Considérons enfin comme l'espace X l'espace l^p $(p\geqslant 1)$ composé de toutes les suites réelles $x=\{x_n\}$ telles que $|x_1|^p+|x_2|^p+\ldots<+\infty$ (Banach [1], p. 12). La norme est définie par la formule $\|x\|=(|x_1|^p+|x_2|^p+\ldots)^{1/p}$; cet espace est séparable. L'ensemble Ξ_0 des fonctionnelles linéaires de la forme $\xi(x)=x_n$, où $n=1,2,\ldots$, est fondamental dans l^p. Comme dans les cas précédents on obtient le

Théorème 17. *Soit $f_n(u)$ une suite de fonctions réelles continues telle que $|f_1(u)|^p+|f_2(u)|^p+\ldots<+\infty$ pour tout $u\in[a,b]$. Il existe alors un ensemble résiduel R tel que pour tout $u_0\in R$ on a*

$$\lim_{u\to u_0}\sum_{n=1}^{\infty}|f_n(u)-f_n(u_0)|^p=0.$$

Ouvrages cités.

Banach, S. [1] *Théorie des opérations linéaires.* Monografie Matematyczne, Warszawa (1932).

— [2] *Über analytisch darstellbare Operationen in abstrakten Räumen.* Fund. Math. **17** (1931), p. 283-295.

Bochner, S. [1] *Integration von Funktionen, deren Werte Elemente eines Vektorraumes sind.* Fund. Math. **20** (1933), p. 262-276.

— [2] *Absolut additive abstrakte Mengenfunktionen.* Fund. Math. **21** (1933), p. 211-213.

Gelfand, I. [1] *Abstrakte Funktionen und lineare Operatoren.* Recueil Math. **4** (**46**) (1938), p. 235-286.

Fichtenholz, G. et Kantorovitch, L. [1] *Sur les opérations linéaires dans l'espace des fonctions bornées.* Studia Math. **5** (1934), p. 69—98.

Hahn, H. [1] *Reelle Funktionen.* Erster Teil: *Punktfunktionen.* Leipzig (1932).

Kempisty, S. [1] *Sur les fonctions quasicontinues.* Fund. Math. **19** (1932), p. 184-197.

Kershner, R. [1] *The continuity of functions of many variables.* Trans. Amer. Math. Soc. **53** (1943), p. 83-100.

Mazur, S. und Orlicz, W. [1] *Grundlegende Eigenschaften der Polynomischen Operationen.* Erste Mitteilung. Studia Math. **5** (1934), p. 50-68.

Montel, P. [1] *Sur les suites infinies de fonctions.* Ann. Sc. de l'Ecole Norm. Sup., 3 sér. **24** (1907), p. 233-334.

Montgomery, D. [1] *Non separable metric spaces.* Fund. Math. **25** (1935), p. 527-533.

Pettis, B. J. [1] *On integration in vector spaces.* Trans. Am. Math. Soc. **44** (1938), p. 277-304.

On the principle of dependent choices.

By

Andrzej Mostowski (Warszawa).

Let us consider the following weakened form of the axiom of choice:

(T) $\left\{\begin{array}{l}\textit{if } R \textit{ is a binary relation and } B \textit{ a set } \neq 0 \textit{ and if for every } x\in B\\ \textit{there is a } y\in B \textit{ such that } xRy\textit{, then there is a sequence } x_1,x_2,\ldots,x_n,\ldots\\ \textit{of elements of } B \textit{ such that } x_nRx_{n+1} \textit{ for } n=1,2,\ldots\end{array}\right.$ [1]).

It will be proved here that *the general axiom of choice* (which we shall denote by (Z)) *is independent of* (T), i. e., cannot be proved from (T) and the usual axioms of set-theory.

An independence-proof has sense only with respect to a well defined formal system whose consistency is either proved or assumed as an hypothesis. Our proof applies only to such systems of set-theory as remain self-consistent after adjunction of the following axiom

(N) *there is a non-denumerable set of elements which are not sets.*

It can be shown without difficulty that the system $\mathfrak{S}$ described in one of my former papers [2]) satisfies this condition. Hence we shall take $\mathfrak{S}$ as a basis for our proof.

In order to prove that (Z) is independent of (T) we have to construct in a self-consistent theory $\mathfrak{S}_1$ a model in which all the axioms of $\mathfrak{S}$ as well as the axiom (T) are fulfilled and in which the axiom of choice is false.

---

[1]) This axiom has been considered by A. Tarski in his recent paper *Axiomatic and algebraic aspects of two theorems on sums of cardinals,* this volume, p. 79—104. Tarski calls (T) *the principle of dependent choices.*

[2]) Fundamenta Mathematicae **32** (1939), pp. 201-252.

We shall take as $\mathfrak{S}_1$ the system $\mathfrak{S}$ enriched by the axioms (N) and (Z) but we shall make free use of many notions known from intuitive set-theory without defining them meticulously with the help of primitive notions of $\mathfrak{S}$. Since it is known that the consistency of $\mathfrak{S}$ implies that of $\mathfrak{S}_1$[3]), our result can be stated as follows:

If $\mathfrak{S}$ is self-consistent, then the implication $(T)\to(Z)$ is not provable in $\mathfrak{S}$.

Let A be a non-denumerable set of pairs

$$\{a_s, b_s\} \qquad s \in S$$

whose elements are not sets. Let K_0 be the set of all a_s's and all b_s's:

$$K_0 = \{\ldots, a_s, b_s, \ldots\}$$

and let K_ξ be defined by induction on ξ as follows

$$K_\xi = \sum_{\eta<\xi} K_\eta + \mathfrak{P}\Big(\sum_{\eta<\xi} K_\eta\Big)$$

($\mathfrak{P}(X)$ = set of subsets of X).

If f is a one-to-one mapping of K_0 onto itself and $m \in K_0$, then $f(m)$ is defined as the value of f for the argument m. Suppose that $f(n)$ is already defined for $n \in \sum_{\eta<\xi} K_\eta$ and that $m \in K_\xi - \sum_{\eta<\xi} K_\eta$. We have then $m \subset \sum_{\eta<\xi} K_\eta$ and can define $f(m)$ as $\underset{f(n)}{E}[n \in m]$. Thus $f(m)$ is defined inductively for every element m of any K_ξ.

A mapping f is called *admissible* if for every $s \in S$

$$f(\{a_s, b_s\}) = \{a_s, b_s\}.$$

We shall say that a set $m \in K_\xi$ is *symmetrical* if there is a denumerable set $S_m \subset S$ such that m and S_m satisfy the following condition $\Phi(S_m, m)$:

$$\Phi(S_m, m) \begin{cases} \textit{if } f \textit{ is an admissible mapping and } f(a_s) = a_s \textit{ for } s \in S_m, \\ \textit{then } f(m) = m. \end{cases}$$

An analogous definition applies to classes all of whose elements belong to one of the sets K_ξ.

[3]) This follows from results of K. Gödel. See his book *The Consistency of the Continuum Hypothesis*. Annals of Mathematics Studies, Number 3, Princeton 1940.

Let M_ξ be the set of those $m \in K_\xi$ which are hereditarily symmetrical, i. e., such that from

$$m_k \in m_{k-1} \in \ldots \in m_1 \in m$$

follows that $m, m_1, m_2, \ldots, m_k$ are symmetrical $(k = 1, 2, \ldots)$.

A set m is called *remarkable*[4]) if it belongs to one of the sets M_ξ. A class X is called remarkable if it is symmetrical and all its elements are remarkable.

Now replace in every axiom of $\mathfrak{S}$ the words

individual, *set*, *class*

by

element of K_0, *remarkable set*, *remarkable class.*

It can be shown without difficulty that every axiom of $\mathfrak{S}$ becomes then a provable proposition of $\mathfrak{S}_1$[5]). The axiom of choice becomes a false proposition since the set A is remarkable and there is no remarkable set W which has exactly one element in common with every pair $\{a_s, b_s\}$.

It remains to show that (T) becomes a provable proposition of the system $\mathfrak{S}_1$.

For this purpose let us consider a remarkable set B and a remarkable binary relation R (i. e., a remarkable set of ordered pairs) and suppose that for every $x \in B$ there is a $y \in B$ such that xRy. It follows from the axiom of choice (which is valid in $\mathfrak{S}_1$) that there is a sequence $x_1, x_2, \ldots, x_n, \ldots$ (i. e. a set of ordered pairs $\langle n, x_n\rangle$, $n = 1, 2, \ldots$) such that $x_n R x_{n+1}$ for $n = 1, 2, \ldots$ We shall show that this sequence is remarkable. It is clearly sufficient to show that this sequence is symmetrical.

Since x_n is symmetrical, there is for any integer n a denumerable set $S_n \subset S$ such that $\Phi(S_n, x_n)$. The set $S_0 = \sum_{n=1}^{\infty} S_n$ is a denumerable subset of S and obviously satisfies the condition $\Phi(S_0, x_n)$ for $n = 1, 2, \ldots$ This proves that the sequence $x_1, x_2, \ldots, x_n, \ldots$ is remarkable and our proof is finished.

We can apply the same method to establish another result of this kind:

[4]) Following R. Doss, Journal of Symbolic Logic **10** (1945), pp. 13-15, we use this word as a translation of the German word „ausgezeichnet" which I have used in my paper referred to in the footnote [2]).

[5]) For details see Fundamenta Mathematicae **32**, pp. 220-235.

Let $Z(\mathfrak{m},\mathfrak{n})$ and $Z^*(\mathfrak{m})$ be the following axioms:

$Z(\mathfrak{m},\mathfrak{n})$ *if A is a set with the cardinal number $\mathfrak{m}$ and if every element of A is a non-void set with cardinal number $\leqslant\mathfrak{n}$, then there is a function f such that $f(X)\in X$ for $X\in A$;*

$Z^*(\mathfrak{m})$ *if A is a set with cardinal number $<\mathfrak{m}$ and if every element of A is a non-void set, then there is a function f such that $f(X)\in X$ for $X\in A$.*

Modifying a little the foregoing proof, we can show that if $\mathfrak{m}$ is a cardinal number definable in the system $\mathfrak{S}_0$ due to Bernays [6]), then the implication $Z^*(\mathfrak{m})\rightarrow Z(\mathfrak{m},2)$ is not provable in $\mathfrak{S}$ (provided that $\mathfrak{S}$ is self consistent).

[6]) P. Bernays, Journal of Symbolic Logic **2** (1937), pp. 65-77 and **6** (1941), pp. 1-17.

Ensembles projectifs et ensembles singuliers [1]).

Par

Casimir Kuratowski (Warszawa).

D'après un résultat fondamental de M. Gödel [2]), l'hypothèse du continu est compatible avec le système d'axiomes de la Théorie des Ensembles. Plus encore: on ne parvient à aucune contradiction en admettant l'hypothèse suivante — que l'on peut nommer *hypothèse projective du continu* (ou tout court, *hypothèse H_p*) — *il existe une relation $x\prec y$ qui range l'ensemble de tous les nombres réels de l'intervalle* 01 *en une suite transfinie du type Ω et de façon que l'ensemble plan $\underset{xy}{E}(x\prec y)$ soit projectif* [3]).

L'hypothèse H_p implique aussitôt l'existence d'ensembles projectifs non mesurables au sens de Lebesgue. On constate, en effet, facilement que l'ensemble $\underset{xy}{E}(x\prec y)$ est non mesurable.

Une autre conséquence remarquable de l'hypothèse H_p est l'existence d'ensembles projectifs indénombrables dépourvus de sous-ensembles parfaits (non vides) [4]).

De façon plus générale, on peut établir — comme je montre dans cette note — des énoncés généraux qui permettent de démontrer qu'en admettant l'hypothèse H_p, les constructions utilisées d'habitude pour prouver l'existence de différents ensembles „singu-

[1]) Le manuscrit de cet ouvrage a été rédigé en Décembre 1939. Détruit par le feu en 1944, il fut reconstruit après la guerre et présenté au Congrès des Mathématiciens Polonais à Wrocław le 13 Déc. 1946.

[2]) Voir ***The consistency of the Continuum Hypothesis,*** Annals of Math. Studies, Princeton 1940. Signalé antérieurement dans Proc. Nat. Acad. of Sciences **24** (1938), p. 556 et **25** (1939).

[3]) On appelle *projectifs* les ensembles (situés dans l'espace euclidien à n dimensions) qui se déduisent à partir des ensembles fermés par l'application de deux opérations: projection et passage au complémentaire, effectuées un nombre fini de fois. Pour plus de détails, voir par exemple, ma *Topologie* I, Monogr. Matem. **3**, Warszawa 1933, § 34.

[4]) Cf. K. Gödel, l. c., Proc. Nat. Acad. of Sc. **25**.