6. Finally we have

**Theorem 6.** The hypotheses of Theorem 2 are satisfied whenever
\( f(x) \) is a fractional integral of positive non-zero order of a Lebesgue integrable function.

**Proof.** Let us take \( f(x) \) to be a fractional integral of order
\( a > 0 \) of a function \( g(x) \). Choose \( p \) so that \( 1/p < 1/(1-a) \). Then
\( p(a-1) > -1 \), and we have by (15)

\[
\int f(x+t) - f(x) = \int \int g(x-u) \left( \mathcal{F}^p g(u) \right) \frac{du}{u^{1+1/p'}}.
\]

Therefore, if \( 1/p + 1/p' = 1 \),

\[
\int f(x+t) - f(x) \leq \int \int g(x-u) \left( \mathcal{F}^p g(u) \right) \frac{du}{u^{1+1/p'}}.
\]

where \( B \) depends on \( g \) and \( p \). Hence

\[
\int \left| f(x+t) - f(x) \right| \leq \int \left| \mathcal{F}^p g(u) \right| \frac{du}{u^{1+1/p'}}.
\]

where \( \beta > 0 \), by Lemma 3. It is now evident that the hypothesis (6) of Theorem 2 is satisfied and the theorem is proved.

**List of References.**


Complete normality of cartesian products.

By

Miroslav Katětov (Praha).

All spaces we consider are Hausdorff spaces.

**Theorem 1.** Let \( m \) be an infinite cardinal. Let \( F \) and \( Q \) be spaces such that \( P \times Q \) is completely normal. Then either every subset of \( Q \) with potency \( \leq m \) is closed or the pseudocharacter \( p \) of every closed subset of \( F \) is \( \leq m \).

**Proof.** Suppose there exists an \( MCQ \) with potency \( \leq m \) and a \( b \in M - M \). Let \( F \times P \) have pseudocharacter \( \geq m \). Let us put

\[
A = F \times M, \quad B = (F - F) \times (b).
\]

Then \( A \times P \times Q, B \times C \times (b) \) where \( A \) and \( B \) are separated. Hence there exists an open \( G \supset A \) such that \( GB = 0 \). For each \( y \in M \) let \( G_y \) denote the set of all \( x \in P \) such that \( (x, y) \in G \). Clearly \( y \in M \) implies \( G_y \subset F \). The potency of the family \( \{G_y\} \) being \( \leq m \), we have \( \left\| G_y \right\| \leq m \). Choose \( c \in F \). For any \( y \in M \) we have then \( (c, y) \in G \), whence \( (c, b) \in G \) implying the contradiction \( GB = 0 \).

1) A topological space is called completely normal if any two separated sets \( A, B \) (i.e. such that \( A \cap B = \emptyset \)) are contained in disjoint open sets.

It is easy to show that a topological space is completely normal if and only if it is hereditary normal, i.e. every subspace is normal.

2) Let \( S \) be a space, let \( MCQ \) and let \( \beta \) be a family of neighborhoods of the set \( M \). The collection \( \beta \) is said to be a complete family of neighborhoods of \( M \). The collection \( \beta \) is said to be a pseudocomplete family of neighborhoods of \( M \) if the intersection of all \( \beta \) is equal to \( M \).

The minimal potency of a complete (pseudocomplete) family of neighborhoods of a set \( M \) in a space \( S \) is called the character (pseudocharacter) of \( M \) in \( S \) and is denoted by \( c(M) \) or \( \chi(M) \) (respectively, by \( \chi_0(M) \) or \( \psi(M) \) or \( \psi_0(M) \)).
Corollary 1. If \( P \times Q \) is completely normal, then either every countable subset of \( Q \) is closed or \( P \) is perfectly normal. 2)

Corollary 2. A compact (i.e. bicompetent) space \( P \) is metrizable if, and only if, the space \( P \times P \times P \) is completely normal.

Proof. The necessity being evident suppose the converse to be true. If \( P \) is infinite, it contains a non-closed countable subset. Hence theorem 1. says that the pseudocharacter of the “diagonal” \( D \) of \( P \times P \) is countable. The characters in a compact space being equal to pseudocharacters, there exists a countable basic system \( \{ \mathcal{H}_n \} \) of neighborhoods of \( D \). For each \( n \) there exist open sets \( \mathcal{G}_n \subset \mathcal{P} \) \((k=1,\ldots,p_n)\) such that

\[
D \subseteq \bigcup_{k=1}^{p_n} \mathcal{G}_n \times \mathcal{G}_n \times \mathcal{H}_n.
\]

It is easy to see that \( (\mathcal{G}_n) \) is an open base of \( P \) so that \( P \) is separable.

I do not know whether, for compact \( P \), the complete normality of \( P \times P \) implies metrizability of \( P \). In theorem 1, the hypothesis of the existence of a non-closed subset of \( Q \) is essential, which is shown in the following

Example 1. Let \( P_1 \) have potency \( m \Rightarrow m_1 \). Let all points of \( P_1 \) be isolated with the exception of a single point \( \infty \) whose neighborhoods are \( \{ \infty \} \times \mathcal{G} \) with \( P_1 - \{ \infty \} \) finite. Then the pseudocharacter of \( \infty \) equals \( m \) so that \( P_1 \) is not perfectly normal. Nevertheless we shall show \( P_1 \times P_1 \) to be completely normal. To this end let us put

\[
A_1 = (\infty) \times P_1, \quad A_1 = P_1 \times (\infty), \quad A_1 = P_1 \times P_2 - A_1 - A_1.
\]

We clearly have: if \( M \cap A_1 \), \( N \cap A_1 \) \((i=j \vee i \neq j)\) and if the set \( M, N \) are separated (in \( P_1 \times P_2 \)), then can be separated by open sets. Suppose now \( M, N \) to be two separated subsets of \( P_1 \times P_1 \). There exist open sets \( \mathcal{G}_M \), \( \mathcal{H}_M \) such that

\[
\mathcal{G}_M \cap \mathcal{M} = \emptyset, \quad \mathcal{H}_M \cap \mathcal{N} = \emptyset, \quad \mathcal{G}_M \cap \mathcal{H}_M = \emptyset.
\]

Putting

\[
G = \bigcup_{i \in I} \mathcal{G}_i, \quad H = \bigcup_{j \in J} \mathcal{H}_j,
\]

we obtain \( G \cap M, H \cap N, GH = \emptyset \).

I do not know whether there exists a space \( P \) such that \( P \times P \) is completely normal, contains a non-closed countable set, and is not perfectly normal.

Theorem 2. If all spaces \( P_1 \times \cdots \times P_n \) \((n=1,2,\ldots)\) are perfectly normal, then the space \( P = \bigoplus_{n=1}^{m} P_n \) is perfectly normal as well.

Proof. Let \( A \subseteq P \) be closed. Let \( \pi_n \) denote the projection of \( P \) onto \( P_1 \times \cdots \times P_n \). There exists a continuous function \( g_n(y) \) on \( P_1 \times \cdots \times P_n \) such that \( 0 \leq g_n(y) \leq 1 \) for all \( y \) and \( g_n(y) = 0 \) if and only if, \( y \in \pi_n(A) \). For \( y \in P \) let us put

\[
f_n(x) = g_n(\pi_n(x)), \quad f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x).
\]

Clearly \( f(x) \) is a continuous function on \( P \) such that \( 0 \leq f(x) \leq 1 \) for all \( x \) and \( f(x) = 0 \) for any \( x \in A \). If \( x \in P - A \), then, for a convenient \( m \), we have \( \pi_m(x) \notin \pi_m(A) \), whence \( f_m(x) > 0 \) and \( f(x) > 0 \). Hence \( f(x) = 0 \) if, and only if, \( x \notin A \). This proves the theorem since, by a well known theorem of Urysohn, \( P \) is perfectly normal if, and only if, there exists, for every closed \( A \subseteq P \), a continuous function \( f \) such that \( f(x) = 0 \) if, and only if, \( x \in A \).

Theorem 3. Let the spaces \( P_n \) \((n=1,2,\ldots)\) contain more than one point. The space \( P = \bigoplus_{n=1}^{m} P_n \) is completely normal if, and only if, it is perfectly normal.

Proof. Urysohn having shown every perfectly normal space to be completely normal, let \( P \) be completely normal. We may suppose \( P \) infinite so that it contains (the discontinuities of Cantor and, therefore) a countable non-closed set. The last holds true for any \( \bigoplus_{n=1}^{m} P_n \). Applying now corollary 1 we see that the spaces \( P_1 \times \cdots \times P_n \) are perfectly normal. By theorem 2, the same must hold true for \( P \).

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Theorem 4. The cartesian product of a countable number of countable regular spaces is perfectly normal.

Proof. Let $P = \prod_{n=1}^{\infty} P_n$, $P_n$ being countable and regular. The spaces $P_1 \times \ldots \times P_n$ ($n = 1, 2, \ldots$) are countable and regular, hence, as shown by Urysohn, perfectly normal and it suffices to apply theorem 2.

Example 2. If the spaces $P_1 \times \ldots \times P_n$ are completely normal, the space $\prod_{n=1}^{\infty} P_n$ need not be completely normal. Choosing $P_n = P_1$ for all $n$, where $P_1$ denotes the space of example 1, we may easily show (analogously as for $P_1 \times P_1$ in example 1) that $P_1 \times \ldots \times P_n$ are completely normal. On the other hand, the space $\prod_{n=1}^{\infty} P_n$, where $P_n = P_1$, is not perfectly normal, for its subspace $P_1$ is not. Hence $\prod_{n=1}^{\infty} P_n$ is not completely normal by theorem 3.

On Area and Length.

By

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1. This paper is concerned with intrinsic definitions of area and of length. Although the definitions are new, they are obtained by combining ideas which are quite familiar to anyone working in this field: the ideas of Banach [1, 2] which have been the basis of researches on area for twenty years [2, 14, 11, 12] and which consist in effect in introducing our intrinsic definitions in a special case (the case of a surface situated in a plane); and the well-known theory of measure of Carathéodory [5, 8]. Moreover the old definitions, based on simplicial approximations, have long been regarded as unsatisfactory: examples of space-filling curves which constitute surfaces of zero area though of positive volume have been known for forty years; the examples recently produced by Besicovitch [3, 4] are even more conclusive.

The value of a particular definition however, depends mainly on its usefulness as a tool, and in this connection the Lebesgue-Fréchet definition of area has rendered great services. It has shown itself quite satisfactory for Lipschitzian surfaces (often misleadingly termed „rectifiable“) and has led to important semi-continuity theorems in the Calculus of Variations. Above all, it has had sufficient depth to serve as background to Banach’s fundamental methods already referred to.

The greater part of these results and methods remain when we adopt instead the present intrinsic definitions. We show in particular that the definitions agree for Lipschitzian surfaces. Moreover the new definitions are framed for the purpose of developing tools which are needed as a preliminary to the study of „generalized