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On some problems of Hausdorff and of Sierpiński.

By

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Definitions. The reader may be familiar with most of the following definitions, which are given here for sake of completeness.

In this paper Λ denotes indiscriminately *any finite set of natural numbers*, not only the empty set[1]). Thus „$E=\Lambda$" means „E is a finite (or maybe empty) set of natural numbers". We shall write „*set of n. n.*" for „set of natural numbers" (natural number = positive integer). Two sets (of n. n.) A and B are *almost-disjoint* if $A\cdot B=\Lambda$. $A<B$ (or $B>A$) means: $A\subset B+\Lambda$. Two sets of n. n. differing in a finite number of elements shall be said to be *equivalent.* If A and B differ in an infinity of elements, i. e., if they are *not equivalent*, we write $A \not\sim B$. For example, $E\not\sim\Lambda$ means: E is an infinite set of n. n.

(More generally: given any order relation $<$, the symbol $\not\sim$ means *non-equivalent* with respect to that relation).

A *dyadic* sequence is a sequence of 0's and 1's.

If $S=(s_1,s_2,\dots,s_n,\dots)$ and $T=(t_1,t_2,\dots,t_n,\dots)$ are two sequences of numbers (e. g., dyadic sequences),

$S\lesssim T$ means: $s_n\leqslant t_n$ for almost all n (i. e., for all $n>n_0$).

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[1]) Cf., Ann. Math. **45** (1944), p. 397. The slightly changed definition enables us to discard the symbol $\sim$. The other symbol, $\not\sim$, is indispensible, but the use of both together is confusing.

A partially ordered set F with, say, the order-relation $<$, is said to have a *gap of type* (Ω,ω^*) if F contains an ordered subset:

$$a^1 < a^2 < \ldots < a^\omega < \ldots a^\alpha < \ldots;\ \ldots < b^n < \ldots b^2 < b^1,$$

(where the a's form an Ω-sequence, and the b's an inversely ordered ω-sequence), such that no $c \in F$ exists which would satisfy the condition that $a^\alpha < c < b^\omega$ for all $\alpha < \Omega$ and $n < \omega$.

Similarly, we define (ω,ω^*)-*gaps*, (Ω,Ω^*)-*gaps*, etc.

F is said to contain an Ω-*limit* if F contains a transfinite sequence

$$a^1 < a^2 < \ldots a^\omega < \ldots a^\alpha < \ldots < a^\Omega, \quad (a^\alpha \not\sim a^\Omega \text{ for } \alpha < \Omega),$$

such that the condition:

$$b \not\sim a^\Omega \quad \text{and} \quad a^\alpha < b < a^\Omega \quad \text{for all } \alpha < \Omega$$

is never satisfied, for any $b \in F$.

In this case a^Ω is the Ω-*limit* of the transfinite sequence of the a^α's.

(Example: The set of all rational numbers, in order of magnitude, has (ω,ω^*)-gaps and ω-limits; the set of all real numbers has no gaps).

A *linear set* is a set of real numbers.

Introduction and Summary. We shall consider the following three partially ordered sets:

1⁰ the family F_1 of all sets of n. n., with the order-relation $<$ (or $>$),

2⁰ the family F_2 of all dyadic sequences, with the relation $\lesssim$,

3⁰ the family F_3 of all sequences of n. n., with the relation $\lesssim$; they are equivalent with respect to gap problems, and also limit problems. More precisely:

Lemma. *Any gap existence theorem or limit existence theorem for the family of all dyadic sequences holds also for the family of all sets of n. n., and vice versa.*

Any gap existence theorem or limit existence theorem for the family of dyadic sequences holds also for the family of all sequences of n. n., and vice versa.

The first part of this lemma is obvious, from the fact that the two families F_1 and F_2 are isomorphic. (Since any dyadic sequence is the „characteristic function" of a set of n. n.).

The second part is less obvious, but has been proved elsewhere [2]).

Without the continuum hypothesis, merely using the axiom of choice, F. Hausdorff [3]) has shown that

The family of all dyadic sequences contains (Ω,Ω^*)-gaps, but it does not have any (ω,ω^*)-gaps, nor any ω-limits.

Whether it has any (Ω,ω^*)-gaps, or Ω-limits, is not known. We shall show (cf. chap. II) that the existence of (Ω,ω^*)-gaps would imply the existence of Ω-limits. Also, the hypothesis that $2^{\aleph_0} < 2^{\aleph_1}$ (which seems weaker than the continuum hypothesis) implies the existence of Ω-limits. (Chap. III).

Other problems, connected with the Ω-limit problem, are the following:

(1) Does a non-denumerable linear set exists, *every* subset of which is a relative F_σ? (Sierpiński and Hausdorff) [4]). Such a set shall be called a *Q-set*.

(2) Does every family F of real functions of power of the continuum necessarily have a denumerable base? (Sierpiński) [5]) i. e., given F, does a sequence $\varphi_1(x),\varphi_2(x),\ldots,\varphi_n(x),\ldots$ exist, such that every function belonging to F is the limit-function of some sub-sequence of $\{\varphi_n(x)\}$?

(3) Let F be the family of *all* real functions defined on a (linear) set X of power $\aleph_1$; does F have a denumerable base?

The answer to (1) and (3) is of course negative unless Lusin's „*second continuum hypothesis*": $2^{\aleph_0} = 2^{\aleph_1}$ is assumed [6]). We shall show that problems (1) and (3) are equivalent, and that the negative answer would imply the existence of Ω-limits. (Even if we assume Lusin's hypothesis, these problems are still unsolved). The positive answer to (4) or (5), below, also implies the existence of Ω-limits.

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[2]) F. Rothberger, Proc. Cam. Phil. Soc., **37** (1941), p. 122 and 126 (only proved for gaps, but it obviously also goes for limits).

[3]) F. Hausdorff, *Summen von $\aleph_1$ Mengen*, Fund. Math. **26** (1936), p. 243-247.

[4]) W. Sierpiński, Fund. Math. **30** (1938), p. 1, proposition P_4. Cf., also F. Hausdorff, Fund. Math. **20** (1933), p. 286, problème 58.

[5]) Fund. Math. **27** (1936), p. 293, problème de M. Sierpiński, also: W. Sierpiński, Pont. Acad. Sci. Acta **4** (1940), p. 211, F. Rothberger, Ann. Math. **45** (1944), p. 397-406.

[6]) N. Lusin, Fund. Math. **25** (1935), p. 130.

(4) Is there a linear set of power $\aleph_1$ and 2nd category? [7])

(5) Is there a linear non-measurable (Lebesgue) set of power $\aleph_1$? i. e., is there a set of power $\aleph_1$ which is not of Lebesgue measure 0?

If Lebesgue-measure is replaced by Banach-measure, the problem is solved:

(Tarski and Sierpiński) [8]). *Every set whose power is less than the continuum, is of Banach measure* 0.

The above problems are dealt with in chapter III. The existence problem of Ω-limits remains open.

Throughout this paper, needless to say, the continuum hypothesis is never assumed, but the multiplicative axiom always is.

The three chapters and appendix that follow are independent of each other and can be read in any order, with the one exception that a theorem of chap. II is used in chap. III.

Chapter I. Ω-limits in a stronger sense.

Apart from the definition of an Ω-limit given in the introduction (and treated in chapters II and III), Hausdorff introduces a second kind of Ω-limit.

Hausdorff [9]) writes: „Es ist nicht bekannt, ob von zwei geordneten Mengen [dyadischer Folgen] $A<B$, zwischen die sich kein x [i. e., keine Folge x] einschalten lässt, die eine von der Mächtigkeit $\aleph_1$, die andere höchstens abzählbar sein kann, z. B. ob $a^0<a^1<a^2<\dots a^\omega<\dots\ a^\xi<\dots<b$ und $b=\lim a^\xi$ in dem Sinne sein kann, dass kein x mit $A<x<b$ $(x \nsim b)$ existiert, oder gar in dem schärferen Sinne, dass zu jedem $x<b$ $(x\nsim b)$ ein $a^\xi>x$ vorhanden ist". (The brackets [ ] are my own. Hausdorff writes $<$ where Sierpiński writes $\lesssim$; we have adopted the latter notation).

This second definition, „im schärferen Sinne", is to be studied in the present chapter. Taking sets of n. n. instead of dyadic sequences, and replacing the relation $\lesssim$ (or $<$) by $>$ (for later convenience), the problem may be stated as follows:

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[7]) Cf., F. Rothberger, Fund. Math. **30** (1938), p. 215, and Proc. Cam. Phil. Soc., loc. cit., p. 112.

[8]) A. Tarski, Fund. Math. **30**, p. 226, Korollar 2.22.

[9]) Fund. Math. **26** (1936), p. 247.

Does there exist a transfinite sequence of sets of n. n.

$$E^1>E^2>\dots E^\omega>\dots E^\alpha>\dots E^\Omega \qquad (E^\alpha \nsim E^\beta) \tag{1.1}$$

such that $A>E^\Omega$ $(A\nsim E^\Omega)$ always implies: $A>E^\alpha$ for some $\alpha<\Omega$?

The answer is negative. (Without loss of generality, we may assume $E^\Omega=\Lambda$; put $A=CE^2$ i. e., the complement of E^2. Then we have $A>\Lambda$ $(A\nsim\Lambda)$, but certainly not $A>E^\alpha$, for any α). Thus there seems to be a misprint, or an oversight, in Hausdorff's definition. However, if we drop the order relation in (1.1) and replace it simply by $E^\alpha>E^\Omega$ (for any $\alpha<\Omega$), the definition is all right. We have the following theorem:

Theorem 1. *The continuum hypothesis is equivalent to the following proposition: There exists a set of power $\aleph_1$ of sets of n. n.:*

$$E^1,E^2,\dots,E^\omega,\dots,E^\alpha,\dots \qquad (E^\alpha\nsim\Lambda \text{ for all } \alpha<\Omega)$$

such that for any $A\nsim\Lambda$, we have $A>E^\alpha$ for some $\alpha<\Omega$.

This theorem is an immediate consequence of the following

Theorem 2. *If a family F of (infinite) sets of n. n. has the property that for any $A\nsim\Lambda$ there exists a set $E\in F$ such that $A>E$, then $\overline{\overline{F}}=2^{\aleph_0}$.*

Proof of theorem 2. It is obvious that $\overline{\overline{F}}\leqslant 2^{\aleph_0}$. In order to show that also $2^{\aleph_0}\leqslant\overline{\overline{F}}$, we require the following well known

Lemma. *A denumerable set (e. g., of n. n.) contains $2^{\aleph_0}$ almost-disjoint infinite subsets.*

(For proof, take the set of all rational numbers; for any real number x, let A_x be a set of rational numbers with x as only limit point: obviously, A_x and A_y are almost-disjoint if $x\neq y$, q. e. d.).

Now let Φ be a family of $2^{\aleph_0}$ almost-disjoint infinite sets of n. n., and let F be a family satisfying the hypothesis of the theorem.

Then, if $A,B\in\Phi$, we have $A>E$, $B>E'$ for certain sets $E,E'\in F$. Besides, if $A\neq B$, we have [10]) $AB=\Lambda$, hence $EE'=\Lambda$, and therefore $E\neq E'$. Thus different elements of Φ correspond to different elements of F, therefore $\overline{\overline{\Phi}}\leqslant\overline{\overline{F}}$ and hence $2^{\aleph_0}\leqslant\overline{\overline{F}}$, q. e. d.

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[10]) This means „AB is a finite set", cf. Introduction.

Chapter II. (Ω, ω^*)-gaps and Ω-limits.

The object of this chapter is to prove the following

Theorem 3. *If the family of all dyadic sequences contains any (Ω, ω^*)-gaps it also contains Ω-limits.*

As usual, „dyadic sequences" may here be replaced by „sets of n. n.", etc. (cf., lemma in the Introduction).

Definition. A set $\boldsymbol{S}$ of sequences of natural numbers is said to be *non-bounded* if no sequence T exists, such that $S \prec T$ for all $S \in \boldsymbol{S}$.

We have the following theorem proved in an earlier paper[11]):

The existence of (Ω, ω^*)-gaps (viz., for dyadic sequences, as above) *is equivalent to the following*:

Proposition non-$\boldsymbol{B}(\aleph_1)$. There exists a set $\boldsymbol{S}$ of sequences of n. n. which is well-ordered of type Ω (rel. $\prec$) and non-bounded.

Hence our theorem 3 is equivalent to the following

Theorem 3^a. *The existence of a non-bounded Ω-sequence of sequences of n. n. implies the existence of an Ω-limit* (for dyadic sequences, or for sets of n. n.).

The idea of our proof of this theorem is, roughly speaking, a projection:

A *strictly increasing sequence* of n. n. $(s_1 < s_2 < \dots s_n < \dots)$ can be represented in the cartesian XY-plane as a set of points with the coordinates: $(1, s_1,)$ $(2, s_2)\dots$ This set is then projected onto the Y-axis, giving there the (infinite) set of points $s_1, s_2, \dots, s_n, \dots$ We are going to construct an unbounded set of sequences which, if thus projected, gives a set of sets with Λ as its Ω-limit.

For this purpose, we need a convenient notation and two lemmas:

Notation. We shall denote strictly increasing sequences by $\{h(n)\}$, $\{k^\alpha(n)\}$, $\{f(n)\}$, etc. instead of $\{h_n\}$, $\{k^\alpha_n\}$, $\{f_n\}$, etc., and the corresponding „projected" sets shall be denoted by the corresponding italic capitals: H, K^α, F, etc. More precisely:

If the sequence $\{h(n)\}$ is given, H is the set whose elements are: $h(1), h(2), \dots h(n), \dots$ Or, conversely, if H is given, $\{h(n)\}$ is the sequence consisting of the elements of H in natural order. (Note that this correspondence works both ways).

[11]) F. Rothberger, Proc. Cam. Phil. Soc. loc. cit., théorème 6, p. 121 and théorème 2, p. 113.

Lemma 1. *Given a strictly increasing sequence $\{h(n)\}$ and a denumerable set of sets (of n. n.)*

$$K^1 > K^2 > \dots K^\omega > \dots K^\alpha > \dots \qquad (K^\alpha \neq \Lambda),\ \alpha < \gamma,$$

there exists a set $K \neq \Lambda$ such that $K < K^\alpha$ (any $\alpha < \gamma$) and $\{h(n)\} \prec \{k(n)\}$.

Proof. Owing to a theorem of Hausdorff (non-existence of ω-limits)[12]), there exists a set $F < K^\alpha$ $(\alpha < \gamma)$, $(F \neq \Lambda)$. Then, for any value of m, we have $f(m) \in F$; in particular, if $m = h(n)$, we get $f(h(n)) \in F$. Putting $f(h(n)) = k(n)$, it follows that $k(n) \in F$, hence $K < F$.

Also, since $\{f(m)\}$ is strictly increasing, we have $m \leqslant f(m)$, hence $h(n) \leqslant f(h(n)) = k(n)$ for all n, and therefore $\{h(n)\} \prec \{k(n)\}$, q. e. d.

Lemma 2. *If $K > H \neq \Lambda$, then $\{k(n)\} \prec \{h(2n)\}$.*

Proof. To begin with, it is easily seen that $H \subset K$ implies $k(n) \leqslant h(n)$ for all n, hence $\{k(n)\} \prec \{h(n)\}$.

Now, if merely $H < K$, there exists a finite set Λ_0 such that $H - \Lambda_0 \subset K$. We may assume that Λ_0 consists of the first n_0 elements of H. The set $H - \Lambda$ corresponds to the sequence $\{h(n+n_0)\}$, (i. e., $h(n_0+1), h(n_0+2), \dots$) and it follows, just as above, that

$$\{k(n)\} \prec \{h(n+n_0)\}. \tag{2.1}$$

Besides, since $n + n_0 \leqslant 2n$ for all $n > n_0$, we also have

$$h(n+n_0) \leqslant h(2n) \text{ for all } n > n_0, \tag{2.2}$$

and therefore $\{k(n)\} \prec \{h(2n)\}$, q. e. d.

Proof of theorem 3^a. We assume the existence of a non-bounded Ω-sequence of sequences (of. n. n.), say

$$\{h^1(n)\} \prec \{h^2(n)\} \prec \dots \{h^\omega(n)\} \prec \dots \{h^\alpha(n)\} \prec \dots \qquad (\alpha < \Omega). \tag{2.3}$$

We assume further, without loss of generality, that (2.3) consists of strictly increasing sequences.

We shall construct an Ω-sequence of sets (of n. n.) K^α $(\alpha < \Omega)$ such that:

$$\{h^\alpha(n)\} \prec \{k^\alpha(n)\} \quad \text{and} \quad K^\alpha > K^\beta \quad \text{for } \alpha < \beta, \tag{2.4}$$

$$\lim_{\alpha \to \Omega} K^\alpha = \Lambda. \tag{2.5}$$

First, we put $K^1 = H^1$; then, supposing the K^{α}'s already defined for all $\alpha < \gamma$ in accordance with (2.4), we define K^{γ} by applying lemma 1 with $h(n) = h^{\gamma}(n)$ and $K = K^{\gamma}$. Thus K^{γ} also satisfies (2.4), and by transfinite induction the same holds for all K^{α}, $\alpha < \Omega$. It remains to show that (2.5) holds.

Now, if (2.5) were false, there would exist a set $K^{\Omega} \dotplus \Lambda$ with $K^{\Omega} < K^{\alpha}$ (for all $\alpha < \Omega$); but, by lemma 2, we would have $\{k^{\alpha}(n)\} \precsim \{k^{\Omega}(2n)\}$, hence, by (2.4), $\{h^{\alpha}(n)\} \precsim \{k^{\Omega}(2n)\}$, against the hypothesis that (2.3) is non-bounded, q. e. d.

Chapter III. Further theorems concerning the existence of Ω-limits.

The object of this chapter is to prove the existence of Ω-limits under the assumption of some or other hypothesis. The strongest of these theorems is theorem 4, below, from which the other ones (theorems 7 a, 7 b, 7 c) follow without too much difficulty.

Definition. A linear set E will be called a Q-set if every subset whatsoever of E is a relative F_{σ}.

Theorem 4. *If there are no Ω-limits in the family of all dyadic sequences, then every linear set of power $\aleph_1$ is a Q-set.*

In other words: the existence of Ω-limits would be established if we could prove the existence of at least one set of power $\aleph_1$ which is not a Q-set.

We require a few lemmas.

Definition. A *finite product* is the product of a finite number of terms (not necessarily a finite set itself).

Lemma 3. *If $X_1, X_2, \ldots, X_n, \ldots$ is a finite or infinite sequence of sets (of n. n.) such that every finite product*

$$\prod_{\nu=1}^{m} X_{n_\nu} \dotplus \Lambda,$$

then there exists a set $A \dotplus \Lambda$ such that $A < X_n$ for any n.

Lemma 4. *(Cf. Hausdorff's „Erster Einschaltungssatz")* [12]. *Given two sequences of sets, $X_1, X_2, \ldots$, and $Y_1, Y_2, \ldots$, such that $X_i < Y_k$ for any i, k, there exists a set A such that $X_i < A < Y_k$ for any i, k.*

These two lemmas are identical respectively to lemmas 2 and 3[b] of my paper „On families of real functions" [1]). We omit the proof here.

[12]) F. Hausdorff, l. c., Fund. Math. **26**, p. 244 (Erster Einschaltungssatz).

Lemma 5. *The following two propositions are equivalent:*

(**1**) *The family of all sets of n. n. does not contain any (Ω, ω^*)-gaps.*

(**2**) *Given a set of $\aleph_1$ sets $X_1, X_2, \ldots, X_{\omega}, \ldots, X_{\alpha}, \ldots$ $(\alpha < \Omega)$, and a set of $\aleph_0$ sets $Y_1, Y_2, \ldots, Y_n, \ldots$ $(n < \omega)$ such that $X_{\alpha} < Y_n$ for any $\alpha < \Omega$ and $n < \omega$, there always exists a set E such that $X_{\alpha} < E < Y_n$ (for any α, n).*

Proof. The implication (**2**)$\to$(**1**) follows immediately from the definition of a gap.

Now, assuming (**1**), we shall prove (**2**) for given sets $X_{\alpha} < Y_n$. First we show the existence of a transfinite sequence of sets A_{α} satisfying the conditions:

(3.1) $\quad X_{\alpha} < A_{\beta} < Y_n$ and $A_{\alpha} < A_{\beta}$ for all $\alpha \leqslant \beta < \Omega$ and all n.

Suppose the A_{β}'s already defined for $\beta < \beta_0$ and satisfying (3.1); they form a denumerable set of sets (β_0 being fixed), hence by lemma 4 there exists a set A_{β_0} such that $A_{\alpha} < A_{\beta_0}$, $X_{\alpha} < A_{\beta_0}$, and $A_{\beta_0} < Y_n$, for all $\alpha \leqslant \beta_0$, i. e., satisfying (3.1) for $\alpha < \beta_0$. Thus, since there is no difficulty in defining A_1, all A_{α}'s are defined, by transfinite induction.

Next, putting $B_n = \prod_{\nu \leqslant n} Y_{\nu}$, we have

(3.2) $\quad A_1 < A_2 < \ldots A_{\omega} < \ldots A_{\alpha} \ldots \ldots < B_n < \ldots < B_2 < B_1.$

Since (3.2) is ordered of type $\Omega + \omega^*$, there exists, because of (**1**), a set E such that $A_{\alpha} < E < B_n$. But $X_{\alpha} < A_{\alpha}$ (by (3.1)) and $B_n < Y_n$, hence $X_{\alpha} < E < Y_n$. Thus (**1**)$\to$(**2**), q. e. d.

Note. The relation $<$ does not necessarily imply $\dotplus$ and if (3.2) contains less than $\aleph_1$ non-equivalent A_{α}'s, or less than $\aleph_0$ non-equivalent B_n's, (3.2) is not strictly speaking of type $\Omega + \omega^*$. But there is no difficulty here; for example, if $B_{n_0} = B_{n_0+1} = B_{n_0+2} = \ldots$, it is sufficient to put $E = B_{n_0}$.

Lemma 6. *Of the following two propositions,* (**1**) *implies* (**3**).

(**1**). *There are no (Ω, ω^*)-gaps in the family of all sets of n. n.*

(**3**). *Given a set of $\aleph_1$ sets $Z_1, Z_2, \ldots, Z_{\omega}, \ldots, Z_{\alpha}, \ldots$ $(\alpha < \Omega)$ such that every finite product*

$$\prod Z_{\alpha_\nu} \dotplus \Lambda,$$

there exists a transfinite sequence of sets B_{α} satisfying the condition:

(3.3) $$Z_{\alpha} > B_{\beta}, \quad B_{\alpha} > B_{\beta} \dotplus \Lambda \quad \textit{for} \quad \alpha < \beta < \Omega.$$

Proof. For any given $\gamma<\Omega$ the set of all Z_α's with $\alpha<\gamma$ is atmost denumerable. Therefore, by lemma 3, there exists a set A_γ such that

(3.4) $$Z_\alpha > A_\gamma \quad \text{for any} \quad \alpha<\gamma<\Omega.$$

In this way we define A_γ for every $\gamma<\Omega$. Next, put $B_1=Z_1$, and suppose the B_β's already defined for $\beta<\beta_0$ satisfying the relations

(3.5) $$Z_\alpha > B_\beta > A_\gamma \quad \text{and} \quad B_\alpha > B_\beta \quad \text{for} \quad \alpha\leqslant\beta<\beta_0 \quad \text{and} \quad \beta_0\leqslant\gamma<\Omega.$$

Now, for fixed β_0 the B_β's satisfying (3.5) form a denumerable set, whereas the set of the A_γ's $(\gamma\geqslant\beta_0)$ is of power $\aleph_1$.

Therefore, by lemma 5, there exists a set E such that

(3.6) $$B_\beta > E > A_\gamma \quad \text{for} \quad \beta<\beta_0\leqslant\gamma<\Omega.$$

Putting $B_{\beta_0}=EZ_{\beta_0}$, we have $Z_{\beta_0}>B_{\beta_0}$ and, by (3.6), $B_\beta>B_{\beta_0}$ for $\beta<\beta_0$. Besides, by (3.4) and (3.6), we have $B_{\beta_0}>A_\gamma$ for $\beta_0<\gamma$. These last inequalities together with (3.5), show that

(3.7) $$Z_\alpha > B_\beta > A_\gamma \quad \text{and} \quad B_\alpha > B_\beta \quad \text{for} \quad \alpha\leqslant\beta<\beta_0+1\leqslant\gamma;$$

thus we find by induction that (3.5) holds for any $\beta_0<\Omega$ and this implies (3.3), q. e. d.

Lemma 7. *The following two propositions are equivalent:*

(**4**) *There are no Ω-limits in the family of all sets of n. n.*

(**5**) *If $Z_1, Z_2, \ldots, Z_\omega, \ldots, Z_\alpha, \ldots$, are $\aleph_1$ sets of n. n. such that every finite product*

$$\prod_{\nu=1}^{n} Z_{\alpha_\nu} \not\sim \Lambda,$$

then there exists a set $D\not\sim\Lambda$ such that $D<Z_\alpha$ for all $\alpha<\Omega$.

(Proposition (**5**), if true, would be a generalization of lemma 3).

Proof. If an Ω-limit exists then there exists a transfinite sequence of sets Z_α with $Z_\alpha>Z_\beta$ for $\alpha<\beta$ and $\lim\limits_{\alpha\to\Omega} Z_\alpha=\Lambda$. Thus every finite product of Z's is $\not\sim\Lambda$, but there can be no $D\not\sim\Lambda$ with $D<Z_\alpha$ for all α, in contradiction to (**5**). Hence *non* (**4**) $\to$ *non* (**5**).

On the other hand, if proposition (**4**) holds there are no (Ω,ω^*)-gaps, (by theorem 3), hence, by lemma 6, there exists a (decreasing) transfinite sequence of sets B_α satisfying (3.3). Then, because of the assumed non-existence of Ω-limits, there exists a set $D\not\sim\Lambda$ such that $D<B_{\alpha+1}<Z_\alpha$ for all α, in accordance with (**5**). Therefore (**4**) $\to$ (**5**), q. e. d.

Definitions. A *denumerable base* for a family F of functions is a sequence of functions such that every function belonging to F is the limit of some subsequence of that sequence.

We shall denote by Φ the (denumerable) set of all functions $\varphi(x)$ satisfying the following conditions:

1^0 $\varphi(x)$ is continuous for all real x excepting possibly a finite number of rational points,

2^0 $\varphi(x)$ takes the values 0 and 1 only.

If the domain of x is restricted to a set E, we shall denote by $\Phi(E)$ the set of all $\varphi(x)$ (as above) restricted to $x\in E$.

Lemma 8. *The proposition* (**5**) (see lemma 7) *implies the following proposition*:

(**6**) *If E is a linear set of power $\aleph_1$ then $\Phi(E)$ is a denumerable base for the family of all dyadic functions on E. (A dyadic function is a function which only takes the values* 0 *and* 1).

(In other words: Every dyadic function on E is the limit of some sequence of functions belonging to $\Phi(E)$).

Proof. Let $x_1, x_2, \ldots, x_\omega, \ldots, x_\alpha, \ldots$ $(\alpha<\Omega)$ be the elements of E, and let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ be the elements of $\Phi(E)$. Given any dyadic function $f(x)$ on E, let Z_α be the set of all n such that $\varphi_n(x_\alpha)=f(x_\alpha)$.

Now, we have $Z_\alpha\not\sim\Lambda$ and even every finite product

$$\prod_{\nu=1}^{n} Z_{\alpha_\nu} \not\sim \Lambda,$$

because $\Phi(E)$ contains infinitely many functions which have any assigned value (0 or 1) at any given point, or even at any given finite set of points $x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}$.

Thus the sets Z_α satisfy the conditions in proposition (**5**), lemma 7. Therefore, if (**5**) holds, there exists an infinite set $D<Z_\alpha$ (for all α); let $n_1, n_2, \ldots, n_k, \ldots$ be the elements of D. We shall show that:

(3.8) $$\lim \varphi_{n_k}(x_\alpha)=f(x_\alpha) \quad \text{for all} \quad \alpha<\Omega.$$

The point x_α being arbitrarily fixed, we have, by definition,

$$\varphi_n(x_\alpha)=f(x_\alpha) \quad \text{for every} \quad n\in Z_\alpha.$$

But since $D<Z_\alpha$, almost all n_k's are elements of Z_α, hence $\varphi_{n_k}(x_\alpha)=f(x_\alpha)$ for almost all k, and this is the same as (3.8).

From (3.8), proposition (**6**) follows immediately. Thus (**5**)→(**6**), q. e. d.

Lemma 9. *The proposition* (**6**) *is equivalent to the following proposition*:

(**7**) *Every linear set of power* $\aleph_1$ *is a Q-set* (i. e., a set every subset of which is a relative F_σ).

Proof. Without loss of generality, we assume that the linear set E of power $\aleph_1$ contains no rational points. Then $\Phi(E)$ is a family of continuous functions on E, and it follows from proposition (**6**) that every dyadic function $f(x)$ on E is a function of class 1 of Baire (rel. E). Therefore the set of the 0's (or of the 1's) of $f(x)$ is an F_σ (rel. E). Hence every subset of E is an F_σ (rel. E). Thus (**6**)→(**7**).

To prove that (**7**)→(**6**), let $f(x)$ be any dyadic function on the Q-set E. Then there exist two sets F_1 and F_2, both of class F_σ and satisfying the following conditions:

$$F_1F_2=0, \quad E\subset F_1+F_2, \quad f(x)=\begin{cases}0 & \text{on } EF_1\\ 1 & \text{on } EF_2.\end{cases}$$

Now it is easily seen that a function which is $=0$ on one F_σ and $=1$ on another (disjoint) F_σ can be approximated by a sequence of functions out of Φ. Hence the arbitrary function $f(x)$ can be thus approximated, and this proves (**6**), on the assumption of (**7**), q. e. d.

Proof of theorem 4. Combining the lemmas 7, 8, 9, we get the following chain of implications: (**4**)→(**5**)→(**6**)→(**7**), hence the non-existence of Ω-limits implies that every $\aleph_1$-set is a Q-set, q. e. d.

We now require some theorems on Q-sets.

Theorem 5^a^. *(Sierpiński and Hausdorff)*[4]. *The existence of a Q-set E of power* $\aleph_1$ *implies that* $2^{\aleph_0}=2^{\aleph_1}$ *(i. e., Lusin's Second continuum hypothesis).*

(Because E has $2^{\aleph_1}$ different subsets; each subset of E being separated from its complement by an F_σ, there would have to be at least $2^{\aleph_1}$ different F_σ's, hence $2^{\aleph_0}=2^{\aleph_1}$).

Theorem 5^b^. *Every Q-set of power* $\aleph_1$ *is of Lebesgue measure* 0.

Proof. Let E be the set in question. We have to show that it is of measure 0. Let $E_1, E_2, \dots, E_n, \dots$ be a sequence of mutually disjoint bounded subsets of E. Since each set E_n is contained in an F_σ which has no points in common with $E-E_n$, there exist Borel sets $B_1, B_2, \dots, B_n, \dots$ satisfying the following conditions:

$$(3.9)\quad \begin{cases} B_i\cdot B_j=0, \quad (\text{for any } i,j,\ i\neq j), \qquad E_n\subset B_n \quad (\text{for any } n)\\ m_e(E_n)=m(B_n), \qquad m_e\Big(\sum_{n=1}^{\infty}E_n\Big)=m\Big(\sum_{n=1}^{\infty}B_n\Big),\end{cases}$$

where $m(\dots)$ is the Lebesgue measure, and $m_e(\dots)$ is the outer measure.

The existence of these B_n satisfying (3.9) is easily verified.

It follows from (3.9) that the set-function m_e, if restricted to E (i. e., to the domain consisting of E and its subsets), is an absolutely additive set-function defined for all bounded subsets of E.

But, by Ulam's theorem[13]), every absolutely additive set-function defined for all subsets of a set of power $\aleph_1$ must vanish identically. Hence $m_e(E)=0$, q. e. d.

Remark. Since Ulam's theorem holds for any power smaller than the first inaccessible cardinal, it follows that theorem 5^b^ still holds if $\aleph_1$ is replaced by any such power.

Theorem 5^c^. *Every Q-set (of any power) is of first category.*

Proof. Let F be the closure (fermeture) of E, and let $D\subset E$ be a denumerable set dense in E. Since every subset of E is a relative F_σ, the set $E-D$ is contained in an F_σ disjoint to D. We may assume this F_σ to be contained in F; then it is necessarily of first category, since its complement is dense. Hence $E-D$ is likewise of first category, and therefore also E, q. e. d.

In this order of ideas, we may mention the following theorem:

Theorem 6. *The following two propositions are equivalent*:

(**8**) *There exists a Q-set of power* $\aleph_1$.

(**9**) *On a set E of power* $\aleph_1$ *there exists a denumerable base for the family of all real functions defined on E.*

Since we do not require this theorem for our immediate purpose, we shall postpone the proof.

From theorem 4 and theorems 5^a^, 5^b^, 5^c^, we have immediately the following theorems 7^a^, 7^b^, 7^c^. Theorem 7^d^ is identical with theorem 3.

[13]) Cf., W. Sierpiński, *Hypothèse du continu* (Monografie Matematyczne 4, 1934), p. 159, or S. Ulam, Fund. Math. **16** (1930), p. 142-143.

***Theorem** 7^a, 7^b, 7^c, 7^d. There exists an Ω-limit in the family of all dyadic sequences (or sets of n. n., or sequences of n. n.), provided at least one of the following propositions is true:*

(a) $2^{\aleph_0} < 2^{\aleph_1}$ *(Negation of Lusin's hypothesis).*

(b) *There exists a linear set of power $\aleph_1$ and positive outer measure.*

(c) *There exists a linear set of second category and power $\aleph_1$.*

(d) *The family of all dyadic sequences contains (Ω, ω^*)-gaps.*

Proof of theorem 6. I have shown in an earlier paper[14]) that we may restrict ourselves, without loss of generality, to dyadic functions in (**9**), instead of real functions generally. (A dyadic function takes the values *0* and *1* only).

Assuming (**9**), let $\psi_1(x), \psi_2(x), \dots, \psi_n(x), \dots$ be a base for the family of all dyadic functions on a certain set X of power $\aleph_1$. We transform the set X into the set $Y = T(X)$, by means of the function

$$y = T(x) = \sum_{n=1}^{\infty} \frac{\psi_n(x)}{2^n}. \tag{3.10}$$

Now, the set Y contains no „dyadically rational" numbers: in fact, for any given $x \in X$ the sequence $\psi_n(x)$ contains an infinity both of *0*'s and *1*'s; because, being a base for all dyadic functions, it contains two subsequences, one tending to *0*, the other one tending to *1*.

From this it follows that X and Y are in one-to-one correspondence, and it is verified without difficulty that the functions

$$\varphi_n(y) = \psi_n(x), \quad \text{where} \quad y = T(x), \tag{3.11}$$

form a base for the dyadic functions on Y, and that they (viz., 3.11)) belong to the class $\Phi(Y)$. (Cf., definition, p. 39). Hence, (cf., proof of lemma 9) the set Y is a Q-set of power $\aleph_1$, therefore (**9**)$\to$(**8**). In order to prove that (**8**)$\to$(**9**), it is sufficient to apply literally the second part of the proof of lemma 9, q. e. d.

Problems. Apart from the problems mentioned in the introduction, the following minor questions might be raised:

Are the implications in theorems 3 and 4 resp., reversible? to prove the inverse of theorem 4, it would be sufficient to prove the inverse of lemma 8, i. e., to prove the implication (**6**)$\to$(**5**); then we would have (**4**) $\equiv$ (**5**) $\equiv$ (**6**) $\equiv$ (**7**), etc.

[14]) Ann. Math. **45** (1944), p. 398, theorem 1.

Many propositions are known that *imply* Lusin's „second continuum hypothesis" but very little is known about *consequences* of Lusin's hypothesis[15]).

Appendix. Direct proofs of theorems 7^a and 7^b.

We give here two direct proofs, that is, independent of chapters II and III; all that is necessary is Lemma 3. The two proofs are independent of each other.

Theorem 7^a may be stated as follows:

If there are no Ω-limits in the family of all sets of n. n., then $2^{\aleph_0} = 2^{\aleph_1}$.

Definition. We shall say the sequence $A_1, A_2, \dots$ *contains the infinite subset B*, if $A_n > B \dotplus A$ for all n. (Similarly for transfinite sequences A_α).

Proof. The proof consists, roughly speaking, in taking the set of all n. n., and splitting it in two, over and over again, altogether Ω times, so that we finally arrive at $2^{\aleph_1}$ different sets of n. n. More precisely:

Let E_0 be the set of all n. n. We proceed by transfinite induction.

First step: Subdivide E_0 into two infinite subsets E_1 and E_1'.

Second step: Subdivide each of the two above sets into two infinite subsets; thus we have four disjoint sets: E_2, E_2', E_2'', E_2'''. (In particular, let $E_2 < E_1$). We shall denote this quadruple of sets by „E_2, *etc.*".

n^{th} step: Subdivide each set of the $(n-1)^{\text{st}}$ step into two infinite subsets. This gives 2^n disjoints sets, to be denoted by E_n, *etc.* In particular, let $E_n < E_{n-1}$.

ω^{th} step: We have arrived at $2^{\aleph_0}$ decreasing sequences of sets, e. g., $E_1, E_2, \dots, E_n, \dots$ By lemma 3, each one of these sequences contains an infinite subset, e. g., E_ω. This gives $2^{\aleph_0}$ almost-disjoint sets E_ω, etc.

α^{th} step (α non-limit number): Subdivide each set $E_{\alpha-1}$, etc., defined in the $(\alpha-1)^{\text{st}}$ step, into two infinite subsets E_α, etc. In particular: $E_\alpha < E_{\alpha-1}$.

[15]) N. Lusin, loc. cit., W. Sierpiński, Fund. Math. **25** (1935), p. 132.

α^{th} step (α limit number): We have $2^{\aleph_0}$ decreasing α-sequences, e. g. $E_1, E_2, \dots, E_\omega, \dots, E_\beta, \dots$ ($\beta < \alpha$), each of which contains (by lemma 3) an infinite subset, e. g., E_α; these $2^{\aleph_0}$ almost-disjoint sets shall be denoted by E_α, etc. In particular, let $E_\alpha < E_\beta$ (for $\beta < \omega$). (The sets covered by the „etc." may remain nameless).

This goes, by transfinite induction, for all $\alpha < \Omega$. Now, assuming the non-existence of Ω-limits, we may add the following:

Ω^{th} step: We have $2^{\aleph_1}$ decreasing Ω-sequences, e. g., $E_1, E_2, \dots, E_\omega, \dots, E_\alpha, \dots$ ($\alpha < \Omega$), each of which contains (by hypothesis) an infinite subset E_Ω, etc.

We have thus constructed $2^{\aleph_1}$ different sets, E_Ω, etc., whereas there are only $2^{\aleph_0}$ sets of n. n. altogether. Therefore, $2^{\aleph_1} \leqslant 2^{\aleph_0}$, hence $2^{\aleph_1} = 2^{\aleph_0}$, q. e. d.

Theorem 7^{b} may be stated as follows:

The existence of a linear set of power $\aleph_1$ and positive outer measure (Lebesgue) implies the existence of an Ω-limit in the family of all sets of n. n.

For the proof we require a few definitions and a lemma.

Definitions. Λ denotes any finite set of n. n. and $C\Lambda$ denotes the complement of a finite set, i. e., a set containing almost all n. n. (CE is the complement of E, generally).

To every set of n. n. corresponds a dyadic sequence (cf., introduction) which may be considered as the dyadic „decimal" expansion of a real number x, where $0 \leqslant x \leqslant 1$. This correspondence between sets of n. n. and real numbers x is a one-to-one correspondence, excepting the case of Λ's, $C\Lambda$'s, and the dyadic rationals, where it is 2 to 1. These $\aleph_0$ exceptions need not bother us.

Thus also every set $\mathfrak{X}$ of sets of n. n. corresponds to a linear set $\mathfrak{X}'$ contained in the interval [0,1].

Definition. We shall say the set $\mathfrak{X}$ (of sets of n. n.) is *of measure* 0 (or of positive measure, or of 1st category, etc.) if and only if the corresponding linear set $\mathfrak{X}'$ is of measure 0 (or of positive measure, or of 1st category, etc.).

Lemma. *Given a set $E \not\sim \Lambda$ and $\not\sim C\Lambda$, the following sets are of measure* 0:

(i) The set of all sets X satisfying the relation $X \subset E$,
(ii) „ „ „ „ „ „ „ „ „ $X < E$,

(iii) The set of all sets X satisfying the relation $X > E$,
(iv) „ „ „ „ „ „ „ „ „ $X > CE$,
(v) „ „ „ „ „ „ „ „ „ $X + E = C\Lambda$,
(vi) „ „ „ „ „ „ „ „ „ $X - E = \Lambda$,
(vii) „ „ „ „ „ „ „ *at least one of two relations:* $X + E = C\Lambda$ or $X - E = \Lambda$.

Proof of Lemma. It is easily seen that the linear set corresponding to (i) is a Cantor discontinuum of measure 0. Hence (ii) is the sum of $\aleph_0$ sets of measure 0, therefore of measure 0 itself. The same holds for (iii) (proved similarly) and also for (iv) (replacing E by CE in (iii)). Now, (v) is identical with (iv), and (vi) is identical with (ii) (since the relations are identical respectively). Finally, (vii) is of measure 0 because it is the sum of (v) and (vi), q. e. d.

Proof of theorem 7^b. Suppose there exists a linear set of power $\aleph_1$ and positive outer measure; let

$$E_1, E_2, \dots, E_\omega, \dots, E_\alpha, \dots \quad (\alpha < \Omega) \tag{5.1}$$

be the corresponding set of sets of n. n. Without loss of generality, we assume that (5.1) contains neither a Λ-set nor a $C\Lambda$-set.

By a process of transfinite induction, we are now going to construct an Ω-sequence of sets A^α with $\lim\limits_{\alpha \to \Omega} A_\alpha = C\Lambda$.

Let $A_1 = E_1$. Next, let E_{α_1} be the *first* set in (5.1) such that

$$E_{\alpha_1} - A_1 \not\sim \Lambda \quad \text{and} \quad A_1 + E_{\alpha_1} \not\sim C\Lambda.$$

Such an E_{α_1} exists, for otherwise (5.1) would be of measure 0, by our last lemma.

Now put $A_2 = A_1 + E_{\alpha_1}$, hence we have $A_1 \not\sim A_2$ and $A_1 \leqslant A_2$.

We continue by transfinite induction. Suppose all A_β have been defined for $\beta < \gamma$ (γ fixed, for the moment), such that

$$A_{\beta_1} \not\sim A_{\beta_2} \not\sim C\Lambda \quad \text{and} \quad A_{\beta_1} < A_{\beta_2} \quad \text{for } \beta_1 < \beta_2 < \gamma.$$

By lemma 3 there exists a set $B_\gamma \not\sim C\Lambda$ such that $A_\beta < B_\gamma$ for all $\beta < \gamma$. Now let E_{α_γ} be the first set in (5.1) such that

$$E_{\alpha_\gamma} - B_\gamma \not\sim \Lambda \quad \text{and} \quad B_\gamma + E_{\alpha_\gamma} \not\sim C\Lambda.$$

Such an E_{α_γ} exists, by the last lemma, just as above.

Now we put $A_\gamma = B_\gamma + E_{\alpha_\gamma}$.

Thus we have an increasing Ω-sequence $A_1, A_2, \ldots, A_\omega, \ldots, A_\alpha, \ldots$ and we have to show that $\lim_{\alpha \to \Omega} A_\alpha = C\Lambda$.

Otherwise there would exist a set $A_\Omega \neq C\Lambda$ (and obviously $\neq \Lambda$) such that $A_\alpha < A_\Omega$ for all $\alpha < \Omega$. But it is easily seen that for every α there is a β such that

$$E_\alpha - A_\beta = \Lambda \quad \text{or} \quad E_\alpha + A_\beta = C\Lambda;$$

hence (for all α),

$$\text{either} \quad E_\alpha - A_\Omega = \Lambda \quad \text{or} \quad E_\alpha + A_\Omega = C\Lambda;$$

it would follow, by the lemma, that (5.1) is of measure 0, contrary to the hypothesis, q. e. d.

Remark. Theorem 7^c can be proved in exactly the same way; all that's necessary, is to replace „measure 0" by „first category", and „positive outer measure" by „second category" in the above proof.

A positive, lower semi-continuous, non-degenerate function on a metric space.

By

Marston Morse (Princeton, N. J., U. S. A.).

§ 1. Introduction. Many terms in analysis such as the order of a zero or pole of an analytic function, the multiplicity of a branch point, a critical point, an extremal, the index of a critical point or extremal, a non-degenerate critical point or extremal, etc. admit useful topological definition. See[1]) Kuratowski (1), Seifert and Threlfall (1), Morse (3), (6). These definitions are not precisely equivalent to the definitions of analysis, but are equivalent in certain well determined consequences. The topological definitions give the principal relations in analysis in the large their proper setting, and lead to the simplest proofs. Artificial distinctions such as those between a critical point of a function of n variables and an extremal of an integral disappear.

It is, however, desirable that a topological theory which is to unify many forms of analysis of historic importance shall choose its definitions and axioms not so much in a subjective mood of abstract generality, as from the viewpoint of ready availability and interpretation in analysis. It is with these ideals in mind that the definitions and axioms necessary for the theory of a positive, lower semi-continuous, non-degenerate function F on a metric space S are presented.

The principal term in analysis to be topologically characterized, both in the small and in its global setting, is that of a non-degenerate critical point or extremal. Historically a critical point of a function f of class C'' of n variables $(x_1, \ldots, x_n)$ is non-degenerate if the Hessian of the function fails to vanish at the critical point. An extremal g which satisfies the self-adjoint conditions of a boundary problem in the large is termed non-degenerate in the sense of analysis

[1]) References are listed at the end of the paper. References to Morse are indicated by the letter M.