Si l'on avait $\beta_k f_k(x) \leqslant g(x)$, où $g(x) \geqslant 0$, il existerait évidemment une suite décroissante d'intervalles fermés δ_k contenus dans $[0,1]$ tels que

$$f_k(x) \geqslant \frac{k}{\beta_k} \quad \text{dans} \quad \delta_k, \quad \text{pour} \quad k=1,2,...$$

En désignant par x_0 un point commun des intervalles δ_k, on aurait alors $g(x_0) \geqslant k$ pour $k=1,2,...$, ce qui est impossible.

Un exemple plus simple, mais moins naturel, fournit l'ensemble des fonctions continues dans l'intervalle $(0,1)$ sauf un nombre fini de points au plus [4]). La suite

$$f_n(x) = \frac{1}{\left(\frac{1}{n} - x\right)^2}$$

montre alors que la condition II n'est pas remplie.

[4]) Je dois cet exemple à M. Cz. Ryll-Nardzewski.

On a characterization of the lattice of all ideals of a Boolean ring [1]).

By

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In this note we shall characterize the ordered system of all ideals of a Boolean ring. This question is related to the corresponding problem for lattices which was previously considered by A. Komatu and more recently by G. Birkhoff and O. Frink [2]).

A *sup-lattice* is an ordered set S such that $x \vee y$ has a meaning for any two $x, y \in S$. An *ideal* over S is a set $I \subset S$ such that $x \in S$, $y \in I$, $x \leqslant y$ imply $x \in I$, and $x, y \in I$ imply $x \vee y \in I$. The set $\mathfrak{J}(S)$ of all ideals over S ordered by set inclusion is a complete lattice. The first and last elements of $\mathfrak{J}(S)$ are the empty set $\emptyset$ and S. If $\{I_\lambda\}$ is a non-empty family of ideals over S, then $\bigwedge_\lambda I_\lambda$ is the set intersection and $\bigvee_\lambda I_\lambda$ is the set of all $x \in S$ for which there exist a finite non-empty family $\{\lambda_i\}$ and $x_i \in I_{\lambda_i}$ such that $x \leqslant \bigvee_i x_i$. If $x \in S$, the set $I(x)$ of all $y \in S$, $y \leqslant x$ is called the *principal* ideal generated by x. Denoting the set of all principal ideals by S^*, we have a natural isomorphism $S \to S^*$. It is clear that $I(x_1 \vee x_2) = I(x_1) \vee I(x_2)$. If, in addition, S is a lattice then $I(x_1 \wedge x_2) = I(x_1) \wedge I(x_2)$.

If L is a complete lattice and $x \in L$, then x is said to be *compact* if, for any non-empty family $\{x_\lambda\} \subset L$ such that $x \leqslant \bigvee_\lambda x_\lambda$, there exists a finite non-empty subfamily $\{x_{\lambda_i}\}$ such that $x \leqslant \bigvee_i x_{\lambda_i}$. Clearly the first element of L is compact.

[1]) Presented to the American Mathematical Society, October 30, 1948 at the New York City meeting.

[2]) A. Komatu, *On a characterization of join homomorphic transformation-lattice*, Proc. Imp. Acad. Tokyo, vol. 19 (1943), pp. 119-124; G. Birkhoff and O. Frink Jr., *Representations of lattices by sets*, Trans. Amer. Math. Soc. vol. **64** (1948), to appear.

The principal ideals over S are characterized as the compact elements of $\mathfrak{J}(S)$ distinct from its first element. In fact, if $x \in S$, $I_\lambda \in \mathfrak{J}(S)$, $I(x) \leqslant \bigvee_\lambda I_\lambda$ then $x \in \bigvee_\lambda I_\lambda$. Therefore there exist a finite non-empty family $\{\lambda_i\}$ and $x_i \in I_{\lambda_i}$ such that $x \leqslant \bigvee_i x_i$. This implies $I(x) \leqslant \bigvee_i I_{\lambda_i}$ and thus $I(x)$ is compact. It is clear that $I(x)$ is non-empty. Conversely let $I \in \mathfrak{J}(S)$ be compact and non-empty. Since $I = \bigvee_{\lambda \in I} I(\lambda)$, the compactness implies the existence of a finite non empty-family $\{\lambda_i\}$ such that $I = \bigvee_i I(\lambda_i) = I(\bigvee_i \lambda_i)$, and I is principal.

It is clear that *S has a first element if and only if $\mathfrak{J}(S)$ has at least two elements and the set of all elements of $\mathfrak{J}(S)$ distinct from its first element has a first element*; and that *S has a last element if and only if $\mathfrak{J}(S)$ has at least two elements and the last element of $\mathfrak{J}(S)$ is compact.* A less obvious result is the following: *S has a last element if and only if $\mathfrak{J}(S)$ has at least two elements and every non-empty chain of elements of $\mathfrak{J}(S)$ distinct from its last element has a supremum distinct from this last element.* The half of the proof in which we assume that S has a last element is clear as the last element of $\mathfrak{J}(S)$ is then compact. To prove the other half, let $\mathfrak{J}'(S)$ be the set of all ideals over S distinct from S: assume that $\mathfrak{J}'(S)$ is non empty, i. e. S is non empty, and that every non-empty chain in $\mathfrak{J}'(S)$ has a supremum in $\mathfrak{J}'(S)$. Assume also that S has no last element. Then $S^* \subset \mathfrak{J}'(S)$. From the fact that every finite non-empty subset of S^* has a supremum in $\mathfrak{J}'(S)$, it follows that S^* has a supremum in $\mathfrak{J}'(S)$ [3], but this is impossible since the supremum of S^* in $\mathfrak{J}(S)$ is S.

Every element of $\mathfrak{J}(S)$ is the supremum of all smaller compact elements. In fact, every ideal over S either is empty and therefore compact, or is the supremum of all smaller principal ideals.

[3] If A is an ordered set, the following properties are equivalent:

a) every non empty chain $C \subset A$ has a supremum,

b) if $X \subset A$ is non empty and every finite non empty subset of X has a supremum in A, then X has a supremum in A.

It is clear that (b) $\rightarrow$ (a). To prove that (a) $\rightarrow$ (b), consider some X as stated in (b). Let Y be the set of all $y \in A$ such that: 1) $y \vee x_1 \vee \ldots \vee x_n$ has a meaning for any $x_1, \ldots, x_n \in X$, 2) every upper bound of X is an upper bound of y. It is clear that $X \subset Y$. Moreover every non empty chain of Y has a supremum in Y. By Zorn's theorem there exists some maximal $m \in Y$. For any $x \in X$ we have $m \vee x \in Y$, therefore $x \leqslant m$ and m is an upper bound of X. Since every upper bound of X is also an upper bound of m, we conclude that m is the supremum of X.

S is a lattice if and only if the infimum of two compact elements of $\mathfrak{J}(S)$ distinct from its first element is compact and distinct from this first element. In fact, if $I_1, I_2 \in \mathfrak{J}(S)$ are compact and non-empty, then $I_1 = I(x_1)$, $I_2 = I(x_2)$ where $x_1, x_2 \in S$. If S is a lattice we have $I_1 \wedge I_2 = I(x_1 \wedge x_2)$, i. e. $I_1 \wedge I_2$ is compact and non-empty. Conversely, if for every $x_1, x_2 \in S$ there exists some $x \in S$ such that $I(x_1) \wedge I(x_2) = I(x)$ then $x = x_1 \wedge x_2$ and therefore S is a lattice.

If S is a lattice, then S is distributive if and only if $\mathfrak{J}(S)$ is a distributive lattice. Assume S distributive. Let $I, I_1, I_2 \in \mathfrak{J}(S)$ and consider some $x \in I \wedge (I_1 \vee I_2)$. Then $x \in I$ and there exist $x_1 \in I_1$, $x_2 \in I_2$ such that $x \leqslant x_1 \vee x_2$. Therefore $x = x \wedge (x_1 \vee x_2) = (x \wedge x_1) \vee (x \wedge x_2)$. But $x \wedge x_1 \in I \wedge I_1$, $x \wedge x_2 \in I \wedge I_2$. Therefore $x \in (I \wedge I_1) \vee (I \wedge I_2)$ and we have proved that $\mathfrak{J}(S)$ is a distributive lattice (the same reasoning shows that the known infinite distributive law $I \wedge \bigvee_\lambda I_\lambda = \bigvee_\lambda I \wedge I_\lambda$ is also valid). Conversely, if $\mathfrak{J}(S)$ is a distributive lattice, we have $I(x) \wedge \{I(x_1) \vee I(x_2)\} = \{I(x) \wedge I(x_1)\} \vee \{I(x) \wedge I(x_2)\}$ from which we may infer that S is distributive too.

S_1, S_2 being two sup-lattices, it is clear that *every isomorphism between S_1^*, S_2^* has a unique extension to an isomorphism between $\mathfrak{J}(S_1)$, $\mathfrak{J}(S_2)$.* Conversely *every isomorphism between $\mathfrak{J}(S_1)$, $\mathfrak{J}(S_2)$ is the extension of a unique isomorphism between S_1^*, S_2^** because every compact element is mapped into a compact element. Therefore, *S_1 and S_2 are isomorphic if and only if $\mathfrak{J}(S_1)$ and $\mathfrak{J}(S_2)$ are isomorphic.* Moreover *the group of isomorphisms of S and the group of isomorphisms of $\mathfrak{J}(S)$ are isomorphic.*

Theorem 1. *A non-empty lattice L is isomorphic to $\mathfrak{J}(S)$ for some sup-lattice S if and only if:*

(1) *L is complete,*

(2) *every element in L is the supremum of all smaller compact elements.*

In this case S is essentially unique. S has a first element if and only if L has at least two elements and the set of all elements of L distinct from its first element has a first element. S has a last element if and only if L has at least two elements and the last element of L is compact; it amounts to the same to say that L has at least two elements and every non-empty chain of elements of L distinct from its last element has a supremum distinct from this last element. S is a lattice if and only if the infimum of two compact elements of L distinct from its first element is compact and distinct from this first element.

Proof. The necessity of the conditions is already clear. To prove the sufficiency, assume that conditions (1) and (2) are satisfied. Let S be the set of all compact elements of L distinct from its first element. Since the case where S is empty is trivial, we shall assume that S is non-empty. If $x, y \in S$ then clearly $x \vee y \in S$: therefore S is a sup-lattice. For any $I \in \mathfrak{J}(S)$ let $\varphi(I)$ be defined as $\bigvee_{t \in I} t$ if I is non empty, and as the first element of L otherwise. For any $x \in L$, let $\psi(x)$ be defined as the set of all $y \in S$, $y \leqslant x$. It is clear that $\varphi(I) \in L$ and $\psi(x) \in \mathfrak{J}(S)$. We have $\varphi\{\psi(x)\} = x$. This is clear if x is the first element of L; otherwise it follows from the fact that x, being the supremum of all smaller compact elements, must also be the supremum of all smaller elements in S. We also have $\psi\{\varphi(I)\} = I$. It is clear that $I \leqslant \psi\{\varphi(I)\}$; on the other hand, if $x \in \psi\{\varphi(I)\}$, we have $x \leqslant \varphi(I) = \bigvee_{t \in I} t$ and by the compactness of x there exist $t_1, \dots, t_n \in I$ such that $x \leqslant t_1 \vee \dots \vee t_n$ which implies $x \in I$. Since $I_1 \leqslant I_2$ implies $\varphi(I_1) \leqslant \varphi(I_2)$ and $x_1 \leqslant x_2$ implies $\psi(x_1) \leqslant \psi(x_2)$, we have proved that $\mathfrak{J}(S)$ and L are isomorphic. The remaining of the theorem follows from the previous results.

It is convenient to notice that several authors make use only of the *proper* ideals, i. e. the non-empty ideals which at the same time are strict subsets of S. The corresponding theorem for these proper ideals is obtained from theorem 1 in an obvious way.

An element $x \in L$ of a lattice L is said to be *prime* in L if $x_1, x_2 \in L$, $x_1 \wedge x_2 \leqslant x$ imply either $x_1 \leqslant x$ or $x_2 \leqslant x$. The element $x \in L$ is called *inf-irreducible* in L if $x_1, x_2 \in L$, $x_1 \wedge x_2 = x$ imply either $x_1 = x$ or $x_2 = x$. Clearly every prime element is inf-irreducible. If L is distributive, every inf-irreducible element is prime; in fact, $x_1 \wedge x_2 \leqslant x$ imply $x = x \vee (x_1 \wedge x_2) = (x \vee x_1) \wedge (x \vee x_2)$: therefore either $x \vee x_1 = x$, i. e. $x_1 \leqslant x$, or $x \vee x_2 = x$, i. e. $x_2 \leqslant x$.

If S is a lattice, then $I \in \mathfrak{J}(S)$ is called a *prime* ideal if $x, y \in S - I$ imply $x \wedge y \in S - I$. *An ideal $I \in \mathfrak{J}(S)$ is prime if and only if I is a prime element of $\mathfrak{J}(S)$.* In fact, let $I \in \mathfrak{J}(S)$ be a prime ideal and $I_1, I_2 \in \mathfrak{J}(S)$ be such that $I_1 \wedge I_2 \leqslant I$. Assume that $I_1 \leqslant I, I_2 \leqslant I$ are false, i. e. there exist $x_1 \in I_1 - I$, $x_2 \in I_2 - I$. Then $x_1 \wedge x_2 \in I_1 \wedge I_2 - I$ which is impossible. Conversely, if I is prime in $\mathfrak{J}(S)$ and $x, y \in S - I$, then $I(x) \leqslant I$, $I(y) \leqslant I$ are false: therefore $I(x) \wedge I(y) = I(x \wedge y) \leqslant I$ is false, i. e. $x \wedge y \in S - I$.

Theorem 2. *A non empty lattice L is isomorphic to the lattice $\mathfrak{J}(R)$ of all ideals of some Boolean ring R*[4]) *if and only if:*

(1) *L is complete,*

(2) *every element in L is the supremum of all smaller compact elements,*

(3) *the infimum of two compact elements is compact,*

(4) *L is a distributive lattice,*

(5) *every inf-irreducible element in L distinct from its last element is a dual atom*[5]).

Then R is essentially unique. R has a unity if and only if the following equivalent conditions are satisfied:

(6) *the last element of L is compact,*

(6′) *every non empty chain of elements of L distinct from its last element has a supremum distinct from this last element.*

Proof. We have to prove only the sufficiency. By the preceding theorem L is isomorphic to the set $\mathfrak{J}'(R)$ of all non empty ideals of a non empty distributive lattice R with first element 0. We hav- to prove that the segment $[0,x]$ is a Boolean algebra for any $x \in R$. Let I be a prime proper ideal over $[0,x]$. By a known result[6]) there exists a prime proper ideal J over R such that $J \wedge [0,x] = I$. Since J is a inf-irreducible element of $\mathfrak{J}'(R)$, it follows that J is a dual atom of $\mathfrak{J}'(R)$, i. e. J is a maximal proper ideal over R. By a known result[7]) I is a maximal proper ideal over $[0,x]$. Thus every prime proper ideal over $[0,x]$ is a maximal proper ideal over $[0,x]$. From this we may conclude that $[0,x]$ is a Boolean algebra[8]) and therefore R is a Boolean ring. The statements concerning the uniqueness of R and the existence of unity are already clear.

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[4]) See M. H. Stone, *The theory of representation of Boolean algebras*, Trans. Amer. Math. Soc., vol. **40** (1936), pp. 37-111; G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Coll. Publ., vol. **25** (1940), New York. It must be noticed that in the ring theory we do not include the empty set among the ring ideals.

[5]) I. e. an element covered by the last element.

[6]) If A is a sublattice of the distributive lattice B, an ideal I over A is prime if and only if there exists a prime ideal J over B such that $J \cap A = I$.

[7]) If A is an ideal over the distributive lattice B, J is a dual atom of $\mathfrak{J}(B)$ and $J \cap A \leq A$ then $J \cap A$ is a dual atom of $\mathfrak{J}(A)$.

[8]) See *Une propriété caractéristique des algèbres booléiennes*, Portugaliae Math., vol. **6** (1947), pp. 115-118; A. Monteiro, *Sur l'Arithmétique des filtres premiers*, Comptes Rendus Paris, t. **225** (1947), pp. 846-848.

The lattice of all open sets of a Hausdorff space satisfies conditions (1),...,(5) of the preceding theorem if and only if the space is locally compact and totally-disconnected. Therefore (by virtue of Stone's theorem on the topological representation of Boolean rings)[9]) theorem 2 gives also the characterization of the lattice of all open sets of a locally compact totally-disconnected space. The compact case is obtained by adding condition (6) or (6′).

[9]) See M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., vol. **41** (1937), pp. 375-481.

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An undecidable arithmetical statement.

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The purpose of this paper is to give an alternative proof of the existence of formally undecidable sentences. Instead of the arithmetization of syntax and the diagonal process which were used by Gödel in his famous paper of 1931[1]), I shall make use of some simple set-theoretic lemmas and of the Skolem-Löwenheim theorem.

My result is in some respect stronger than that of Gödel. The sentence constructed by his method ceases to be undecidable if one enlarges the underlying logic by a new rule of proof, in the simplest case by the rule of infinite induction[2]). The undecidability of the sentence to be constructed here is, on the contrary, independent of whether we accept the absolute notion of integers or the relative (axiomatic) one[2a]).

On the other hand the proof of undecidability to be given below is unlike that of Gödel non-finitary. It rests on the axioms of the Zermelo-Fraenkel set-theory including the axiom of choice and an additional axiom ensuring the existence of at least one inaccessible aleph[3]). Finally the method of Gödel gives undecidable sentences expressed in terms of the arithmetic of natural numbers whereas we shall obtain here a sentence from the arithmetic of reals.

[1]) Gödel [4]. Numbers in brackets refer to the bibliography on p. 163.

[2]) Tarski [12].

[2a]) Other such sentences have been constructed by Rosser [10] and Tarski [15]. My method is different from theirs.

[3]) Tarski [14]. Using Tarski's terminology we would have to say that $\aleph_\xi$ is weakly inaccessible.