Telle est la famille de tous les ensembles $\{\mathrm{E}x, \mathrm{E}2x, \mathrm{E}3x, \dots\}$, *où* x *parcourt tous les nombres réels* $\geqslant 1$ (et où $\mathrm{E}t$ désigne l'entier le plus grand $\leqslant t$).

Démonstration. Soient A et B deux ensembles infinis de nombres naturels équivalents par décomposition finie. On a donc des décompositions en ensembles disjoints:

$$A = A_1 + A_2 + \dots + A_m, \quad B = B_1 + B_2 + \dots + B_m,$$

où $A_i \simeq B_i$ pour $i = 1, 2, \dots, m$.

Admettons que pour $i = 1, 2, \dots, p$ les ensembles A_i et B_i sont superposables par translation, soit $B_i = A_i(a_i)$ pour $i = 1, 2, \dots, p$, et que, pour $i = p+1, p+2, \dots, m$, il le sont par rotation. Les nombres a_i $(i = 1, 2, \dots, p)$ sont évidemment des entiers, et les ensembles A_i $(i = p+1, p+2, \dots, m)$ sont finis (deux ensembles infinis de nombres naturels ne pouvant pas être superposables par rotation).

n étant un nombre naturel donné, désignons par $P^{(n)}$ le nombre de tous les nombres de l'ensemble de nombres naturels P qui sont $\leqslant n$. On voit sans peine que pour tout a entier on a $|P^{(n)}(a) - P^{(n)}| \leqslant |a|$. Il en résulte tout de suite que $|B_i^{(n)} - A_i^{(n)}| \leqslant |a_i|$ pour $i = 1, 2, \dots, p$, et on en conclut que

$$|A^{(n)} - B^{(n)}| \leqslant q, \quad \text{où} \quad q = |a_1| + |a_2| + \dots + |a_p| + \overline{\overline{A_{p+1}}} + \overline{\overline{A_{p+2}}} + \dots + \overline{\overline{A_m}}$$

est un entier indépendant de n.

Soient x et y deux nombres réels distincts $\geqslant 1$, p. e. $x < y$, et posons

$$A = \{\mathrm{E}x, \mathrm{E}2x, \mathrm{E}3x, \dots\} \quad \text{et} \quad B = \{\mathrm{E}y, \mathrm{E}2y, \mathrm{E}3y, \dots\}.$$

On vérifie facilement que

$$\mathrm{E}\frac{n+1}{x} - 1 \leqslant A^{(n)} \leqslant \mathrm{E}\frac{n+1}{x} \quad \text{et} \quad \mathrm{E}\frac{n+1}{y} - 1 \leqslant B^{(n)} \leqslant \mathrm{E}\frac{n+1}{y},$$

d'où

$$A^{(n)} > \frac{n+1}{x} - 2, \quad B^{(n)} \leqslant \frac{n+1}{y}, \quad \text{donc} \quad A^{(n)} - B^{(n)} > (n+1)\frac{y-x}{xy} - 2,$$

ce qui donne

$$\lim_{n=\infty} (A^{(n)} - B^{(n)}) = +\infty,$$

contrairement à l'inégalité $|A^{(n)} - B^{(n)}| \leqslant q$ trouvée plus haut dans l'hypothèse que $A \overset{f}{=} B$. Les ensembles A et B ne peuvent donc être équivalents par décomposition finie et le théorème 4 se trouve démontré.

On the inducing of homomorphisms by mappings.

By

Roman Sikorski (Warszawa).

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two fields of subsets of sets $\mathfrak{X}$ and $\mathfrak{Y}$ respectively and let f be a homomorphism (i. e. an additive and complementative transformation) of $\boldsymbol{Y}$ in $\boldsymbol{X}$. I say that the homomorphism f is *induced* by a mapping $y = \varphi(x)$ of $\mathfrak{X}$ into $\mathfrak{Y}$ if

(*) $$f(Y) = \varphi^{-1}(Y) \quad \text{for every} \quad Y \in \boldsymbol{Y}\text{ }^{1)}.$$

More generally, if $\boldsymbol{I}$ and $\boldsymbol{J}$ are two ideals of sets of the fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively and f is a homomorphism of the quotient algebra $\boldsymbol{Y}/\boldsymbol{J}$ in $\boldsymbol{X}/\boldsymbol{I}$, I say that a mapping φ of $\mathfrak{X}$ into $\mathfrak{Y}$ *induces* the homomorphism f if

(**) $$\varphi^{-1}(Y) \in \boldsymbol{X} \quad \text{and} \quad f([Y]) = [\varphi^{-1}(Y)] \quad \text{for every} \quad Y \in \boldsymbol{Y}.$$

$[Y]$ denotes here the element of $\boldsymbol{X}/\boldsymbol{J}$ (i. e. a class of sets belonging to $\boldsymbol{Y}$) which contains the set $Y \in \boldsymbol{Y}$ and $[\varphi^{-1}(Y)]$ denotes the element of $\boldsymbol{X}/\boldsymbol{I}$ which contains the set $\varphi^{-1}(Y) \in \boldsymbol{X}$.

This paper contains several theorems [2]) on the inducing of homomorphisms by mappings [3]).

Terminology and notation. A mapping f of a Boolean algebra $\boldsymbol{A}$ in a Boolean algebra $\boldsymbol{B}$ is called a *homomorphism*, if

$$f(A_1 + A_2) = f(A_1) + f(A_2) \quad \text{and} \quad f(A_1') = (f(A_1))',$$

for every $A_1, A_2 \in \boldsymbol{A}$. The symbols „+" and „′" denote here the Boolean operations of addition and complementation which correspond to the operations of addition and complementation of sets in the general theory of sets.

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[1]) Hence $\varphi^{-1}(Y) \in \boldsymbol{X}$ for any $Y \in \boldsymbol{Y}$. Conversely, if $\varphi^{-1}(Y) \in \boldsymbol{X}$ for every $Y \in \boldsymbol{Y}$, then the formula (*) defines a homomorphism f of $\boldsymbol{Y}$ in $\boldsymbol{X}$.

[2]) An application of one of these theorems (th. 3.1) to the theory of the integral is given in my paper [1].

[3]) This paper was presented by me at a session of the Warsaw Section of the Polish Mathematical Society on December 12, 1947.

A similar problem on the inducing of isomorphisms between fields of sets has been considered by E. Marczewski in paper [2].

A one-one homomorphism is called an *isomorphism*. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are σ-complete Boolean algebras, a homomorphism f of $\boldsymbol{A}$ in $\boldsymbol{B}$ is called a *σ-homomorphism* if

$$f(A_1+A_2+\ldots)=f(A_1)+f(A_2)+\ldots$$

for every sequence $\{A_n\}$ of elements of $\boldsymbol{A}$. Every isomorphism between two σ-complete Boolean algebras is a σ-homomorphism.

Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be Boolean algebras. If f is a homomorphism of $\boldsymbol{A}$ in $\boldsymbol{B}$ and g is a homomorphism of $\boldsymbol{B}$ in $\boldsymbol{C}$, gf denotes the composed homomorphism $C=g(f(A))$ of $\boldsymbol{A}$ in $\boldsymbol{C}$. If f is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$, f^{-1} denotes the inverse isomorphism of $\boldsymbol{B}$ on $\boldsymbol{A}$.

Let $\mathscr{X}$ be an abstract non-empty set. A class $\boldsymbol{X}$ of subsets of $\mathscr{X}$ is called a *field* if $X_1+X_2 \in \boldsymbol{X}$ and $\mathscr{X}-X_1 \in \boldsymbol{X}$ for every $X_1, X_2 \in \boldsymbol{X}$. A field $\boldsymbol{X}$ is called a *σ-field* if the condition $X_n \in \boldsymbol{X}$ $(n=1,2,3,\ldots)$ implies $X_1+X_2+X_3+\ldots \in \boldsymbol{X}$.

Let $\boldsymbol{X}$ be a σ-field. A non-empty class $\boldsymbol{I} \subset \boldsymbol{X}$ is called a *σ-ideal* of $\boldsymbol{X}$ if the conditions: $X_n \in \boldsymbol{I}$ $(n=1,2,3,\ldots)$ and $X_0 \subset X_1$ imply: $X_1+X_2+X_3+\ldots \in \boldsymbol{I}$ and $X_0 \in \boldsymbol{I}$.

If $\boldsymbol{X}$ is a σ-field of subsets of $\boldsymbol{X}$ and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{X}$, the σ-complete Boolean algebra $\boldsymbol{X}/\boldsymbol{I}$ is called a *σ-quotient algebra* of $\mathscr{X}$. Elements of $\boldsymbol{X}/\boldsymbol{I}$ are disjoint classes of sets $X \in \boldsymbol{X}$ such that two sets X_1, X_2 belong to the same class if and only if

(i) $$(X_1-X_2)+(X_2-X_1) \in \boldsymbol{I}.$$

The class containing a set $X \in \boldsymbol{X}$ is denoted by $[X]$. The condition $[X_1]=[X_2]$ is equivalent to (i). The Boolean operations in $\boldsymbol{X}/\boldsymbol{I}$ are defined by the formulas:

$$[X]'=[\mathscr{X}-X], \quad \sum_n [X_n]=[\sum_n X_n].$$

I. Homomorphisms between fields of sets.

1. ***Homomorphisms between finitely additive fields.*** In this section $\boldsymbol{X}$ and $\boldsymbol{Y}$ are two fields of subsets of two non-empty sets $\mathscr{X}$ and $\mathscr{Y}$ respectively. f denotes a homomorphism of $\boldsymbol{Y}$ in $\boldsymbol{X}$.

For every $y \in \mathscr{Y}$ let $f(y)$ denote the product of all sets $f(Y)$ such that $y \in Y \in \boldsymbol{Y}$.

1.1. *In order that a homomorphism f of $\boldsymbol{Y}$ in $\boldsymbol{X}$ be induced by a mapping φ of $\mathscr{X}$ into $\mathscr{Y}$, it is necessary and sufficient that*

(i) $$\mathscr{X}=\sum_{y \in \mathscr{Y}} f(y).$$

The necessity follows from the fact that $\varphi^{-1}(y) \subset f(y)$, and therefore

$$\mathscr{X}=\sum_{y \in \mathscr{Y}} \varphi^{-1}(y) \subset \sum_{y \in \mathscr{Y}} f(y) \subset \mathscr{X}.$$

In order to prove the sufficiency suppose that (i) is true. From the definition of $f(y)$ it follows that for $y_1 \neq y_2$ either $f(y_1)=f(y_2)$ or $f(y_1)\cdot f(y_2)=0$. Hence we infer that there exists a (transfinite) sequence $\{y_\alpha\}$ of elements of $\mathscr{Y}$ such that

$$\sum_\alpha f(y_\alpha)=\mathscr{X} \quad \text{and} \quad f(y_{\alpha'})\cdot f(y_{\alpha''})=0 \quad \text{for} \quad \alpha' \neq \alpha''.$$

For every $x \in \mathscr{X}$ there exists exactly one ordinal number α such that $x \in f(y_\alpha)$. Let $\varphi(x)=y_\alpha$. φ is a mapping of $\mathscr{X}$ into $\mathscr{Y}$. Now let Y be any element of $\boldsymbol{Y}$, let $\{y'_\alpha\}$ be the sequence formed of all terms of $\{y_\alpha\}$ which belong to Y and let $\{y''_\alpha\}$ be the sequence formed of all remaining terms of $\{y_\alpha\}$. We have

(ii) $$\mathscr{X}=\sum_\alpha f(y'_\alpha)+\sum_\alpha f(y''_\alpha) \quad \text{and}$$

(iii) $$\sum_\alpha f(y'_\alpha)\cdot \sum_\alpha f(y''_\alpha)=0.$$

Since $y'_\alpha \in Y$, we have $f(y'_\alpha) \subset f(Y)$ and consequently

$$\sum_\alpha f(y'_\alpha) \subset f(Y).$$

Analogously

$$\sum_\alpha f(y''_\alpha) \subset f(\mathscr{Y}-Y)=\mathscr{X}-f(Y).$$

It follows now from (iii) and (ii) that

$$f(Y)=\sum_\alpha f(y'_\alpha)=\varphi^{-1}(Y)$$

and thus the theorem is proved.

A *two-valued measure* on a field $\boldsymbol{Y}$ is a function m defined for every $Y \in \boldsymbol{Y}$, such that: 1^0 m assumes only the numbers 0 and 1; 2^0 $m(\mathscr{Y})=1$; 3^0 $m(Y_1+Y_2)=m(Y_1)+m(Y_2)$ for any two disjoint sets $Y_1, Y_2 \in \boldsymbol{Y}$.

A two-valued measure m is called *trivial* if there exists $y \in \mathscr{Y}$ such that $y \in Y \in \boldsymbol{Y}$ implies $m(Y)=1$. Then, obviously, the condition $y \in Y$ is equivalent to $m(Y)=1$. Conversely, if there exists $y \in \mathscr{Y}$ such that $m(Y)=1$ implies $y \in Y$ (for $Y \in \boldsymbol{Y}$), then m is trivial.

1.2. *In order that every homomorphism f of $\boldsymbol{Y}$ in $\boldsymbol{X}$ be induced by a mapping φ of $\mathfrak{X}$ into $\mathcal{Y}$, it is necessary and sufficient that every two-valued measure m on $\boldsymbol{Y}$ be trivial.*

Necessity. Let m be any two-valued measure on $\boldsymbol{Y}$ and let

$$f(y)=\mathfrak{X} \quad \text{if} \quad m(Y)=1$$

and

$$f(Y)=0 \quad \text{if} \quad m(Y)=0,$$

for every $Y \epsilon \boldsymbol{Y}$. f is a homomorphism of $\boldsymbol{Y}$ in $\boldsymbol{X}$. From the assumption, there exists a mapping φ of $\mathfrak{X}$ into $\mathcal{Y}$ such that $\varphi^{-1}(Y)=f(Y)$ for every $Y \epsilon \boldsymbol{Y}$. Let $x_0 \epsilon \mathfrak{X}$ and $y_0=\varphi(x_0)$. If $m(Y)=1$, then $f(Y)=\mathfrak{X}=\varphi^{-1}(Y)$, hence $y_0=\varphi(x_0) \epsilon Y$. Thus the measure m is trivial.

Sufficiency. Suppose that a homomorphism f is not induced by a mapping φ. From 1.1 it follows that there exists an x_0 such that

(i) $$x_0 \epsilon \mathfrak{X}-\sum_{y \epsilon \mathcal{Y}} f(y).$$

Let us put for $Y \epsilon \boldsymbol{Y}$:

$$m(Y)=1 \quad \text{if} \quad x_0 \epsilon f(Y)$$

and

$$m(Y)=0 \quad \text{if} \quad x_0 \text{ non } \epsilon f(Y).$$

m is a two-valued measure on $\boldsymbol{Y}$. By hypothesis m is trivial, i. e. there exists an element $y_0 \epsilon Y$ such that $y_0 \epsilon Y \epsilon \boldsymbol{Y}$ implies $m(Y)=1$. Hence $x_0 \epsilon f(y_0)$ which contradicts the formula (i). The sufficiency is proved.

The field of all subsets of a set $\mathcal{Y}$ will be denoted by $\mathfrak{S}(\mathcal{Y})$.

1.3. *If $\mathcal{Y}$ is finite, every homomorphism of $\mathfrak{S}(\mathcal{Y})$ in a field $\boldsymbol{X}$ is induced by a mapping φ of $\mathfrak{X}$ into $\mathcal{Y}$. If $\mathcal{Y}$ is infinite, then for every field $\boldsymbol{X}$ there exists a homomorphism f of $\mathfrak{S}(\mathcal{Y})$ in $\boldsymbol{X}$ which is not induced by a mapping φ of $\mathfrak{X}$ into $\mathcal{Y}$.*

The first part results immediately from 1.1, the second from 1.2 since for every infinite set $\mathcal{Y}$ there exists a two-valued non-trivial measure on $\mathfrak{S}(\mathcal{Y})$.

1.4. *For every field $\boldsymbol{Y}$ of subsets of $\mathcal{Y}$ there exists a set $\mathcal{Z}$ and a field $\boldsymbol{Z}$ of subsets of $\mathcal{Z}$ such that:*

a) $\mathcal{Y} \subset \mathcal{Z}$;

b) *the mapping $g(Z)=Z\mathcal{Y}$ (for $Z \epsilon \boldsymbol{Z}$) is an isomorphism of $\boldsymbol{Z}$ on $\boldsymbol{Y}$;*

c) *every two-valued measure on $\boldsymbol{Z}$ is trivial;*

d) *for every homomorphism f of $\boldsymbol{Y}$ in a field $\boldsymbol{X}$ (of subsets of $\mathfrak{X}$) there exists a mapping φ of $\mathfrak{X}$ into $\mathcal{Z}$ such that*

$$f(Y)=\varphi^{-1}(g^{-1}(Y)) \quad \textit{for every} \quad Y \epsilon \boldsymbol{Y}.$$

Let $M(Y)$ denote (for $Y \epsilon \boldsymbol{Y}$) the set of all non-trivial two-valued measures m on $\boldsymbol{Y}$ such that $m(Y)=1$ and let

$$\mathcal{Z}=\mathcal{Y}+M(\mathcal{Y}),$$

and

$$h(Y)=Y+M(Y) \quad \text{for} \quad Y \epsilon \boldsymbol{Y}.$$

The class $\boldsymbol{Z}$ of all sets $h(Y)$ (where $Y \epsilon \boldsymbol{Y}$) is a field of subsets of $\mathcal{Z}$ and h is an isomorphism of $\boldsymbol{Y}$ on $\boldsymbol{Z}$. The properties a) and b) are obvious. g is the inverse isomorphism of h.

If m is a two-valued measure on $\boldsymbol{Z}$, mh is the same on $\boldsymbol{Y}$. If mh is trivial, then there exists an $y \epsilon \mathcal{Y}$ such that $y \epsilon Y \epsilon \boldsymbol{Y}$ implies $mh(Y)=1$. Since the conditions $y \epsilon Y$ and $y \epsilon h(Y)$ are equivalent, m is trivial. Suppose now that mh is a non-trivial measure on $\boldsymbol{Y}$, i. e. $mh \epsilon M(\mathcal{Y}) \subset \mathcal{Z}$. If $mh \epsilon Z=h(Y) \epsilon \boldsymbol{Z}$, i. e. if $mh \epsilon M(Y)$, then $mh(Y)=1$, hence $m(Z)=1$. Therefore m is also trivial. The property c) is proved.

fg is a homomorphism of $\boldsymbol{Z}$ in $\boldsymbol{X}$. $\boldsymbol{Z}$ having the property c), on account of theorem 1.2 there exists a mapping φ of $\mathfrak{X}$ into $\mathcal{Z}$ such that

$$fg(Z)=\varphi^{-1}(Z) \quad \text{for every} \quad Z \epsilon \boldsymbol{Z},$$

i. e.

$$f(Y)=\varphi^{-1}(g^{-1}(Y)) \quad \text{for every} \quad Y \epsilon \boldsymbol{X},$$

q. e. d.

As Stone has proved [4]), every Boolean algebra may be considered as a field of sets. On account of theorems 1.2 and 1.4 every homomorphism between two Boolean algebras may be considered as a homomorphism f between two fields of sets which is induced by a point mapping φ.

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[4]) See Stone [1], p. 106.

2. ***σ-homomorphisms.*** A two-valued measure m of a σ-field $\boldsymbol{Y}$ of subsets of $\mathcal{Y}$ is called a *two-valued σ-measure* if

$$m(Y_1+Y_2+\ldots)=m(Y_1)+m(Y_2)+\ldots$$

for every infinite sequence $\{Y_n\}$ of disjoint sets which belong to $\boldsymbol{Y}$.

The following theorems on σ-homomorphisms are analogous to those on finitely additive homomorphisms in § 1.

2.1. *In order that every σ-homomorphism f of a σ-field $\boldsymbol{Y}$ (of subsets of $\mathcal{Y}$) in a σ-field $\boldsymbol{X}$ (of subsets of $\mathcal{X}$) be induced by a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$, it is necessary and sufficient that every two-valued σ-measure on $\boldsymbol{Y}$ be trivial.*

2.2. *For every σ-field $\boldsymbol{Y}$ of subsets of a set $\mathcal{Y}$ there exists a σ-field $\boldsymbol{Z}$ of subsets of a set $\mathcal{Z}$ such that:*

a) $\mathcal{Y}\subset\mathcal{Z}$;

b) $g(Z)=\mathcal{Y}Z$ *(for $Z\in\boldsymbol{Z}$) is an isomorphism of $\boldsymbol{Z}$ on $\boldsymbol{Y}$;*

c) *every two-valued σ-measure on $\boldsymbol{Z}$ is trivial,*

d) *for every σ-homomorphism f of $\boldsymbol{Y}$ in a σ-field $\boldsymbol{X}$ (of subsets of $\mathcal{X}$) there exists a mapping φ of $\mathcal{X}$ into $\mathcal{Z}$ such that*

$$f(Y)=\varphi^{-1}g^{-1}(Y) \quad \textit{for any} \quad Y\in\boldsymbol{Y}.$$

The proofs of 2.1 and 2.2 are analogous to those of 1.2 and 1.4.

We say that a cardinal number $\mathfrak{n}$ is of *two-valued measure zero* if every two-valued σ-measure on $\mathfrak{S}(\mathcal{Y})$, where $\mathcal{Y}$ is a set of potency $\mathfrak{n}$, is trivial. If $\mathfrak{n}$ is less than the first aleph inaccessible in the strict sense, $\mathfrak{n}$ is of two-valued measure zero [4a]).

By this definition and theorem 2.1 we obtain:

2.3. *In order that every σ-homomorphism f of a field $\mathfrak{S}(\mathcal{Y})$ in a σ-field $\boldsymbol{X}$ (of subsets of $\mathcal{X}$) be induced by a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$, it is necessary and sufficient that the cardinal number $\overline{\overline{\mathcal{Y}}}$ be of two-valued measure zero.*

The following theorem is another consequence of theorem 2.1:

2.4. *In order that every σ-homomorphism of a field of all Borel subsets of a topological metric space $\mathcal{Y}$ in a σ-field $\boldsymbol{X}$ (of subsets of $\mathcal{X}$) be induced by a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$, it is necessary and sufficient that the cardinal number $\overline{\overline{\mathcal{Y}}}$ be of two-valued measure zero.*

This follows from the fact that every two-valued σ-measure on the field of all Borel subsets of a metric space $\mathcal{Y}$ is trivial if and only if $\overline{\overline{\mathcal{Y}}}$ is of two-valued measure zero [5]).

[4a]) See Ulam [1], p. 150.

[5]) See Marczewski and Sikorski [1], p. 139.

II. σ-homomorphisms between σ-quotient algebras.

3. ***Homomorphisms defined on fields of Borel sets.*** In § 3 and § 4 we shall study σ-homomorphisms of a σ-field $\boldsymbol{Y}$ (of subsets of a set $\mathcal{Y}$) in a σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ (of a set $\mathcal{X}$). According to the definition (**) we say that a σ-homomorphism f of $\boldsymbol{Y}$ in $\boldsymbol{X}/\boldsymbol{I}$ is *induced* by a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$ if

(***) $\quad \varphi^{-1}(Y)\in\boldsymbol{X}$ and $[\varphi^{-1}(Y)]=f(Y)$ for every $Y\in\boldsymbol{Y}$ [6]).

We begin with a study of the case where $\boldsymbol{Y}$ is the field of all Borel sets of real numbers. $\mathcal{R}$ will denote always the set of all real numbers together with $\pm\infty$, and W will denote the set of all (finite) rational numbers. For $r\in W$ the symbol $R(r)$ will denote the set of all real numbers greater than r.

Lemma. *Let $X(r)$ be (for every $r\in W$) a subset of a set $\mathcal{X}$. If*

$$X(r_0)=\sum_{\substack{r>r_0\\ r\in W}}X(r)$$

for every $r_0\in W$, then there exists a mapping φ of $\mathcal{X}$ into $\mathcal{R}$ such that

(i) $$X(r)=\varphi^{-1}(R(r)) \quad \textit{for every} \quad r\in W.$$

Let $\varphi(x)$ denote (for every $x\in\mathcal{X}$) the upper bound of all rational numbers r such that $x\in X(r)$. In particular $\varphi(x)=-\infty$ if x non $\in X(r)$ for every $r\in W$ and $\varphi(x)=+\infty$ if $x\in X(r)$ for every $r\in W$. The function φ satisfies the condition (i). The easy proof is omitted.

If $\mathcal{T}$ is a topological space, the symbol $\mathfrak{B}(\mathcal{T})$ will denote the σ-field of all Borel subsets of $\mathcal{T}$.

3.1. *Every σ-homomorphism f of $\mathfrak{B}(\mathcal{R})$ in a σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ of a set $\mathcal{X}$ is induced by a mapping φ of $\mathcal{X}$ into $\mathcal{R}$.*

For any $r\in W$ let $X^0(r)$ be a subset of $\mathcal{X}$ such that

(i) $$X^0(r)\in\boldsymbol{X} \quad \text{and} \quad [X^0(r)]=f(R(r))$$

and let

(ii) $$X^0=\sum_{r'}\Big(\big(X^0(r')-\sum_{r>r'}X^0(r)\big)+\big(\sum_{r>r'}X^0(r)-X^0(r')\big)\Big),$$

(iii) $$X(r)=X^0(r)-X^0$$

[6]) Conversely, if the first of the conditions (***) is satisfied, the second defines a homomorphism f of $\boldsymbol{Y}$ in $\boldsymbol{X}/\boldsymbol{I}$.

(r, r' denote always rational numbers). Since $R(r')=\sum_{r>r'} R(r)$, we have

(iv) $$[X^0(r')]=f(R(r'))=f\Big(\sum_{r>r'} R(r)\Big)=\Big[\sum_{r>r'} X^0(r)\Big].$$

$\boldsymbol{I}$ being a σ-ideal, we obtain from (ii) and (iv) that

(v) $$X^0 \epsilon \boldsymbol{I}$$

and from (i), (iii) and (v) that

(vi) $$X(r) \epsilon \boldsymbol{X} \quad \text{and} \quad f(R(r))=[X(r)] \quad \text{for every } r \epsilon W.$$

It follows easily from (ii) and (iii) that the sets $X(r)$ satisfy the assumptions of the above proved lemma. Thus there exists a real function φ defined for all $x \epsilon \mathfrak{X}$ such that

(vii) $$\varphi^{-1}(R(r))=X(r) \quad \text{for every } r \epsilon W.$$

From (vi) and (vii):

(viii) $$\varphi^{-1}(R(r)) \epsilon \boldsymbol{X} \quad \text{and} \quad [\varphi^{-1}(R(r))]=f(R(r)) \quad \text{for every } r \epsilon W.$$

Let $\mathfrak{B}$ be the class of all sets $Y \epsilon \mathfrak{B}(\mathcal{R})$ such that the condition (***) is satisfied. By (viii) $R(r) \epsilon \mathfrak{B}$ for every $r \epsilon W$. It is easy to prove that $\mathfrak{B}$ is a σ-field. Consequently $\mathfrak{B}(\mathcal{R})=\mathfrak{B}$, q. e. d.

3.2. *Let* $0 \neq \mathcal{Z} \subset \mathcal{R}$. *In order that every σ-homomorphism f of $\mathfrak{B}(\mathcal{Z})$ in any σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ (of a set $\mathfrak{X}$) be induced by a mapping φ of $\mathfrak{X}$ in $\mathcal{Z}$, it is necessary and sufficient that $\mathcal{Z} \epsilon \mathfrak{B}(\mathcal{R})$ (i. e. that $\mathcal{Z}$ be a linear Borel set).*

Sufficiency. The homomorphism $f(\mathcal{Z}R)$ defined for $R \epsilon \mathfrak{B}(\mathcal{R})$ maps $\mathfrak{B}(\mathcal{R})$ in $\boldsymbol{X}/\boldsymbol{I}$. By 3.1 there exists a mapping φ_0 of $\mathfrak{X}$ into $\mathcal{R}$ such that

(i) $$\varphi_0^{-1}(R) \epsilon \boldsymbol{X} \quad \text{and} \quad [\varphi_0^{-1}(R)]=f(\mathcal{Z}R) \quad \text{for every } R \epsilon \mathfrak{B}(\mathcal{R}).$$

Suppose that $\mathcal{Z} \epsilon \mathfrak{B}(\mathcal{R})$. Consequently $\mathcal{R}-\mathcal{Z} \epsilon \mathfrak{B}(\mathcal{R})$ and by (i):

$$\varphi_0^{-1}(\mathcal{R}-\mathcal{Z}) \epsilon \boldsymbol{X} \quad \text{and} \quad [\varphi_0^{-1}(\mathcal{R}-\mathcal{Z})]=f(\mathcal{Z}(\mathcal{R}-\mathcal{Z}))=0,$$

hence

(ii) $$\varphi_0^{-1}(\mathcal{R}-\mathcal{Z}) \epsilon \boldsymbol{I}.$$

Let $z_0 \epsilon \mathcal{Z}$ and let

$$\varphi(x)=\varphi_0(x) \quad \text{for } x \epsilon \varphi_0^{-1}(\mathcal{Z})$$

and

$$\varphi(x)=z_0 \quad \text{for} \quad x \epsilon \varphi_0^{-1}(\mathcal{R}-\mathcal{Z}).$$

It results from (ii) that

(iii) $$\varphi^{-1}(R) \epsilon \boldsymbol{X} \quad \text{and} \quad [\varphi^{-1}(R)]=[\varphi_0^{-1}(R)] \text{ for every } R \epsilon \mathfrak{B}(\mathcal{R}).$$

If $Z \epsilon \mathfrak{B}(\mathcal{Z})$, then $Z \epsilon \mathfrak{B}(\mathcal{R})$ and therefore by (i) and (iii):

$$\varphi^{-1}(Z) \epsilon \boldsymbol{X} \quad \text{and} \quad [\varphi^{-1}(Z)]=f(Z) \quad \text{for every } Z \epsilon \mathfrak{B}(\mathcal{Z}).$$

Since $\varphi(\mathfrak{X}) \subset \mathcal{Z}$, the sufficiency is proved.

Necessity. Suppose now that every σ-homomorphism of $\mathfrak{B}(\mathcal{Z})$ in any σ-quotient algebra is induced by a mapping.

Let $\mathfrak{X}=\mathcal{R}-\mathcal{Z}$, $\boldsymbol{X}=\mathfrak{B}(\mathfrak{X})$ and $\boldsymbol{I}=\mathfrak{B}(\mathfrak{X})\cdot\mathfrak{B}(\mathcal{R})$ ($\boldsymbol{I}$ is the ideal of all Borel subsets of $\mathcal{R}$ which are contained in $\mathfrak{X}$).

If $Z \epsilon \mathfrak{B}(\mathcal{Z})$, then $Z=\mathcal{Z}R$ where $R \epsilon \mathfrak{B}(\mathcal{R})$. Obviously $X=R\mathfrak{X} \epsilon \mathfrak{B}(\mathfrak{X})$. Thus for every $Z \epsilon \mathfrak{B}(\mathcal{Z})$ there exists a set X such that

(i) $$X \epsilon \boldsymbol{X} \quad \text{and} \quad Z+X \epsilon \mathfrak{B}(\mathcal{R}).$$

If X_1 is another set such that $X_1 \epsilon \mathfrak{X}$ and $Z+X_1 \epsilon \mathfrak{B}(\mathcal{R})$, then

$$(X-X_1)+(X_1-X)=((Z+X)-(Z+X_1))+((Z+X_1)-(Z+X)) \epsilon \mathfrak{B}(\mathcal{R}).$$

Since the set $(X-X_1)+(X_1-X)$ is contained in $\mathfrak{X}$, it belongs to $\boldsymbol{I}$. Consequently

$$[X]=[X_1].$$

Let $f(Z)=[X]$ for every $Z \epsilon \mathfrak{B}(\mathcal{Z})$, where X fulfils the condition (i). From the above considerations it follows easily that f is a σ-homomorphism of $\mathfrak{B}(\mathcal{Z})$ in the σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$.

By the assumption, there exists a mapping φ of $\mathfrak{X}$ in $\mathcal{Z}$ such that

(ii) $$\varphi^{-1}(Z) \epsilon \boldsymbol{X}=\mathfrak{B}(\mathfrak{X}) \quad \text{and} \quad [\varphi^{-1}(Z)]=f(Z) \quad \text{for every } Z \epsilon \mathfrak{B}(\mathcal{Z}).$$

From (ii) and from the definition of f

(iii) $$Z+\varphi^{-1}(Z) \epsilon \mathfrak{B}(\mathcal{R}) \quad \text{for every} \quad Z \epsilon \mathfrak{B}(\mathcal{Z}).$$

Let $\psi(x)=x$ for $x\in\mathcal{Z}$ and $\psi(x)=\varphi(x)$ for $x\in\mathfrak{X}$. The mapping ψ maps $\mathcal{R}$ on $\mathcal{Z}$. From (iii) we obtain that

$$\psi^{-1}(Z)=Z+\varphi^{-1}(Z)\in\mathfrak{B}(\mathcal{R})\quad\text{for every}\quad Z\in\mathfrak{B}(\mathcal{Z}),$$

i. e. that ψ is a Baire function. Consequently

$$\mathcal{Z}=\underset{x}{E}\,(\psi(x)=x)\in\mathfrak{B}(\mathcal{R}),$$

q. e. d.

4. *Homomorphisms defined on a simple field.* Theorem 3.2 can be generalized for an important kind of fields of sets which I shall call simple fields. A field $\boldsymbol{Y}$ of subsets of $\mathcal{Y}$ is *simple* if there exists an enumerable sequence $\{Y_n\}$ of subsets of $\mathcal{Y}$ such that $\boldsymbol{Y}$ is the least σ-field (of subsets of $\mathcal{Y}$), containing all the sets Y_n $(n=1,2,3,...)$.

$\mathcal{C}$ will denote always Cantor's discontinuous set [7]).

4.1. *A σ-field is simple if and only if it is isomorphic to a field of Borel subsets of a separable metric space.*

The sufficiency follows from the fact that every field of Borel subsets of a metric separable space is simple. The necessity follows from the following theorem:

4.2. *Every simple field $\boldsymbol{Y}$ of subsets of $\mathcal{Y}$ possesses the following properties:*

a) *there exists a set $\mathcal{Z}\subset\mathcal{C}$ such that $\boldsymbol{Y}$ is isomorphic to $\mathfrak{B}(\mathcal{Z})$;*

b) *every two-valued σ-measure on $\boldsymbol{Y}$ is trivial;*

c) *every σ-homomorphism f of $\boldsymbol{Y}$ in a σ-field $\boldsymbol{X}$* (of subsets of $\mathfrak{X}$) *is induced by a mapping φ of $\mathfrak{X}$ in $\mathcal{Y}$.*

Proof. a) The characteristic function [8]) c of $\{Y_n\}$ maps the set $\mathcal{Y}$ on a set $\mathcal{Z}\subset\mathcal{C}$ and induces an isomorphism h of $\mathfrak{B}(\mathcal{Z})$ on $\boldsymbol{Y}$ [9]).

[7]) This means: the set of all real numbers t which can be represented in the form $t=\sum_{n=1}^{\infty}\frac{t_n}{3^n}$ where $t_n=0$ or 2.

[8]) The characteristic function of a sequence $\{Y_n\}$ is the function

$$c(y)=\sum_{n=1}^{\infty}\frac{c_n(y)}{3^n}$$

where $c_n(y)=2$ if $y\in Y_n$ and $c_n(y)=0$ if y non $\in Y_n$. See Marczewski [1], p. 211.

[9]) See Marczewski [1], p. 212.

b) Let m be any two-valued σ-measure on $\boldsymbol{Y}$. Then $m_0=mc^{-1}$ is a two-valued σ-measure on $\mathfrak{B}(\mathcal{Z})$, thus it is trivial [10]). Let z_0 be an element of $\mathcal{Z}$ such that $z_0\in Z\in\mathfrak{B}(\mathcal{Z})$ implies $m_0(Z)=1$, and let y_0 be an element of $\mathcal{Y}$ such that $c(y_0)=z_0$. It is easy to prove that $y_0\in Y\in\boldsymbol{Y}$ implies $m(Y)=1$. Hence the measure m is trivial too.

c) is an immediate consequence of b) and 2.2.

A topological space $\mathcal{T}$ is called a *Borel space* if it is homeomorphic to a Borel subset of the Hilbert cube.

4.3. *In order that every σ-homomorphism f of a simple field $\boldsymbol{Y}$ (of subsets of $\mathcal{Y}$) in an arbitrary σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ (of a set $\mathfrak{X}$) be induced by a mapping φ of $\mathfrak{X}$ in $\mathcal{Y}$, it is necessary and sufficient that $\boldsymbol{Y}$ be isomorphic to a field $\mathfrak{B}(\mathcal{T})$ of Borel subsets of a Borel space $\mathcal{T}$.*

Sufficiency. Let c, h and $\mathcal{Z}$ have the same meaning as in the proof of 4.2. From the assumption that $\boldsymbol{Y}$ is isomorphic to $\mathfrak{B}(\mathcal{T})$ it follows that $\mathfrak{B}(\mathcal{T})$ and $\mathfrak{B}(\mathcal{Z})$ are isomorphic. Hence there exists a generalized homeomorphism (in the sense defined by Kuratowski [11]) between $\mathcal{T}$ and $\mathcal{Z}$ [12]). Since $\mathcal{T}$ is a Borel space, $\mathcal{Z}\in\mathfrak{B}(\mathcal{R})$.

The σ-homomorphism fh maps $\mathfrak{B}(\mathcal{Z})$ in $\boldsymbol{X}/\boldsymbol{I}$. On account of 3.2 there exists a mapping ψ of $\mathfrak{X}$ in $\mathcal{Z}$ inducing the homomorphism fh. Let $\varkappa(z)$ denote (for every $z\in\mathcal{Z}$) an element of $\mathcal{Y}$ such that $c\varkappa(z)=z$. The mapping $\varphi=\varkappa\psi$ of $\mathfrak{X}$ in $\mathcal{Y}$ induces the homomorphism f, q. e. d.

Necessity. Suppose that every σ-homomorphism of $\boldsymbol{Y}$ in $\boldsymbol{X}/\boldsymbol{I}$ is induced by a mapping of $\mathfrak{X}$ into $\mathcal{Y}$. Let $\mathcal{Z}$, h, c have the same meaning as in the proof of theorem 4.2, and let g be a σ-homomorphism of $\mathfrak{B}(\mathcal{Z})$ in $\boldsymbol{X}/\boldsymbol{I}$. Then $f=gh^{-1}$ is a σ-homomorphism of $\boldsymbol{Y}$ in $\boldsymbol{X}/\boldsymbol{I}$, induced by a mapping φ of $\mathfrak{X}$ into $\mathcal{Y}$. It is easy to show that $\psi=c\varphi$ is a mapping of $\mathfrak{X}$ in $\mathcal{Z}$ inducing the homomorphism g. $\boldsymbol{X}$, $\boldsymbol{I}$, g being arbitrary, $\mathcal{Z}$ is a Borel space since (by 3.2) $\mathcal{Z}$ is a Borel subset of $\mathcal{R}$. The necessity is proved.

As an immediate consequence of 4.3 we obtain the following characterization of Borel spaces (which is expressed only in terms of the theory of Boolean algebras and of the general theory of sets):

4.4. *In order that a separable metric space $\mathcal{T}$ be a Borel space it is necessary and sufficient that every σ-homomorphism of $\mathfrak{B}(\mathcal{T})$ in any σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ (of $\mathfrak{X}$) be induced by a mapping of $\mathfrak{X}$ into $\mathcal{T}$.*

[10]) See Marczewski and Sikorski [1], p. 139.

[11]) See Kuratowski [1], p. 221.

[12]) See Marczewski [2], p. 138.

If $\boldsymbol{Y}$ is a field of subsets of $\mathcal{Y}$ and $y \in \mathcal{Y}$, then $\mathrm{At}(y)$ will denote the product of all sets Y such that $y \in Y \in \boldsymbol{Y}$. The sets $\mathrm{At}(y)$ will be called *atoms* of $\boldsymbol{Y}$. Obviously (for every $Y \in \boldsymbol{Y}$) $\mathrm{At}(y) \subset Y$ if and only if $y \in Y$. If $y_1 \neq y_2$, then either $\mathrm{At}(y_1) = \mathrm{At}(y_2)$ or $\mathrm{At}(y_1) \cdot \mathrm{At}(y_2) = 0$.

4.5. *Let $\boldsymbol{Y}$ be a simple field of subsets of $\mathcal{Y}$, let $\boldsymbol{X}/\boldsymbol{I}$ be a σ-quotient algebra of $\mathcal{X}$ and let φ_1 and φ_2 be two mappings of $\mathcal{X}$ into $\mathcal{Y}$ such that $\varphi_i^{-1}(Y) \in \boldsymbol{X}$ for every $Y \in \boldsymbol{Y}$, $i=1,2$. The mappings φ_1 and φ_2 induce the same homomorphism of $\boldsymbol{Y}$ in $\boldsymbol{X}/\boldsymbol{I}$ if and only if*

(i) $$\underset{x}{E}\,(\mathrm{At}(\varphi_1(x)) \dotplus \mathrm{At}(\varphi_2(x)) \in \boldsymbol{I}.$$

If every atom of $\boldsymbol{Y}$ contains only one point (for instance if $\boldsymbol{Y} = \mathfrak{B}(\mathcal{T})$ where $\mathcal{T}$ is a separable metric space), the condition (i) can be expressed in the form:

(ii) $$\underset{x}{E}\,(\varphi_1(x) \dotplus \varphi_2(x)) \in \boldsymbol{I}.$$

Necessity. Let

$$X = \underset{x}{E}\,\big(\mathrm{At}(\varphi_1(x)) \dotplus \mathrm{At}(\varphi_2(x))\big)$$

and let $\{Y_n\}$ be a sequence of sets such that $\boldsymbol{Y}$ is the least σ-field which contains all Y_n. It is easy to show that for every pair of different atoms $\mathrm{At}(y_1)$, $\mathrm{At}(y_2)$ there exists an integer n such that either

$$\mathrm{At}(y_1) \subset Y_n \quad \text{and} \quad \mathrm{At}(y_2) \cdot Y_n = 0$$

or

$$\mathrm{At}(y_1) \cdot Y_n = 0 \quad \text{and} \quad \mathrm{At}(y_2) \subset Y_n.$$

Therefore

$$X = \sum_n (\varphi_1^{-1}(Y_n) - \varphi_2^{-1}(Y_n)) + \sum_n (\varphi_2^{-1}(Y_n) - \varphi_1^{-1}(Y_n)).$$

If φ_1 and φ_2 induce the same homomorphism, we have

$$[\varphi_1^{-1}(Y_n)] = [\varphi_2^{-1}(Y_n)]$$

i. e.

$$\varphi_1^{-1}(Y_n) - \varphi_2^{-1}(Y_n) \in \boldsymbol{I}$$

and

$$\varphi_2^{-1}(Y_n) - \varphi_1^{-1}(Y_n) \in \boldsymbol{I}.$$

Hence $X \in \boldsymbol{I}$, q. e. d.

Sufficiency. Let X have the same meaning as in the proof of the necessity. If $x \in \mathcal{X} - X$, then $\mathrm{At}(\varphi_1(x)) = \mathrm{At}(\varphi_2(x))$. Since the conditions

$$\varphi_i(x) \in Y \quad \text{and} \quad \mathrm{At}(\varphi_i(x)) \subset Y$$

($i=1,2$, $Y \in \boldsymbol{Y}$) are equivalent,

$$\varphi_1^{-1}(Y) - X = \varphi_2^{-1}(Y) - X$$

for every $Y \in \boldsymbol{Y}$. If $X \in \boldsymbol{I}$, then

$$[\varphi_1^{-1}(Y)] = [\varphi_1^{-1}(Y) - X] = [\varphi_2^{-1}(Y) - X] = [\varphi_2^{-1}(Y)],$$

q. e. d.

5. *Homomorphisms between σ-quotient algebras.* By (**) and (***) a mapping φ induces a homomorphism f of a σ-quotient algebra $\boldsymbol{Y}/\boldsymbol{J}$ in a σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ if and only if φ induces the homomorphism h of $\boldsymbol{Y}$ in $\boldsymbol{X}/\boldsymbol{I}$ defined by the formula

$$h(Y) = f([Y]) \quad \text{for} \quad Y \in \boldsymbol{Y}.$$

On account of 4.3 and 4.5 we obtain the following theorem:

5.1. *Let $\boldsymbol{X}/\boldsymbol{I}$ and $\boldsymbol{Y}/\boldsymbol{J}$ be σ-quotient algebras of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then:*

a) *in order that a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$ induce a σ-homomorphism f of $\boldsymbol{Y}/\boldsymbol{J}$ in $\boldsymbol{X}/\boldsymbol{I}$, it is necessary and sufficient that*

(i) $\varphi^{-1}(Y) \in \boldsymbol{X}$ *for every* $Y \in \boldsymbol{Y}$ *and* $\varphi^{-1}(Y) \in \boldsymbol{I}$ *for every* $Y \in \boldsymbol{J}$;

b) *if $\boldsymbol{Y}$ is simple, two mappings φ_1 and φ_2 (of $\mathcal{X}$ into $\mathcal{Y}$) satisfying the condition* (i), *induce the same homomorphism f of $\boldsymbol{Y}/\boldsymbol{J}$ in $\boldsymbol{X}/\boldsymbol{I}$ if and only if*

(ii) $$\underset{x}{E}\,\big(\mathrm{At}(\varphi_1(x)) \dotplus \mathrm{At}(\varphi_2(x))\big) \in \boldsymbol{I};$$

c) *if $\boldsymbol{Y}$ is isomorphic to the field of Borel subsets of a Borel space, every σ-homomorphism f of $\boldsymbol{Y}/\boldsymbol{J}$ in $\boldsymbol{X}/\boldsymbol{I}$ is induced by a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$.*

If every atom of $\boldsymbol{Y}$ contains only one element, the condition (ii) is identical with the condition:

(iii) $$\underset{x}{E}\,(\varphi_1(x) \dotplus \varphi_2(x)) \in \boldsymbol{I}.$$

Generalizing the definition given in par. 4 we shall say that a σ-complete Boolean algebra $\boldsymbol{A}$ is *simple* if there exists an enumerable sequence $\{A_n\}$ of elements of $\boldsymbol{A}$ such that the least σ-complete subalgebra of $\boldsymbol{A}$ containing all A_n $(n=1,2,3,\ldots)$ is identical with $\boldsymbol{A}$.

5.2 *Every simple Boolean algebra $\boldsymbol{A}$ is isomorphic to a σ-quotient algebra $\mathfrak{B}(\mathcal{C})/\boldsymbol{J}$ of Cantor's discontinuous set $\mathcal{C}$.*

$\boldsymbol{A}$ is isomorphic to a σ-quotient algebra $\boldsymbol{Y}/\boldsymbol{I}$ of a set $\mathcal{Y}$[13]. Let h be an isomorphism of $\boldsymbol{Y}/\boldsymbol{I}$ on $\boldsymbol{A}$ and let Y_n be a set of $\boldsymbol{Y}$ such that

$$A_n=h([Y_n]).$$

The formula

$$f(Z)=h([c^{-1}(Z)]) \quad \text{for} \quad Z\in\mathfrak{B}(\mathcal{C})$$

(where c is the characteristic function of the sequence $\{Y_n\}$) defines a σ-homomorphism of $\mathfrak{B}(\mathcal{C})$ on $\boldsymbol{A}$. Let $\boldsymbol{J}$ be the ideal of all $Z\in\mathfrak{B}(\mathcal{C})$ such that $f(Z)=0$. $\boldsymbol{A}$ is isomorphic to $\mathfrak{B}(\mathcal{C})/\boldsymbol{J}$.

Let now $\boldsymbol{A}$ be a simple Boolean algebra and let $\boldsymbol{B}$ be any σ-complete Boolean algebra. $\boldsymbol{B}$ being isomorphic to a σ-quotient algebra[13], on account of 5.2 and 5.1 c) every σ-homomorphism f of $\boldsymbol{A}$ in $\boldsymbol{B}$ may be considered as a homomorphism between two σ-quotient algebras which is induced by a point mapping φ.

6. *Isomorphisms.* In this section we shall consider two Borel spaces $\mathcal{X}$ and $\mathcal{Y}$ and two σ-ideals $\boldsymbol{I}$ and $\boldsymbol{J}$ of $\mathfrak{B}(\mathcal{X})$ and $\mathfrak{B}(\mathcal{Y})$ respectively such that $\mathfrak{B}(\mathcal{X})/\boldsymbol{I}$ is isomorphic to $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$.

Under these assumptions:

6.1. *For every isomorphism h of $\mathfrak{B}(\mathcal{X})/\boldsymbol{I}$ on $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$ there exist two Borel sets: $X_0\in\boldsymbol{I}$ and $Y_0\in\boldsymbol{J}$ and a generalized homeomorphism*[14] *φ_0 of $\mathcal{X}-X_0$ on $\mathcal{Y}-Y_0$ such that*

$$h([X])=[\varphi_0(X)] \quad \textit{for every} \quad X\in\mathfrak{B}(\mathcal{X})\,^{15}).$$

$\mathcal{Y}$ being a Borel space, on account of theorem 5.1 c) there exists a mapping ψ of $\mathcal{Y}$ into $\mathcal{X}$ which induces the σ-homomorphism h of $\mathfrak{B}(\mathcal{X})/\boldsymbol{I}$ on $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$. Analogously there exists a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$ which induces the σ-homomorphism h^{-1} of $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$ on $\mathfrak{B}(\mathcal{X})/\boldsymbol{I}$.

The mapping $\psi\varphi$ of $\mathcal{X}$ into $\mathcal{X}$ induced the σ-homomorphism $h^{-1}h$ of $\mathfrak{B}(\mathcal{X})/\boldsymbol{I}$ in $\mathfrak{B}(\mathcal{X})/\boldsymbol{I}$. Since $h^{-1}h([X])=[X]$ for every $X\in\mathfrak{B}(\mathcal{X})$, the homomorphism $h^{-1}h$ is also induced by the identical mapping $\varkappa(x)=x$. By theorem 5.1 b) the set

$$X_0=\mathop{E}_{x}(\psi\varphi(x)\neq x)$$

[13]) See Loomis [1], p. 757 and Sikorski [2], p. 256.

[14]) In the sense defined by Kuratowski. See footnote [11]).

[15]) $[X]$ is an element of $\boldsymbol{X}/\boldsymbol{I}$ and $[\varphi_0(X)]$ is an element of $\boldsymbol{Y}/\boldsymbol{J}$. $\varphi_0(X)$ is the set of all $y=\varphi_0(x)$ where $x\in\boldsymbol{X}-X_0$.

belongs to the ideal $\boldsymbol{I}$. Analogously the set

$$Y_0=\mathop{E}_{y}(\varphi\psi(y)\neq y)$$

belongs to $\boldsymbol{J}$. Let A and B denote the geometrical images of φ and ψ respectively, i. e. the subsets

$$\mathop{E}_{xy}(y=\varphi(x)) \quad \text{and} \quad \mathop{E}_{xy}(x=\psi(y))$$

of the cartesian product $\mathcal{X}\times\mathcal{Y}$. $\mathcal{X}-X_0$ and $\mathcal{Y}-Y_0$ being projections of AB on the „axes" $\mathcal{X}$ and $\mathcal{Y}$ respectively, we infer that the mapping $\varphi_0=\varphi|\mathcal{X}-X_0$[16]) is a generalized homeomorphism of $\mathcal{X}-X_0$ on $\mathcal{Y}-Y_0$ and that the mapping $\psi|\mathcal{Y}-Y_0$ is the inverse mapping to φ_0. Therefore for any $X\in\mathfrak{B}(\mathcal{X})$:

$$\varphi_0(X)=\psi^{-1}(X)-Y_0.$$

Hence

$$[\varphi_0(X)]=[\psi^{-1}(X)]-[Y_0]=h([X])$$

for every $X\in\mathfrak{B}(\mathcal{X})$ since ψ induces h and $Y_0\in\boldsymbol{J}$, q. e. d.

Let $\mathfrak{p}(\boldsymbol{I})$ $(\mathfrak{p}(\boldsymbol{J}))$ denote the greatest of the numbers $0,1,2,3,\ldots,\aleph_0,2^{\aleph_0}$, which is the power of a set $X\in\boldsymbol{I}$ $(Y\in\boldsymbol{J})$.

6.2. *Let h be an isomorphism of $\mathfrak{B}(\mathcal{X})/\boldsymbol{I}$ on $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$. In order that there exists a generalized homeomorphism φ of $\mathcal{X}$ on $\mathcal{Y}$ such that*

$$[\varphi(X)]=h([X]) \quad \textit{for every} \quad X\in\mathfrak{B}(\mathcal{X}),$$

it is necessary and sufficient that $\mathfrak{p}(\boldsymbol{I})=\mathfrak{p}(\boldsymbol{J})$.

The necessity follows from 5.1 b), since φ induces h^{-1} and φ^{-1} induces h.

Suppose that $\mathfrak{p}(\boldsymbol{I})=\mathfrak{p}(\boldsymbol{J})=\mathfrak{p}$. Let X_1 and Y_1 be two sets such that $X_1\in I$, $Y_1\in J$ and $\overline{\overline{X}}_1=\overline{\overline{Y}}_1=\mathfrak{p}$, and let φ_0, X_0, Y_0 have the same meaning as in theorem 6.1. The Borel sets

$$X_2=X_0+X_1+\varphi_0^{-1}(Y_1) \quad \text{and} \quad Y_2=Y_0+Y_1+\varphi_0(X_1)$$

are of potency $\mathfrak{p}$ and

$$\varphi_0(\mathcal{X}-X_2)=\mathcal{Y}-Y_2.$$

[16]) This means: φ_0 is defined only on the set $\mathcal{X}-X_0$ and $\varphi_0(x)=\varphi(x)$ for $x\in\mathcal{X}-X_0$.

Let φ_2 be a generalized homeomorphism of X_2 on Y_2 and let

$$\varphi(x)=\varphi_2(x) \quad \text{for} \quad x \in X_2$$

and

$$\varphi(x)=\varphi_0(x) \quad \text{for} \quad x \in \mathfrak{X}-X_2.$$

φ is a generalized homeomorphism of $\mathfrak{X}$ on $\mathcal{Y}$ and

$$[\varphi(X)]=[\varphi_2(X)]+[\varphi_0(X-X_2)]=[\varphi_0(X)]-[\varphi_0(X_2)]=[\varphi_0(X)]=h([X])$$

for every $X \in \mathfrak{B}(\mathfrak{X})$ since $\varphi_2(X)\subset Y_2 \in \boldsymbol{J}$ and $\varphi_0(X_2)\subset Y_2 \in \boldsymbol{J}$.

References.

Kuratowski C. [1] *Topologie I.* Warszawa—Lwów 1933.

Loomis L. H. [1] *On the representation of σ-complete algebras.* Bull. Am. Math. Soc. **53** (1947) pp. 757-760.

Marczewski (Szpilrajn) E. [1] *The characteristic function of a sequence of sets and some of its applications.* Fund. Math. **31** (1938), pp. 207-223.

— [2] *On the isomorphism and the equivalence of classes and sequences of sets.* Fund. Math. **32** (1939), pp. 122-144.

Marczewski E. and Sikorski R. [1] *Measures in non-separable spaces.* Coll. Math. **1** (1948), pp. 119-125.

Sikorski R. [1] *The integral in a Boolean algebra.* Coll. Math. **2**, 1 (1949).

— [2] *On the representation of Boolean algebras as fields of sets.* Fund. Math. **35** (1948), pp. 247-258.

Stone M. H. [1] *The theory of the representation for Boolean algebras.* Trans. Am. Math. Soc. **40** (1936), pp. 37-111.

Ulam S. [1] *Zur Masstheorie in der allgemeinen Mengenlehre,* Fund. Math. **16** (1930), pp. 140-150.

On joins of spherical mappings.

By

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1. Introdution. In a recent work of G. W. Whitehead, [4], an important generalization of H. Freudenthal's Einhängung, [3], has been introduced which seems to be one of the essential intruments for the attacking of the unsolved problem of calculating the homotopy groups of spheres. For each pair of elements $\alpha \in \pi^p(S^m)$, $\beta \in \pi^q(S^n)$, a unique element $\alpha \vee \beta \in \pi^{p+q+1}(S^{m+n+1})$ is determined, which will be called the join of α and β. If $q=n$ and β is of degree $+1$, then $\alpha\vee\beta$ is the $(n+1)$-fold Einhängung of α.

The object of the present paper is to give a detailed investigation of this joining operation. Instead of considering it as an operation on the homotopy groups, we shall present it by an imbedding of the product space $(S^m)^{S^p}\times(S^n)^{S^q}$ into the space $(S^{m+n+1})^{S^{p+q+1}}$, where Y^X denotes, as usual, the space of all mappings (i. e. continuous transformations) of X into Y.

In another recent work of G. W. Whitehead, [5], it has been proved in a quite complicate way that the Einhängung of a Whitehead product, [6], is always inessential. By using our methods, we are able to prove its generalization that the join $\alpha\vee\beta$ is inessential if at least one of the elements α,β is a product.

2. The imbedding by means of joining. For the sake of briefness, we shall denote by $\{p,m\}$ the space $(S^m)^{S^p}$. In the present paragraph, we shall define an imbedding of the product space $\{p,m\}\times\{q,n\}$ into the space $\{p+q+1, m+n+1\}$ which forms the kernel of the whole investigation.

Let R^{p+1}, R^{q+1} be two euclidean spaces with coordinates systems $(x_0,x_1,\dots,x_p)$, $(y_0,y_1,\dots,y_q)$ respectively. Let

$$R^{p+q+2}=R^{p+1}\times R^{q+1},$$