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Approximation to functions by trigonometric polynomials (II).

By

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1. The object of this paper is to give some criteria for the convergence and strong summability of certain trigonometric polynomials introduced by Marcinkiewicz and Zygmund[1]), and defined in the following way. Suppose

$$x_i = \frac{2\pi i}{2n+1}, \qquad i=0,1,2,\dots,2n, \tag{1}$$

and that $\varphi_{2n+1}(u)$ ist a non-decreasing step function with jumps $2\pi/(2n+1)$ at the $2n+1$ equidistant points x_i. We define

$$I_{n,u}(x,f) = \frac{1}{2n+1}\sum_{i=0}^{2n} f(x_i+u)\frac{\sin(n+\frac{1}{2})(x-x_i-u)}{\sin\frac{1}{2}(x-x_i-u)} \tag{2}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}\,d\varphi_{2n+1}(t-u),$$

so that $I_{n,u}(x,f)$ is equal to $f(x)$ at the $2n+1$ points x_i+u. If $u=0$ they become the ordinary interpolation polynomials which we denote by $I_n(x,f)$. We prove

Theorem 1. *Let $f(x)$ be periodic of period 2π and write*

$$\Delta f = \tfrac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}. \tag{3}$$

Then, if $r>1$, and if $f(x)$ satisfies either of the following conditions

$$\int_0^{2\pi}\frac{dt}{t^2}\int_0^{2\pi}|\Delta f|^r\,dx<\infty, \tag{4}$$

[1]) Marcinkiewicz and Zygmund, 4.

or

$$\int_0^{2\pi} \frac{dt}{t^{1+1/r}} \left(\int_0^{2\pi} |\Delta f|^r \, dx \right)^{1/r} < \infty, \tag{5}$$

the polynomials $I_{n,u}(x,f)$ *converge to* $f(x)$ *at almost all points of the square* $0 \leqslant x \leqslant 2\pi$, $0 \leqslant u \leqslant 2\pi$.

Theorem 2. *If* $f(x)$ *is such that*

$$\int_0^{2\pi} \frac{dt}{t} \int_0^{2\pi} |\Delta f|^r \, dx < \infty \tag{6}$$

for some $r > 1$, *then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |I_{n,u}(x,f) - f(x)| = 0, \tag{7}$$

at almost all points of the square $0 \leqslant x \leqslant 2\pi$, $0 \leqslant u \leqslant 2\pi$.

If further $\{n_k\}$ *is a sequence of positive integers such that* $n_{k+1}/n_k \geqslant \gamma$, *where* γ *is a fixed number exceeding* 1, *then* $I_{n,u}(x,f)$ *converges to* $f(x)$ *at almost all points of the square* $0 \leqslant x \leqslant 2\pi$, $0 \leqslant u \leqslant 2\pi$.

In § 5 we give some further conditions which imply the hypotheses of Theorems 1 and 2. In particular, we show that the hypothesis of Theorem 1 is satisfied when $f(x)$ is the fractional integral of order $1/r$ of a function of the Lebesgue class L^r. We refer the reader to § 5 for further details.

2. To prove these theorems we make use of the following result [2]). Let us write

$$\Delta_n = \frac{2\pi}{2n+1}$$

and

$$f_n(x) = \frac{1}{2\Delta_n} \int_{x-\Delta_n}^{x+\Delta_n} f(t) \, dt,$$

then we have

Theorem A. *The interpolating polynomials* $I_n(x,f_n)$, *which take the values* $f_n(x_i)$ *at the* $2n+1$ *points* x_i *of* (1), *converge to* $f(x)$ *at almost all points.*

[2]) Offord 5, p. 508. Actually Δ_n is defined as $2K_n\pi/(2n+1)$ where K_n is an integer, which may depend on n, but which is such that Δ_n tends to zero Here we use only the case $K_n = 1$.

Since

$$I_{n,u}(x,f) = I_n(x-u, g),$$

where $g(t) = g_u(t) = f(t+u)$,

$$I_{n,u}(x,f_n) = I_n(x-u, g_n),$$

where

$$g_n(t) = f_n(t+u) = \frac{1}{2\Delta_n} \int_{t-\Delta_n}^{t+\Delta_n} f(v+u) \, dv,$$

and so Theorem A implies

Theorem A'. *The interpolating polynomials* $I_{n,u}(x,f_n)$ *which take the values* $f_n(x+u)$ *at the* $2n+1$ *points* $x_i + u$ *converge to* $f(x)$ *almost everywhere.*

Theorem 1 is included in the following result.

Theorem 3. *If either of the conditions* (4) *or* (5) *of Theorem 1 is satisfied with* $r > 1$, *then*

$$\sum_1^{\infty} |I_{n,u}(x,f) - I_{n,u}(x,f_n)|^r$$

is convergent.

Theorem 2 is included in the following result.

Theorem 4. *If* $f(x)$ *satisfies condition* (6) *of Theorem 2 with* $r > 1$, *then*

$$\sum_1^{\infty} \frac{1}{n} |I_{n,u}(x,f) - I_{n,u}(x,f_n)|^r$$

and

$$\sum_1^{\infty} |I_{n_k,u}(x,f) - I_{n_k,u}(x,f_{n_k})|^r$$

are convergent.

3. We require the following lemmas.

Lemma 1. *If* $r > 1$, *there exists a number* B_r, *depending only on* r, *such that*

$$\int_0^{2\pi} \int_0^{2\pi} |I_{n,u}(x,f)|^r \, du \, dx \leqslant B_r^r \int_0^{2\pi} |f(x)|^r \, dx.$$

This lemma is due to Marcinkiewicz and Zygmund [3]).

[3]) Marcinkiewicz and Zygmund 4, p. 155 eqn. (5) et seq.

Lemma 2. *If*

$$\psi_n(x) = \frac{1}{2\Delta_n}\int_{x-\Delta_n}^{x+\Delta_n} f(t)\,dt - f(x), \tag{9}$$

and if $r > 1$, then there exist absolute constants A_1, A_2, A_3 and A_4 such that

$$\sum_1^N \int_0^{2\pi} |\psi_n(x)|^r dx \leqslant A_1 \int_0^{2\pi} \frac{dt}{t^2} \int_0^{2\pi} |\Delta f|^r dx, \tag{10}$$

$$\left(\sum_1^N \int_0^{2\pi} |\psi_n(x)|^r dx\right)^{1/r} \leqslant A_2 \int_0^{2\pi} \frac{dt}{t^{1+1/r}} \left(\int_0^{2\pi} |\Delta f|^r dx\right)^{1/r}, \tag{11}$$

and

$$\sum_1^N \frac{1}{n} \int_0^{2\pi} |\psi_n(x)|^r dx \leqslant A_3 \int_0^{2\pi} \frac{dt}{t} \int_0^{2\pi} |\Delta f|^r dx. \tag{12}$$

If further $\{n_k\}$ is a sequence of positive integers such that $n_{k+1}/n_k \geqslant \gamma > 1$, where γ is some fixed number exceeding 1, then

$$\sum_{n_k \leqslant N} \int_0^{2\pi} |\psi_{n_k}(x)|^r dx \leqslant A_4 \int_0^{2\pi} \frac{dt}{t} \int_0^{2\pi} |\Delta f|^r dx. \tag{13}$$

Proof. We have

$$|\psi_n(x)| \leqslant \frac{1}{2\Delta_n} \int_0^{\Delta_n} |f(x+t) + f(x-t) - 2f(x)|\,dt,$$

and so, writting Δf for the integrand, by Hölder's inequality,

$$|\psi_n(x)|^r \leqslant \frac{1}{2\Delta_n} \int_0^{\Delta_n} |\Delta f|^r dt.$$

Whence

$$\int_0^{2\pi} |\psi_n(x)|^r dx \leqslant \frac{1}{2\Delta_n} \int_0^{\Delta_n} dt \int_0^{2\pi} |\Delta f|^r dx,$$

and writing $\alpha(t)$ for $\sum_{\Delta_n \geqslant \pi t} (2\Delta_n)^{-1}$, and $\alpha_n = \alpha(\Delta_n)$, we have

$$\begin{aligned}\sum_1^N \int_0^{2\pi} |\psi_n(x)|^r dx &\leqslant \sum_1^N (\alpha_n - \alpha_{n-1}) \int_0^{\Delta_n} dt \int_0^{2\pi} |\Delta f|^r dx \\ &= \sum_1^N \alpha_n \int_{\Delta_{n+1}}^{\Delta_n} dt \int_0^{2\pi} |\Delta f|^r dx + \alpha_N \int_0^{\Delta_{N+1}} dt \int_0^{2\pi} |\Delta f|^r dx \\ &\leqslant \int_0^{2\pi} \alpha(t)\,dt \int_0^{2\pi} |\Delta f|^r dx.\end{aligned}$$

Since $\alpha(t) \leqslant At^{-2}$ this is (10). The proofs of (12) and (13) are similar.

To prove (11) we employ Minkowski's inequality which in its simplest asserts that, for $a_{m,n} \geqslant 0$ and $r \geqslant 1$,

$$\left(\sum_m \left(\sum_n a_{m,n}\right)^r\right)^{1/r} \leqslant \sum_n \left(\sum_m a_{m,n}^r\right)^{1/r}.$$

We use a form with mixed Σ and $\int$.

Write

$$g_n(x,t) = \begin{cases} |\Delta f|/\Delta_n, & 0 \leqslant t \leqslant \Delta_n, \\ 0, & t > \Delta_n. \end{cases}$$

Then

$$|\psi_n(x)| \leqslant \int_0^{2\pi} g_n(x,t)\,dt,$$

and

$$\begin{aligned}\left(\sum_{n=1}^N \int_0^{2\pi} |\psi_n(x)|^r dx\right)^{1/r} &\leqslant \left(\sum_{n=1}^N \int_0^{2\pi} dx \left(\int_0^{2\pi} g_n(x,t)\,dt\right)^r\right)^{1/r} \\ &\leqslant \int_0^{2\pi} dt \left(\sum_{n=1}^N \int_0^{2\pi} g_n^r(x,t)\,dx\right)^{1/r}.\end{aligned}$$

Now

$$\sum_{n=1}^N g_n^r(x,t) = \sum_{\Delta_n \geqslant t} \frac{|\Delta f|^r}{\Delta_n^r} \leqslant A^r |\Delta f|^r t^{-r-1},$$

where A is an absolute constant.

Hence we obtain

$$\left(\sum_{n=1}^{N}\int_0^{2\pi}|\psi_n(x)|^r\,dx\right)^{1/r}\leqslant A\int_0^{2\pi}\frac{dt}{t^{1+1/r}}\left(\int_0^{2\pi}|\Delta f|^r\,dx\right)^{1/r},$$

the desired result.

4. Proofs of Theorems 1 and 3. Write

$$f_n(x)=\frac{1}{2\Delta_n}\int_{x-\Delta_n}^{x+\Delta_n}f(t)\,dt.$$

Then

$$I_{n,u}(x,f)-I_{n,u}(x,f_n)=I_{n,u}(x,f-f_n)$$
$$=-I_{n,u}(x,\psi_n),$$

and, by Lemma 1,

$$\int_0^{2\pi}\int_0^{2\pi}|I_{n,u}(x,\psi_n)|^r\,du\,dx\leqslant B_r^r\int_0^{2\pi}|\psi_n(x)|^r\,dx.$$

Therefore, by Lemma 2,

$$\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{n=1}^{N}|I_{n,u}(x,f)-I_{n,u}(x,f_n)|^r\right)dx\,du$$
$$=\sum_{n=1}^{N}\int_0^{2\pi}\int_0^{2\pi}|I_{n,u}(x,\psi_n)|^r\,du\,dx$$
$$\leqslant B_r^r\sum_{n=1}^{N}\int_0^{2\pi}|\psi_n(x)|^r\,dx$$
$$\leqslant B_r^r\int_0^{2\pi}\frac{dt}{t^2}\int_0^{2\pi}|\Delta f|^r\,dx,$$

and the second member is (4). Hence, since the integrand on the left is a monotone increasing function of N, we infer, by Lebesgue's theorem on the integration of monotone sequences, that

$$\sum_{n=1}^{\infty}|I_{n,u}(x,f)-I_{n,u}(x,f_n)|^r$$

converges for almost all x and u. This is the conclusion of Theorem 3. We infer also that

$$I_{n,u}(x,f)-I_{n,u}(x,f_n)$$

tends to zero for almost all x and u. But, by Theorem A′, $I_{n,u}(x,f_n)$ converges to $f(x)$ almost everywhere, and hence so must also $I_{n,u}(x,f)$. This proves Theorem 1 under condition (4).

From (11) of Lemma 2, by a very similar argument, we can show that Theorem 1 and Theorem 3 hold under condition (5).

The proofs of Theorems 2 and 4 are similar. We have, as in the proof of Theorem 3,

$$\sum_1^N\frac{1}{n}\int_0^{2\pi}\int_0^{2\pi}|I_{n,u}(x,f)-I_{n,u}(x,f_n)|^r\,dx\,du$$
$$\leqslant B_r^r\sum_1^N\frac{1}{n}\int_0^{2\pi}|\psi_n(x)|^r\,dx\leqslant B_r^r\int_0^{2\pi}\frac{dt}{t}\int_0^{2\pi}|\Delta f|^r\,dx$$

and so the first conclusion of Theorem 4 follows. The second conclusion follows similarly from (13) of Lemma 2.

Theorem 2 follows from Theorem 4 and Theorem A′, since

$$\frac{1}{N}\sum_1^N|I_{n,u}(x,f)-f(x)|$$
$$\leqslant\frac{1}{N}\sum_1^N|I_{n,u}(x,f)-I_{n,u}(x,f_n)|\quad+o(1)$$
$$\leqslant\left(\frac{1}{N}\sum_1^N|I_{n,u}(x,f)-I_{n,u}(x,f_n)|^r\right)^{1/r}+o(1),$$

by Hölders inequality.

5. We shall now obtain some simpler sufficient conditions for the convergence and summability of the trigonometric polynomials $I_{n,u}(x,f)$. For this we make use of integrals of fractional order defined by Weyl[4]). Throughout this section we take $g(x)$ to be a periodic function of period 2π and mean value zero, and we write $\sum_{-\infty}^{\infty}c_n e^{inx}$, with $c_0=0$ for the Fourier series of $g(x)$.

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[4]) Zygmund 6, p. 222. Zygmund considers functions of period 1 and so there is a slight formal difference. For the relation between the fractional integral due to Weyl and that due to Riemann and Liouville see Zygmund 6, p. 224.

The fractional integral of order a, $0<a\leqslant 1$ of $g(x)$ is the function

(14) $$G_a(x)=\frac{1}{\Gamma(a)}\int_{-\infty}^{x}(x-t)^{a-1}g(t)\,dt.$$

It is known that this integral converges for almost all x. Indeed

$$G_a(x)=\frac{1}{\Gamma(a)}\int_{0}^{\infty}t^{a-1}g(x-t)\,dt,$$

and since $g(t)$ is periodic of period 2π and mean value zero, this may be written

(15) $$G_a(x)=\int_{0}^{2\pi}g(x-t)\,\Psi_a(t)\,dt,$$

where

(16) $$\Psi_a(t)=\frac{1}{\Gamma(a)}\lim_{n\to\infty}\left\{t^{a-1}+(t+2\pi)^{a-1}+\ldots+(t+2n\pi)^{a-1}-\frac{n^a}{a}\right\},$$

for $0<t<2\pi$, and $\Psi_a(t)$ is periodic of period 2π outside this range. It follows at once from (16) that $\Psi_a(t)$ satisfies the following inequalities

(17) $$|\Psi_a(t)|\leqslant At^{a-1},\quad |\Psi_a(t+h)-\Psi_a(t)|\leqslant A_1ht^{a-2}.$$

It is also easy to see that the Fourier series of $G_a(x)$ is

(18) $$\sum_{-\infty}^{\infty}\frac{c_n}{(in)^a}e^{inx},\quad c_0=0.$$

We prove the following theorem.

Theorem 5. *The hypotheses of Theorem 1 are satisfied whenever $f(x)$ is a fractional integral of order $1/r$ of a function of L^r.*

Corollary. *This result holds in particular if the Fourier coefficients c_n of $f(x)$ satisfy $\sum_{-\infty}^{\infty}|n|^{p-1}|c_n|^p<\infty$, $1<p\leqslant 2$.*

It was proved by Hardy and Littlewood [5]) that the Fourier series of a function which satisfies the conditions of Theorem 5 converges almost everywhere. We have therefore established a similar property for the polynomials $I_{n,u}(x,f)$. The case $p=2$ of the Corollary is due to Marcinkiewicz and Zygmund [6]).

We require the following lemmas.

Lemma 3. *If $1\leqslant p<1/(1-a)$, $0<a<1$,*

$$\int_{0}^{2\pi}|\Psi_a(h+t)-\Psi_a(t)|^p\,dt\leqslant B_{a,p}h^{p(a-1)+1}.$$

Proof. This lemma is well known [7]). It follows easily from (17). Write

$$\int_{0}^{2\pi}|\Psi_a(h+t)-\Psi_a(t)|^p\,dt=\int_{0}^{h}+\int_{h}^{2\pi-h}+\int_{2\pi-h}^{2\pi}=I_1+I_2+I_3.$$

By the first inequality of (17)

$$I_1\leqslant A\int_{0}^{h}t^{p(a-1)}\,dt\leqslant B_{a,p}\,h^{p(a-1)+1},$$

and similarly, since $\Psi_a(t)$ is periodic,

$$I_3\leqslant B_{a,p}\,h^{p(a-1)+1}.$$

Also, by the second inequality of (17),

$$I_2\leqslant Ah^p\int_{h}^{2\pi-h}t^{p(a-2)}\,dt\leqslant B_{a,p}\,h^{p(a-1)+1}.$$

This completes the proof.

Lemma 4 [8]). *If $f(x)$ is a fractional integral of order a, $0<a<1$ of a function $g(x)$ of L^q, where $q\geqslant 2$, then*

$$\int_{0}^{2\pi}\frac{dt}{t^{1+aq}}\int_{0}^{2\pi}|\Delta f|^q\,dx\leqslant B_a\int_{0}^{2\pi}|g(x)|^q\,dx.$$

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5) Hardy and Littlewood 1, p. 606 and p. 613.
6) Marcinkiewicz and Zygmund 4, p. 166.
7) cf. Zygmund 6, p. 227 eqn. (1) et seq.
8) The case $a=1$ of this lemma is due to Marcinkiewicz 3.

Proof. It is sufficient to prove the lemma for a bounded function $g(x)$, for, if g is not bounded, we can write

$$g_n = \begin{cases} g(x) & |g| \leqslant n \\ n & |g| > n \end{cases}$$

and then, having proved the lemma for g_n (and the corresponding f_n), the desired result follows on taking the limit.

Consider the integral

$$I_{q,\varepsilon} = \left(\int_{\varepsilon}^{2\pi} \int_{0}^{2\pi} \left| \frac{\Delta f}{t^\alpha} \right|^q d\left(\log \frac{1}{t}\right) dx \right)^{1/q}.$$

We show that, when $q=2$,

$$I_{2,\varepsilon} \leqslant B_\alpha \left(\int_0^{2\pi} |g(x)|^2 \, dx \right)^{1/2},$$

and when $q=\infty$

$$I_{\infty,\varepsilon} \leqslant B_\alpha \sup |g(x)|,$$

where the numbers B are independent of ε. Now $\Delta f \cdot t^{-\alpha}$ is a linear functional of g defined for $0 \leqslant x \leqslant 2\pi$, $0 \leqslant t \leqslant 2\pi$. We can therefore infer by M. Riesz's convexity theorem that

$$I_{q,\varepsilon} \leqslant B_\alpha \left(\int_0^{2\pi} |g(x)|^q \, dx \right)^{1/q}$$

for $2 \leqslant q \leqslant \infty$, B being independent of ε. Allowing ε to tend to zero we get the required result.

In the case $q=2$, we have from (18) that, if $\sum_{-\infty}^{\infty} c_n e^{inx}$, $c_0=0$ is the Fourier series of $g(x)$, then the Fourier series of $f(x)$ is

$$\sum_{-\infty}^{\infty} \frac{c_n}{(in)^\alpha} e^{inx}, \quad c_0 = 0,$$

and so that of Δf is

$$\sum_{-\infty}^{\infty} \frac{c_n}{(in)^\alpha} e^{inx} (\cos nt - 1).$$

Hence

$$\int_0^{2\pi} |\Delta f|^2 \, dx = 2\pi \sum_{-\infty}^{\infty} \frac{|c_n|^2}{n^{2\alpha}} (\cos nt - 1)^2.$$

Now

$$\int_2^{2\pi} \frac{(\cos nt - 1)^2}{t^{1+2\alpha}} dt < n^{2\alpha} \int_0^{\infty} \frac{(1-\cos\theta)^2}{\theta^{1+2\alpha}} d\theta.$$

Hence

$$I_{2,\varepsilon}^2 \leqslant B_\alpha \sum_{-\infty}^{\infty} |c_n|^2 = B_\alpha \int_0^{2\pi} |g(x)|^2 \, dx$$

as desired.

Again

$$I_{q,\varepsilon} \leqslant A \left(\log \frac{1}{\varepsilon}\right)^{1/q} \sup \left| \frac{\Delta f}{t^\alpha} \right|$$

and so

$$\lim_{q\to\infty} I_{q,\varepsilon} \leqslant A \sup \left| \frac{\Delta f}{t^\alpha} \right|$$

where A is independent of ε. Now, by the case $p=1$ of Lemma 3,

$$|f(x+t) - f(x)| \leqslant \int_0^{2\pi} |g(x-u)| \, |\Psi_\alpha(u+t) - \Psi_\alpha(u)| \, du$$
$$\leqslant B_\alpha t^\alpha \sup |g(x)|.$$

Hence

$$\sup \left| \frac{\Delta f}{t^\alpha} \right| \leqslant B_\alpha \sup |g(x)|,$$

as desired. The lemma is therefore proved.

Proof of Theorem 5. If $r \geqslant 2$, we have only to put $q=r$ and $\alpha = 1/r$ in Lemma 4 and the result follows. If $1 < r \leqslant 2$, let q be defined by $\frac{1}{2} + 1/q = 1/r$ and let $G_{1/2}(x)$ be the fractional integral of order $\frac{1}{2}$ of $g(x)$. Then, in view of (18), $f(x)$ will be the fractional integral of order $1/q$ of $G_{1/2}(x)$. But, by a theorem of Hardy and Littlewood [9], since $g(x)$ belongs to L^r and $1/r > \frac{1}{2}$, $G_{1/2}(x)$ belongs to L^q. Hence $f(x)$ is a fractional of order $1/q$ of a function $G_{1/2}(x)$ of L^q. Since $q > 2$, the desired result again follows by Lemma 4.

The Corollary follows, since the convergence of $\sum_{-\infty}^{\infty} (|n|^{1/p'} |c_n|)^p$ where $1/p + 1/p' = 1$, implies that $n^{1/p'} c_n$ are the Fourier coefficients of a function of $L^{p'}$, $p' \geqslant 2$, and the fractional integral of order $1/p'$ of this function is $f(x)$.

[9]) cf. Zygmund 6, p. 227.

6. Finally we have

Theorem 6. *The hypotheses of Theorem 2 are satisfied whenever $f(x)$ is a fractional integral of positive non-zero order of a Lebesgue integrable function.*

Proof. Let us take $f(x)$ to be a fractional integral of order $\alpha>0$ of a function $g(x)$. Choose p so that $1<p<1/(1-\alpha)$. Then $p(\alpha-1)>-1$, and we have by (15)

$$f(x+t)-f(x)=\int_0^{2\pi} g(x-u)\{\Psi_\alpha(u+t)-\Psi_\alpha(u)\}\,du.$$

Therefore, if $1/p+1/p'=1$,

$$|f(x+t)-f(x)|^p \leqslant \Big(\int_0^{2\pi}|g(x-u)|\,du\Big)^{p/p'}\int_0^{2\pi}|g(x-u)|\,|\Psi_\alpha(u+t)-\Psi_\alpha(u)|^p\,du$$
$$\leqslant B\int_0^{2\pi}|g(x-u)|\,|\Psi_\alpha(u+t)-\Psi_\alpha(u)|^p\,du,$$

where B depends on g and p only. Hence

$$\int_0^{2\pi}|f(x+t)-f(x)|^p\,dx \leqslant B_1\int_0^{2\pi}|\Psi_\alpha(u+t)-\Psi_\alpha(u)|^p\,du \leqslant B_2\,t^\beta,$$

where $\beta>0$, by Lemma 3. It is now evident that the hypothesis (6) of Theorem 2 is satisfied and the theorem is proved.

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Complete normality of cartesian products.

By

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All spaces we consider are Hausdorff spaces.

Theorem 1. *Let $\mathfrak{m}$ be an infinite cardinal. Let P and Q be spaces such that $P\times Q$ is completely normal*[1]*). Then either every subset of Q with potency $\leqslant\mathfrak{m}$ is closed or the pseudocharacter*[2]*) of every closed subset of P is $\leqslant\mathfrak{m}$.*

Proof. Suppose there exists an $M\subset Q$ with potency $\leqslant\mathfrak{m}$ and a $b\in\overline{M}-M$. Let $F\subset P$ have pseudocharacter $>\mathfrak{m}$. Let us put

$$A=F\times M,\quad B=(P-F)\times(b).$$

Then $\overline{A}\subset F\times Q$, $\overline{B}\subset P\times(b)$ whence A and B are separated. Hence there exists an open $G\supset A$ such that $\overline{G}B=0$. For each $y\in M$ let G_y denote the set of all $x\in P$ such that $(x,y)\in G$. Clearly $y\in M$ implies G_y open, $G_y\supset F$. The potency of the family $\{G_y\}$ being $\leqslant\mathfrak{m}$ we have $\prod_{y\in M}G_y\neq F$. Choose $c\in\prod G_y-F$. For any $y\in M$ we have then $(c,y)\in G$, whence $(c,b)\in\overline{G}$ implying the contradiction $\overline{G}B\neq 0$.

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[1]) A topological space is called *completely normal* if any two separated sets A,B (i. e. such that $A\overline{B}+\overline{A}B=0$) are contained in disjoint open sets.

It is easy to show that a topological space is completely normal if and only if it is hereditarily normal, i. e. every subspace is normal.

[2]) Let S be a space, let $M\subset S$ and let $\mathfrak{A}$ be a family of neighborhoods of the set M. The collection $\mathfrak{A}$ is said to be a *complete family* of neighborhoods of M if there exists, for any neighborhood H of the set M, a set $A\in\mathfrak{A}$ such that $M\subset A\subset H$. The collection $\mathfrak{A}$ is said to be a *pseudocomplete family* of neighborhoods of M if the intersection of all $A\in\mathfrak{A}$ is equal to M.

The minimal potency of a complete (pseudocomplete) family of neighborhoods of a set M in a space S is called the *character (pseudocharacter)* of M in S and is denoted by $\chi(M)$ or more explicitly by $\chi_S(M)$ (respectively, by $\psi(M)$ or $\psi_S(M)$).