic,}$$

$$(16) \quad \textit{the set } [s] = C(\epsilon_s) \textit{ is complete.}$$

It is easy to see that (13) and (15) entail the equality

$$(17) \quad [s] = C(\epsilon_s) = s.$$

s is evidently an at most denumerable subset of a i. e. of $t_{\xi+1}$. Hence if $m \in s$, then $m \in t_{\xi+1}$, i. e. $m \subset t_\xi$. Since $m \in s$ implies $m \subset s$ because of (16) and (17), it follows that every element m of s is at

[15]) Cf. Kuratowski [6].

most denumerable and hence $m \in t_\xi$ according to the definition of inaccessible ordinals. This proves that

(18) $$s \subset t_\xi$$

and we obtain $s \in t_\xi$ and finally

(19) $$s \in a.$$

Applying theorem 7 to the formula $\Phi(\epsilon)$ and using (13), (15), and (17), we obtain the equivalence $\Phi_s(\epsilon_s) \equiv \Phi_y(\epsilon_y)$. Theorem 6 shows that we may omit the subscripts s and y staying by the letter ϵ. In view of (14) we obtain from the modified equivalence $\Phi_s(\epsilon)$ and further (since $s = st_\xi = sa$ according to (18)) $\Phi_{s \cdot a}(\epsilon)$. This gives in virtue of theorem 5

(20) $$[\Phi_s(\epsilon)]_a.$$

It follows further from (17) that

$$y \in s \equiv y \in [s]$$
$$\equiv (y \in s) \cdot \sum_t \{(t \in s) \cdot [(y \in t) + (t \in y)]\}.$$

Since $(t \in s) \rightarrow (t \in a)$ we may replace the quantifier $\sum_t$ by $\sum_{t \epsilon a}$ and obtain the equivalence

$$(y \in s) \equiv (y \in s) \cdot \sum_{t \epsilon a} \{(t \in s) \cdot [(y \in t) + (t \in y)]\}.$$

This equivalence being valid for every y, we infer that

(21) $$\prod_{y \epsilon a} [(y \in s) \equiv (y \in s) \cdot \sum_{t \epsilon a} \{(t \in s) \cdot [(y \in t) + (t \in y)]\}].$$

From (16) and (17) we see that $(y \in s) \cdot (t \in y) \rightarrow (t \in s)$ which proves that

(22) $$\sum_{y \epsilon a} \prod_{t \epsilon a} [(y \in s) \cdot (t \in y) \rightarrow (t \in s)].$$

Consider now the conjunction of (19), (20), (21), and (22) and put the quantifier $\sum_s$ before it. We obtain thus

$$\sum_{s \epsilon a} \{[\Phi_s(\epsilon)]_a \cdot \prod_{y \epsilon a} [(y \in s) \equiv (y \in s) \cdot \sum_{t \epsilon a} \{(t \in s) \cdot [(y \in t) + (t \in y)]\}] \cdot$$
$$\cdot \prod_{y \epsilon a} \prod_{t \epsilon a} [(y \in s) \cdot (t \in y) \rightarrow (t \in s)]\}$$

from which follows the validity of the weaker formula

$$\sum_{x \epsilon a} \sum_{s \epsilon a} \{[\Phi_s(\epsilon)]_a \cdot \prod_{y \epsilon a} [(y \in s) \equiv (y \in x) \cdot \sum_{t \epsilon a} \{(t \in x) \cdot [(y \in t) + (t \in y)]\}] \cdot$$
$$\cdot \prod_{y \epsilon a} \prod_{t \epsilon a} [(y \in s) \cdot (t \in y) \rightarrow (t \in s)]\}.$$

This formula is identical with the formula $\Delta(\epsilon)$ relativized to a, i. e. with $\Delta_a(\epsilon)$. Since $[a] = a$, we obtain $\Delta_{[a]}(\epsilon)$ which together with (11) yields $\Phi_{[a]}(\epsilon) \cdot \Delta_{[a]}(\epsilon)$. Theorem 12 is thus proved.

Theorem 13. *There is a set b such that $\Phi_{[b]}(\epsilon) \cdot [\Delta_{[b]}(\epsilon)]'$.*

Proof. Looking at the proof of theorem 12 we see that there are sets $w \in t_{\xi+2}$ such that

(23) $$\Phi_{[w]}(\epsilon),$$

(24) $$\text{if} \quad y \in [w], \quad \text{then} \quad y \subset [w].$$

E. g. $t_{\xi+1}$ is such a set. Let k be the set of all these sets w. By axiom $D1$ there is a set b such that $b \in k$ but no element of b is in k.

From $b \in k$ it follows in virtue of (23)

(25) $$\Phi_{[b]}(\epsilon).$$

Suppose now that $\Delta_{[b]}(\epsilon)$. It follows according to (10) that there are sets x and s such that

(26) $$x \in [b] \quad \text{and} \quad s \in [b],$$

(27) $$[\Phi_s(\epsilon)]_{[b]},$$

(28) $$\prod_{y \epsilon [b]} [(y \in s) \equiv (y \in x) \cdot \sum_{t \epsilon [b]} \{(t \in x) \cdot [(y \in t) + (t \in y)]\}],$$

(29) $$\prod_{y \epsilon [b]} \prod_{t \epsilon [b]} [(y \in s) \cdot (t \in y) \rightarrow (t \in s)].$$

(28) and (29) can be rewritten thus

$$\prod_y [(y \in s \cdot [b]) \equiv (y \in x \cdot [b]) \cdot \sum_t \{(t \in x \cdot [b]) \cdot [(y \in t) + (t \in y)]\}],$$
$$\prod_y \prod_t [(y \in s \cdot [b]) \cdot (t \in y \cdot [b]) \rightarrow (t \in s)].$$

According to (24) and (26) we have $x \subset [b]$, $s \subset [b]$, and (for $y \in [b]$) $y \subset [b]$; we may therefore omit the letter b in square brackets thorought these formulae and obtain thus

$$\prod_y [(y \in s) \equiv (y \in x) \cdot \sum_t \{(t \in x) \cdot [(y \in t) + (t \in y)]\}],$$
$$\prod_y \prod_t [(y \in s) \cdot (t \in y) \rightarrow (t \in s)].$$

The first formula shows that

(30) $$s = [x]$$

and therefore the second is equivalent to

(31) $$y \epsilon [x] \rightarrow y \subset [x].$$

Apply now theorem 5 to the formula (27). We obtain $\Phi_{s \cdot [b]}(\epsilon)$ what gives $\Phi_s(\epsilon)$ since $s \subset [b]$. Replace here s by $[x]$ according to (30); it comes

(32) $$\Phi_{[x]}(\epsilon).$$

Comparing (31) and (32) with (23) and (24), we see that $x \epsilon k$. On the other hand (26) proves that $x \epsilon b$. We arrive thus at a contradiction since no element of b is in k.

This contradiction shows that it cannot be $\Delta_{[b]}(\epsilon)$ and the theorem 13 follows from (25).

From theorems 12 and 13 we infer easily the following

Theorem 14. *The formula $\Delta(\epsilon)$ is neither demonstrable nor refutable in (S).*

Proof[16]. If $\Delta(\epsilon)$ were demonstrable in (S), then every relation satisfying the formula $\Phi(\epsilon)$ would satisfy the formula $\Delta(\epsilon)$. Now there is by theorem 13 a set b such that all the axioms of the system (S) remain true if sets and classes are interpreted as elements of $[b]$ and ϵ as ϵ_b; by the same interpretation $\Delta(\epsilon)$ is carried over into a false statement. $\Delta(\epsilon)$ is therefore non-demonstrable in (S).

Using theorem 12 we show in the same manner that $\Delta'(\epsilon)$ is non demonstrable in (S) i. e. that $\Delta(\epsilon)$ is not refutable, q. e. d.

The formula $\Delta(\epsilon)$ yields thus an instance of an undecidable formula. Observe now that according to (9) the formulae (10) and (10*) are equivalent. Applying theorem 10 to the formula (10*) we infer that the formula $\Delta(\epsilon)$ is equivalent to an arithmetical formula of the form $\sum_{A \subset N} \prod_{B \subset N} \mathcal{G}(A,B)$, where $\mathcal{G}(A,B)$ is an elementary arithmetical formula. This equivalence being provable in (S), we obtain

Theorem 15. *There is an arithmetical formula $\mathcal{F}$ of the form $\sum_{A \subset N} \prod_{B \subset N} \mathcal{G}(A,B)$ where $\mathcal{G}(A,B)$ is an elementary arithmetical formula such that $\mathcal{F}$ is undecidable in (S).*

[16]) In this proof use is made of some notions from the general methodology of deductive systems.

6. Remarks. (1) We compare here the undecidability of the formula $\mathcal{F}$ (cf. Theorem 15) with the undecidability of formulae constructed by the method of Gödel, Rosser, and Tarski[17]). The latter formulae assert their own undecidability, their intuitive truth is therefore evident. They become decidable after the introduction of suitable axioms or of a suitable rule of proof.

In the case of the Gödel's formula it is sufficient to adjoin the rule of infinite induction, i. e. to replace the axiomatic (relative) notion of integers by the absolute one.

To decide formulae constructed by Rosser and Tarski we have to adjoin to the system a number of intuitively obvious axioms stating certain properties of the notion of truth for sentences containing exclusively variables whose types do not surpass a fixed type n (in Rosser's case $n=2$).

The intuitive truth or falsity of the formula $\mathcal{F}$ from the theorem 15 is not evident unless one assumes the existence of inaccessible limit numbers. No „reasonable" rule of proof seems to exist which would be sufficient to decide within (S) whether $\mathcal{F}$ is true of false.

We see thus that the undecidability of $\mathcal{F}$ is caused by other circumstances than the undecidability of formulae constructed by the method of Gödel.

On the other hand, if we define the „absolute" undecidability as the undecidability irrespective of any assumption concerning the existence of sufficiently high cardinals or ordinals[18]), we see immediately that $\mathcal{F}$ is not absolutely undecidable since it follows from the axiom $F1$. A mathematical Platonist who believes in the existence of „any" cardinal and „any" ordinal would therefore consider $\mathcal{F}$ as incontestably true.

The surprising property of $\mathcal{F}$ is that its truth cannot be established without presupposing the existence of inaccessible ordinals. Also it can be decided neither within arithmetic nor within the theory of function nor within any theory translatable into a subsystem of (S). Yet $\mathcal{F}$ expresses a fact concerning real numbers: it states that a ***CA***-set is non void.

[17]) Gödel [4], Rosser [10], Tarski [15].

[18]) I owe the acquaintance with this notion to conversations with Tarski. The explanation given in the text is of course very vague and it is doubtful whether an exact definition of the notion of absolute undecidability will ever be found. Cf. Tarski [14], p. 87.

We note still one (though unimportant) difference between the undecidable formulae of Gödel and the undecidable formula $\mathcal{F}$. The formulae defined by Gödel are so long that it is practically impossible to write them down explicitly. The formula $\mathcal{F}$ is very long too, but it occupies not more than one or two pages.

(2) The unprovability of $\Delta(\epsilon)$ within (S) could have been proved as follows. The arithmetization of meta-mathematics enables us to express as an elementary arithmetical formula the following meta-mathematical statement: (S) *is a self-consistent system.* Let $\mathcal{W}$ be the arithmetical formula obtained in this way. It is easy to see that $\Delta(\epsilon)\to\mathcal{W}$ is provable in (S). Indeed $\Delta(\epsilon)$ says that there is a model satisfying all the axioms of (S) and therefore $\Delta(\epsilon)$ implies that (S) is self-consistent.

Now $\mathcal{W}$ has been shown by Gödel[19]) to be unprovable within (S) and hence $\Delta(\epsilon)$ is also unprovable.

We remark however that $\mathcal{W}$ is decidable if one adjoins the rule of infinite induction. The above proof gives therefore less than the former proof since it leaves open the possibility that $\Delta(\epsilon)$ may become provable after assuming the rule of infinite induction.

(3) If the existence of inaccessible numbers were provable in (S), then $\Delta(\epsilon)$ would be provable too (cf. theorem 11). Hence it is impossible to prove within (S) the existence of these numbers[20]).

(4) The following remark concerns the theorem 9 of Skolem-Löwenheim. It might seem that the hypothesis of theorem 9 is unnecessarily strong. Indeed one can prove[21]) the following theorem 9* which we propose to call „theorem of Skolem-Gödel": *if $\Psi(\epsilon)$ is a non-contradictory*[22]) *formula, then there is a relation R with at most denumerable field C such that $\Psi_C(R)$.*

Theorem 9* however is neither stronger nor weaker than theorem 9. Its hypothesis is indeed considerably weaker than that of theorem 9 but its conclusion is weaker too since it cannot be ascertained that R is well founded. As a matter of fact it can be proved that for several formulae $\Psi(\epsilon)$ the R of the theorem 9* cannot be well founded.

[19]) Gödel [4], Theorem XI, p. 196.

[20]) This has been proved by Kuratowski [6]. Cf. also Firestone and Rosser [2].

[21]) Gödel [2].

[22]) I. e. such that taking $\Psi(\epsilon)$ as an axiom and applying all the rules of functional calculus one obtains never a contradiction.

In theorem 10 we used theorem 3 which could be proved only for well-founded R. This shows that theorem 9* would be useless for our purpose.

(5) We shall show here that the axiom of choice could have been eliminated from the above proof. We have used this axiom only in the proof of theorem 9. The following theorem is however provable without this axiom:

Theorem 9.** *If x is a well ordered set, $a\in x,\dots,m\in x$ and $\Phi_x(\epsilon,a,\dots,m)$, then there is an at most denumerable set y such that $a\in y,\dots,m\in y$, $E(y)$, $C(\epsilon_y)=y$, and $\Phi_y(\epsilon,a,\dots,m)$.*

Let $\Phi(\epsilon)$ be again the conjunction of the axioms $A1$—$E1$.

Gödel[23]) has defined a transfinite sequence of well ordered sets $m_0,m_1,\dots,m_\xi,\dots$ such that if ξ is a suitable ordinal[24]), then $\Phi_{m_{\xi+1}}(\epsilon)$. Now the proofs of theorems 10-15 can be repeated using theorem 9** instead of 9.

(6) The finiteness of the axiom-system $A1$—$E1$ has enabled us to carry out the proof of undecidability without using any semantical notion. To extend our-construction to systems based on an infinite number of axioms, one has to take the semantical notion of satisfaction[25]) into consideration and the proof becomes much more complicated.

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Closure algebras.

By

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This paper treats of σ-complete Boolean algebras on which there is defined a closure operation satisfying the well-known axioms of Kuratowski [1]). A σ-complete Boolean algebra with a closure operation is called a closure algebra.

Almost all topological theorems which can be expressed in terms of the theory of Boolean algebras hold also for closure algebras. The proof of these theorems on closure algebras is often the same as the proof of analogous theorems on topological spaces. C. Kuratowski has worked out a method for the proof of topological theorems, the so-called topological calculus [2]). This method is especially suitable for generalizing topological theorems to the case of closure algebras. In general, in order to obtain a proof of a theorem on closure algebras it is sufficient to replace the term: „a subset of a topological space" by the term „an element of a closure algebra" in Kuratowski's proof of an analogous theorem on topological spaces. Therefore I shall omit proofs of many theorems on closure algebras.

The specification of all topological theorems which hold for closure algebras is not the purpose of this paper. I shall show only the method and the direction of generalizing and I shall cite many examples of topological theorems (given in the work Kuratowski [1]) which can be generalized.

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[1]) Finitely additive Boolean algebras and lattices with a closure operation were examined by many writers. See e. g. Mc Kinsey and Tarski [1]; Monteiro and Ribeiro [1]; Nöbeling [1]; Terasaka [1].

[2]) Kuratowski [1] and Kuratowski [2]. See also S. Janiszewski, *Thèse*, Journ. Ec. Polytechn. (1911).