Soit, en effet, $U_1, U_2, U_3, \ldots$ la suite des ensembles donnés. En appliquant le théorème de séparabilité simple aux ensembles ouverts disjoints $U = U_1$ et $V = U_2 + U_3 + \ldots$, on trouve un ensemble ouvert à construction dénombrable $\boldsymbol{U}_1$ tel que $\boldsymbol{U}_1 \supset U_1$, $\boldsymbol{U}_1 \cdot V = 0$ (donc $\boldsymbol{U}_1 \cdot U_2 = \boldsymbol{U}_1 \cdot U_3 = \ldots = 0$). En posant maintenant

$$U = U_2, \quad V = \boldsymbol{U}_1 + U_3 + U_4 + \ldots,$$

on trouve, moyennant le même théorème, un ensemble ouvert à construction dénombrable $\boldsymbol{U}_2$ tel que

$$\boldsymbol{U}_2 \supset U_2, \quad \boldsymbol{U}_2 \cdot \boldsymbol{U}_1 = \boldsymbol{U}_2 \cdot U_3 = \boldsymbol{U}_2 \cdot U_4 + = \ldots = 0$$

et ainsi de suite.

La famille des ensembles $\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3, \ldots$ ainsi obtenus jouit évidemment des propriétés demandées.

La généralisation au cas non dénombrable est privée de sens grâce à un résultat de M. Edward Szpilrajn-Marczewski [6]) qui a montré qu'on ne peut trouver dans un produit topologique d'espaces satisfaisant au second axiome de dénombrabilité plus qu'une quantité dénombrable d'ensembles ouverts disjoints.

Notes.

[1]) Cf. A. Tychonoff. Math. Annalen **102** (1930), p. 546.

[2]) Pour éviter l'encombrement des indices nous omettrons dans la suite les indices supérieurs dans v_i^{α} (en écrivant simplement v_i).

[3]) Nous dirons, en général, qu'un système d'ensembles est *majoré par un ensemble* A si A renferme tous les ensembles du système.

[4]) Parce que deux ensembles ouverts élémentaires dans T: $\mathcal{O}_{\alpha_1 \ldots \alpha_m}(v_{i_1}, \ldots, v_{i_m})$ et $\mathcal{O}_{\beta_1 \ldots \beta_n}(v_{j_1}, \ldots, v_{j_n})$ sont disjoints dans le cas, et dans ce cas seulement, où au moins un α_r est égal à un β_s et l'on a pour ce r et ce s: $v_{i_r} \cdot v_{j_s} = 0$.

[5]) Dans le cas où la puissance du système fondamental d'ensembles ouverts dans T_α n'est pas dénombrable, mais est $\leqslant \mathfrak{m} > \aleph_0$, on démontre de même la proposition analogue, dans laquelle les sommes au plus dénombrables d'ensembles ouverts élémentaires sont remplacées par des sommes de puissance $\leqslant \mathfrak{m}$.

[6]) C. R. (Doklady) de l'Acad. des Sciences de l'URSS, **31**, N° 6 (1941).

On the representation of Boolean algebras as fields of sets.

By

Roman Sikorski (Warszawa).

The subject of this paper is the problem of the representation of an $\mathfrak{m}$-complete Boolean algebra as an $\mathfrak{m}$-additive field of sets[1]). Stone has proved that every Boolean algebra is isomorphic to a field of sets. A σ-complete Boolean algebra however can be no isomorph of a σ-field of sets. An easy analysis of Stone's representation theorem permits one to obtain simple necessary and sufficient conditions that an $\mathfrak{m}$-complete Boolean algebra be isomorphic to an $\mathfrak{m}$-additive field of sets (§ 1). With the help of these conditions I shall formulate several criteria for a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ (where $\boldsymbol{X}$ is an $\mathfrak{m}$-additive field of sets and $\boldsymbol{I}$ is an $\mathfrak{m}$-additive ideal of sets) to be isomorphic to an $\mathfrak{m}$-additive field of sets (§ 2) and I shall give an answer to two questions posed respectively by Professor E. Marczewski (§ 3) and by Professor A. Mostowski (§ 4).

The final part of this paper (§ 5) contains a proof of the theorem that every σ-complete Boolean algebra is isomorphic to a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ and $\boldsymbol{I}$ are respectively a σ-field and a σ-ideal of sets[2]). In contrast to this theorem, an $\mathfrak{m}$-complete Boolean algebra, where $\mathfrak{m} \geqslant 2^{\aleph_0}$, can be no isomorph of a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ and $\boldsymbol{I}$ are respectively an $\mathfrak{m}$-additive field and an $\mathfrak{m}$-additive ideal of sets.

[1]) The definitions of an $\mathfrak{m}$-complete Boolean algebra and an $\mathfrak{m}$-additive field of sets etc. are given in *Terminology and notation* on p. 248. $\mathfrak{m}$ denotes always an infinite cardinal number.

[2]) This theorem was presented by me at the Polish Mathematical Congress in Kraków in May 1947. Another proof of this theorem was given by Loomis. See Loomis [1], p. 757. An application of this theorem to the theory of the integral is given in my paper [1].

Terminology and notation. Let $\boldsymbol{A}$ be a Boolean algebra [3]), $A_1, A_2 \in \boldsymbol{A}$. A_1+A_2, A_1-A_2, A_1A_2, and A_1' will denote the Boolean operations of addition, substraction, multiplication, and complementation which correspond to the analogous operations on sets in the general theory of sets. If $A_1+A_2=A_2$, we write $A_1 \subset A_2$ and we say that A_1 is *contained* in A_2. 0 and $|\boldsymbol{A}|$ will denote respectively the least and the greatest element of $\boldsymbol{A}$ (i. e. $0 \subset A \subset |\boldsymbol{A}|$ for any $A \in \boldsymbol{A}$). An element $A \in \boldsymbol{A}$ $(A \neq 0)$ is called an *atom* if $0 \neq A_1 \subset A$ implies $A_1 = A$. $\boldsymbol{A}$ is called *atomic* if every element $A(\neq 0)$ of $\boldsymbol{A}$ contains an atom. $\boldsymbol{A}$ is called $\mathfrak{m}$-*complete* if for every non-empty class $\boldsymbol{A}_0 \subset \boldsymbol{A}$ of potency $\leqslant \mathfrak{m}$ there exists an $A_0 \in \boldsymbol{A}$ (the *sum* of all $A \in \boldsymbol{A}_0$) such that

$$A \subset A_0 \quad \text{for every} \quad A \in \boldsymbol{A}_0$$

and:

$$\text{if } A \subset A_1 \in \boldsymbol{A} \text{ or any } A \in \boldsymbol{A}_0, \text{ then } A_0 \subset A_1.$$

$\boldsymbol{A}$ is called *complete* if it is $\mathfrak{m}$-complete for every cardinal number $\mathfrak{m}$. $\aleph_0$-complete Boolean algebras are called also *σ-complete* algebras.

If $\boldsymbol{A}$ is $\mathfrak{m}$-complete, $A_\tau \in \boldsymbol{A}$ for every $\tau \in T$ where T is an abstract set of potency $\leqslant \mathfrak{m}$, then the sum of all A_τ is denoted by $\sum_{\tau \in T} A_\tau$. $\prod_{\tau \in T} A_\tau$ denotes the product of all A_τ, i. e. the element $(\sum_{\tau \in T} A_\tau')' \in \boldsymbol{A}$. The meaning of the symbols $\sum_{n=1} A_n$ and $\prod_{n=1} A_n$ (in case of a σ-complete Boolean algebra) is clear.

A non-empty class $\boldsymbol{I}$ of elements of a Boolean algebra $\boldsymbol{A}$ is called an *ideal* (of $\boldsymbol{A}$) if the conditions $A_1, A_2 \in \boldsymbol{I}$, $A \subset A_1$, $A \in \boldsymbol{A}$ imply $A_1+A_2 \in \boldsymbol{I}$ and $A \in \boldsymbol{I}$. An ideal $\boldsymbol{I}$ is called *prime* if 1° $|\boldsymbol{A}|$ non $\in \boldsymbol{I}$, 2° for every $A \in \boldsymbol{A}$ either $A \in \boldsymbol{I}$ or $A' \in \boldsymbol{I}$. An ideal $\boldsymbol{I}$ of an $\mathfrak{m}$-complete Boolean algebra is called $\mathfrak{m}$-*additive* if $A_\tau \in \boldsymbol{I}$ $(\tau \in T,\ \overline{\overline{T}} \leqslant \mathfrak{m})$ implies $\sum_{\tau \in T} A_\tau \in \boldsymbol{I}$. $\aleph_0$-additive ideals will be called also *σ-ideals*.

A class $\boldsymbol{X}$ of subsets of an abstract set $\mathcal{X}$ is called a *field* if the condition $X_1, X_2 \in \boldsymbol{X}$ implies $X_1+X_2 \in \boldsymbol{X}$ and $\mathcal{X}-X_1 \in \boldsymbol{X}$. A field $\boldsymbol{X}$ is called $\mathfrak{m}$-*additive* (or: *totally additive*) if for every subclass $\boldsymbol{X}_0 (\subset \boldsymbol{X})$ of potency $\leqslant \mathfrak{m}$ (or: of an arbitrary potency) the sum (union) of all $X \in \boldsymbol{X}_0$ belongs to $\boldsymbol{X}$. Every field of sets is a Boolean algebra, every $\mathfrak{m}$-additive field is an $\mathfrak{m}$-complete Boolean algebra [4]). $\aleph_0$-additive fields of sets will be called also *σ-fields*.

A mapping h of a Boolean algebra $\boldsymbol{A}$ on a Boolean algebra $\boldsymbol{B}$ is called a *homomorphism* (of $\boldsymbol{A}$ on $\boldsymbol{B}$) if $h(A_1+A_2)=h(A_1)+h(A_2)$ and $h(A_1')=(h(A_1))'$ for every $A_1, A_2 \in \boldsymbol{A}$. A homomorphism h is called an *isomorphism* if it is one-one, i. e. if $h(A)=0$ implies $A=0$. Two Boolean algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are *isomorphic* if there exists an isomorphism h of $\boldsymbol{A}$ on $\boldsymbol{B}$.

[3]) Boolean algebras will be denoted always by the letters $\boldsymbol{A}$ and $\boldsymbol{B}$, their elements by $A, B, \ldots$ Abstract sets will be denoted by $\mathcal{X}, \mathcal{Y}, \ldots$, their elements by $x, y, \ldots$ and their subsets by $X, Y, \ldots$ Fields of sebsets of $\mathcal{X}, \mathcal{Y}, \ldots$ will be denoted by $\boldsymbol{X}, \boldsymbol{Y}, \ldots$

[4]) However, a field of sets can be an $\mathfrak{m}$-complete Boolean algebra, without being an $\mathfrak{m}$-additive field of sets.

§ 1. The two following theorems on representation are well known [5]):

1.1. (Tarski's theorem) *A complete Boolean algebra is isomorphic to a totally additive field of sets if and only if it is atomic.*

1.2. (Stone's theorem) *Every Boolean algebra is isomorphic to a field of sets.*

Namely let $\mathcal{S}$ denote the set of all prime ideals of a Boolean algebra $\boldsymbol{A}$ and let $\mathfrak{s}(A)$ denote (for $A \in \boldsymbol{A}$) the set of all prime ideals $\boldsymbol{I}$ such that A non $\in \boldsymbol{I}$. The class $\boldsymbol{S}$ of all sets $\mathfrak{s}(A)$ where $A \in \boldsymbol{A}$ is a field of subsets of $\mathcal{S}$ and the mapping $S=\mathfrak{s}(A)$ is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{S}$.

The basis of this proof is the fact that $\mathfrak{s}(A) \neq 0$ for $A \neq 0$, i. e. that for every $A \neq 0$ there exists a prime ideal $\boldsymbol{I}$ which does not contain the element A. $\mathcal{S}$, $\boldsymbol{S}$, and $\mathfrak{s}$ will be called respectivelly Stone's set, field, and isomorphism of $\boldsymbol{A}$.

The question arises now whether every $\mathfrak{m}$-complete Boolean algebra is isomorphic to an $\mathfrak{m}$-additive field of sets. The answer is given in the following theorem:

1.3. *An $\mathfrak{m}$-complete Boolean algebra is isomorphic to an $\mathfrak{m}$-additive field of sets if and only if for every $A \neq 0$ $(A \in \boldsymbol{A})$ there exists an $\mathfrak{m}$-additive prime ideal $\boldsymbol{I}$ such that A non $\in \boldsymbol{I}$* [6]).

The proof of the sufficiency is the same as that of 1.2. It is sufficient to add the words: „$\mathfrak{m}$-additive" before the words: „ideal" and „field". The necessity follows from the fact that the condition given in 1.3 is invariant under isomorphism and is fulfilled in every $\mathfrak{m}$-additive field $\boldsymbol{X}$. In fact, when $0 \neq X_0 \in \boldsymbol{X}$, let $x_0 \in X_0$. The class $\boldsymbol{I}$ of all $X \in \boldsymbol{X}$ such that x_0 non $\in X$ is an $\mathfrak{m}$-additive prime ideal and X_0 non $\in \boldsymbol{I}$.

A function m defined for all elements of an $\mathfrak{m}$-complete Boolean algebra $\boldsymbol{A}$ is said to be a *two-valued $\mathfrak{m}$-additive measure* on $\boldsymbol{A}$ provided that: 1° m assumes only the numbers 0 and 1; 2° $m(|\boldsymbol{A}|)=1$; 3° if $A_\tau A_{\tau'}=0$ for $\tau \neq \tau'$, then

$$m\Big(\sum_{\tau \in T} A_\tau\Big) = \sum_{\tau \in T} m(A_\tau)$$

where T is any abstract set of potency $\leqslant \mathfrak{m}$.

[5]) See Tarski [1], p. 198, Stone [1], p. 98 and p. 106 and Stone [2].

[6]) This theorem was presented by me at a session of the Warsaw Section of the Polish Mathematical Society on May 10, 1946. See Sikorski [2], p. 246.

If m is a two-valued $\mathfrak{m}$-additive measure on $\boldsymbol{A}$, the set $\boldsymbol{I}_m$ of all $A \in \boldsymbol{A}$ such that $m(A)=0$ is an $\mathfrak{m}$-additive prime ideal. Since the correspondance $m \to \boldsymbol{I}_m$ is one-one, we obtain immediately:

1.4. *An $\mathfrak{m}$-complete Boolean algebra $\boldsymbol{A}$ is isomorphic to an $\mathfrak{m}$-additive field of sets if and only if for every $A \neq 0$ $(A \in \boldsymbol{A})$ there exists a two-valued $\mathfrak{m}$-additive measure m on $\boldsymbol{A}$ such that $m(A)=1$.*

An ideal $\boldsymbol{I}$ of a $\mathfrak{m}$-complete Boolean algebra $\boldsymbol{A}$ is called $\mathfrak{m}$-*regular* if it is the common part (product) of a class of $\mathfrak{m}$-additive prime ideals of $\boldsymbol{A}$ [7]). By definition every $\mathfrak{m}$-regular ideal of $\boldsymbol{A}$ is $\mathfrak{m}$-additive.

Theorem 1.3 can be expressed also in the following form:

1.5. *The necessary and sufficient condition for an $\mathfrak{m}$-complete Boolean algebra $\boldsymbol{A}$ to be isomorphic to an $\mathfrak{m}$-additive field of sets is that the „null" ideal* (0) [8]) *be $\mathfrak{m}$-regular.*

It is well known that every ideal $\boldsymbol{I}$ of a Boolean algebra $\boldsymbol{A}$ divides the set $\boldsymbol{A}$ into mutually exclusive classes in such a way that two elements A, A_1 belong to the same class if and only if $AA_1'+A_1A' \in \boldsymbol{I}$. The class containing the element A is denoted by $[A]$. The set of these classes constitutes a Boolean algebra denoted by $\boldsymbol{A}/\boldsymbol{I}$ and called a *quotient algebra*. If $\boldsymbol{A}$ is $\mathfrak{m}$-complete and $\boldsymbol{I}$ is $\mathfrak{m}$-additive, then $\boldsymbol{A}/\boldsymbol{I}$ is also $\mathfrak{m}$-complete. By definition:

$$[A]'=[A'], \quad \sum_{\tau\in T}[A_\tau]=[\sum_{\tau\in T} A_\tau] \quad \text{and} \quad \prod_{\tau\in T}[A_\tau]=[\prod_{\tau\in T} A_\tau]\ ^{9)}.$$

1.6. *Let $\boldsymbol{I}$ be an $\mathfrak{m}$-additive ideal of an $\mathfrak{m}$-complete Boolean algebra $\boldsymbol{A}$. The quotient algebra $\boldsymbol{A}/\boldsymbol{I}$ is isomorphic to an $\mathfrak{m}$-additive field of sets if and only if $\boldsymbol{I}$ is $\mathfrak{m}$-regular.*

---

[7]) As Stone proved, every ideal $\boldsymbol{I}$ is the common part of all prime ideals containing $\boldsymbol{I}$. However, an $\mathfrak{m}$-additive ideal is not, in general, the product of a class of $\mathfrak{m}$-additive prime ideals.

[8]) I. e. the ideal containing only the element $0 \in \boldsymbol{A}$.

[9]) Here $\sum_{\tau\in T}[A_\tau]$ does not denote the union of the classes $[A_\tau]$ but the Boolean sum of the elements $[A_\tau] \in \boldsymbol{A}/\boldsymbol{I}$. Similarly for the complement $[A]'$ and product $\sum_{\tau\in T}[A_\tau]$.

The necessity of this condition follows from 1.5 and the fact that if $\boldsymbol{I}_0$ is an $\mathfrak{m}$-additive prime ideal of $\boldsymbol{A}/\boldsymbol{I}$, the class of all $A \in \boldsymbol{A}$ such that $[A] \in \boldsymbol{I}_0$ is an $\mathfrak{m}$-additive prime ideal of $\boldsymbol{A}$. The sufficiency follows from 1.5 and the fact that, if $\boldsymbol{I}_0$ is an $\mathfrak{m}$-additive prime ideal (of $\boldsymbol{A}$) containing $\boldsymbol{I}$, the class of all $[A] \in \boldsymbol{A}/\boldsymbol{I}$ where $A \in \boldsymbol{I}_0$ is an $\mathfrak{m}$-additive prime ideal of $\boldsymbol{A}/\boldsymbol{I}$.

An ideal $\boldsymbol{I}$ is called *principal* if it is formed of all elements $A \subset A_0$, where A_0 is a given element of $\boldsymbol{A}$. By 1.5 and 1.6 we obtain:

1.7. *An $\mathfrak{m}$-complete Boolean algebra $\boldsymbol{A}$ is isomorphic to an $\mathfrak{m}$-additive field of sets if and only if every principal ideal of $\boldsymbol{A}$ is $\mathfrak{m}$-regular.*

For, if $\boldsymbol{A}$ is isomorphic to an $\mathfrak{m}$-additive field of sets and $\boldsymbol{I}$ is a principal ideal, $\boldsymbol{A}/\boldsymbol{I}$ is also isomorphic to an $\mathfrak{m}$-additive field of sets.

Every $\mathfrak{m}$-complete Boolean algebra which is not an isomorph of an $\mathfrak{m}$-additive field of sets is not atomic since:

1.8. *Every $\mathfrak{m}$-complete atomic Boolean algebra $\boldsymbol{A}$ is the isomorph of an $\mathfrak{m}$-additive field of subsets of the sets of all atoms of $\boldsymbol{A}$.*

Let $\mathrm{At}(A)$ denote the set of all atoms contained in $A \in \boldsymbol{A}$. The class of all sets $\mathrm{At}(A)$ is an $\mathfrak{m}$-additive field of subsets of $\mathrm{At}(|\boldsymbol{A}|)$ which is isomorphic to $\boldsymbol{A}$.

§ 2. In this paragraph $\boldsymbol{X}$ will denote an $\mathfrak{m}$-additive field of subsets of a set $\mathfrak{X}$.

An ideal $\boldsymbol{I}$ of $\boldsymbol{X}$ is called *semi-principal* [10]) if it is formed of all sets $X \in \boldsymbol{X}$ which are contained in a given set $X_0 \subset \mathfrak{X}$ (X_0 can belong to $\boldsymbol{X}$ or not). Obviously every principal ideal of $\boldsymbol{X}$ is semi-principal and every semi-principal ideal of $\boldsymbol{X}$ is $\mathfrak{m}$-additive.

For every ideal $\boldsymbol{I}$ of $\boldsymbol{X}$ the symbol $|\boldsymbol{I}|$ will denote the sum of all sets $X \in \boldsymbol{I}$. If $\boldsymbol{I}$ is semi-principal, the conditions: $X \in \boldsymbol{I}$ and $X \subset |\boldsymbol{I}|$ $(X \in \boldsymbol{X})$ are equivalent.

2.1. *If $\boldsymbol{I}$ is a semi-principal ideal of an $\mathfrak{m}$-additive field $\boldsymbol{X}$, $\boldsymbol{X}/\boldsymbol{I}$ is isomorphic to the $\mathfrak{m}$-additive field of all sets $X - |\boldsymbol{I}|$ where $X \in \boldsymbol{X}$. Thus $\boldsymbol{I}$ is $\mathfrak{m}$-regular.*

---

[10]) The property: „$\boldsymbol{I}$ is semi-principal" is not invariant under isomorphisms between fields of sets. The above defined semi-principal ideals do not coincide with semi-principal ideals in the sense defined by Stone in paper [1].

Obviously the class of all sets $X-|\boldsymbol{I}|$ where $X \in \boldsymbol{X}$ is an $\mathfrak{m}$-additive field of subsets of $\mathfrak{X}-|\boldsymbol{I}|$ and the mapping

$$h([X])=X-|\boldsymbol{I}| \quad (\text{for } X \in \boldsymbol{X})$$

is a homomorphism. Since $h([X])=0$ implies $X\subset|\boldsymbol{I}|$, i. e. $X \in \boldsymbol{I}$ and consequently $[X]=0$, h is an isomorphism.

The second part of 2.1 follows from the first and theorem 1.6.

A two-valued $\mathfrak{m}$-additive measure m on $\boldsymbol{X}$ is called *trivial* if there exists an $x_0 \in \mathfrak{X}$ such that $x_0 \in X \in \boldsymbol{X}$ implies $m(X)=1$ [11]). Clearly a two-valued $\mathfrak{m}$-additive measure m on $\boldsymbol{X}$ is trivial if and only if the prime ideal $\boldsymbol{I}_m$ is semi-principal.

2.2. *Let $\boldsymbol{I}$ be an $\mathfrak{m}$-additive ideal of an $\mathfrak{m}$-additive field $\boldsymbol{X}$. If every two-valued $\mathfrak{m}$-additive measure on $\boldsymbol{X}$ is trivial (i. e. if every $\mathfrak{m}$-additive prime ideal of $\boldsymbol{X}$ is semi-principal), then in order that $\boldsymbol{X}/\boldsymbol{I}$ be isomorphic to an $\mathfrak{m}$-additive field of sets (i. e. that $\boldsymbol{I}$ be $\mathfrak{m}$-regular) it is necessary and sufficient that $\boldsymbol{I}$ be semi-principal* [12]).

The sufficiency follows from 2.1; the necessity follows from 1.6 and the fact that the common part (product) of an arbitrary class of semi-principal ideals is a semi-principal ideal.

$\mathfrak{S}(\mathfrak{X})$ will denote always the field of all subsets of a set $\mathfrak{X}$.

A cardinal number $\mathfrak{n}$ is said to be *of two-valued measure zero* [13]) provided that every $\aleph_0$-additive two-valued measure on $\mathfrak{S}(\mathfrak{X})$ (where $\mathfrak{X}$ is a set of potency $\mathfrak{n}$) is trivial. Ulam has proved [14]) that the class of all cardinal numbers of two-valued measure zero contains: 1^0 $\aleph_0$; 2^0 with $\mathfrak{n}$ the number $2^{\mathfrak{n}}$; 3^0 with $\mathfrak{n}$ every cardinal number $\mathfrak{p}<\mathfrak{n}$; 4^0 with $\mathfrak{n}$ every sum $\sum_{\tau\in T}\mathfrak{n}_\tau$ where $\mathfrak{n}_\tau$ are of two-valued measure zero and $\overline{\overline{T}}=\mathfrak{n}$. In particular every cardinal number which is less than the first inaccessible (in the strict sense) aleph [15]), is of two-valued measure zero.

[11]) The property: „m is trivial" is not invariant under isomorphism between fields of sets. See Sikorski [3], theorem 2.2.

[12]) The assumption that every $\mathfrak{m}$-additive measure is trivial is essential.

[13]) This term is due to Professor E. Marczewski. See Marczewski and Sikorski [1], p. 134 and p. 138.

[14]) Ulam [1], p. 146 and p. 150.

[15]) A cardinal $\mathfrak{n}<\aleph_0$ is called *inaccessible* (in the strict sense) if the conditions: $\mathfrak{n}_\tau<\mathfrak{n}$ for every $\tau \in T$ and $\overline{\overline{T}}<\mathfrak{n}$ imply: $\prod_{\tau\in T}\mathfrak{n}_\tau<\mathfrak{n}$. See Tarski [2], p. 69 and p. 72.

2.3. *Let $\boldsymbol{I}$ be a σ-ideal of $\mathfrak{S}(\mathfrak{X})$. If $\overline{\overline{\mathfrak{X}}}$ is of two-valued measure zero, then $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ is isomorphic to a σ-field if and only if $\boldsymbol{I}$ is principal.*

This follows from 2.2 since semi-principal ideals coincide in $\mathfrak{S}(\mathfrak{X})$ with principal ideals.

For every topological space $\mathfrak{T}$ the symbol $\mathfrak{B}(\mathfrak{T})$ denotes the σ-field of all Borel subsets of $\mathfrak{T}$.

2.4. *Let $\mathfrak{T}$ be a metric space and let $\boldsymbol{I}$ be a σ-ideal of $\mathfrak{B}(\mathfrak{T})$. If $\overline{\overline{\mathfrak{T}}}$ is of two-valued measure zero, then $\mathfrak{B}(\mathfrak{T})/\boldsymbol{I}$ is isomorphic to a σ-field of sets if and only if $\boldsymbol{I}$ is semi-principal. In this case $\mathfrak{B}(\mathfrak{T})/\boldsymbol{I}$ is isomorphic to $\mathfrak{B}(\mathfrak{T}-|\boldsymbol{I}|)$.*

The first part of this theorem results from 2.2 and the fact that every $\aleph_0$-additive measure on $\mathfrak{B}(\mathfrak{T})$ is trivial [16]). The second part follows from 2.1.

§ 3. A σ-complete Boolean algebra $\boldsymbol{A}$ satisfies by definition the condition (M) if for any $A \in \boldsymbol{A}$ $(A\neq 0)$ and for any dyadic system $\{A_{i_1 i_2 \ldots i_n}\}$ $(n=1,2,\ldots,\ i_n=0$ or $1)$ of elements of $\boldsymbol{A}$ such that

(i) $$A=A_0+A_1$$

(ii) $$A_{i_1 i_2\ldots i_n}=A_{i_1 i_2\ldots i_n,0}+A_{i_1 i_2\ldots i_n,1}$$

there exists an infinite sequence $\{j_n\}$ of the numbers 0 and 1 such that

$$\prod_{n=1} A_{j_1 j_2 \ldots j_n}\neq 0.$$

Professor E. Marczewski has remarked [17]) that the condition (M) is necessary for $\boldsymbol{A}$ to be isomorphic to a σ-field of sets, and he has posed the question whether it is sufficient too. The answer is negative. Let $\mathfrak{X}$ be an abstract set of potency $2^{2^{\aleph_0}}$ and let $\boldsymbol{I}$ be the ideal of all subsets X of $\mathfrak{X}$ of potency $\leqslant 2^{\aleph_0}$. The Boolean algebra $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ is σ-complete and is not isomorphic to any σ-fields of sets. $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ satisfies also the condition (M) on account of the following theorem:

3.1. *Let $\boldsymbol{X}$ be a $2^{\aleph_0}$-additive field of sets and let $\boldsymbol{I}$ be a $2^{\aleph_0}$-additive ideal of $\boldsymbol{X}$. The quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ satisfies the condition (M).*

[16]) See Marczewski and Sikorski [1], p. 125.

[17]) See Marczewski [1], p. 243.

Let $A=[X]\neq 0$ be any element of $\boldsymbol{X}/\boldsymbol{I}$ and let $\{A_{i_1 i_2 \dots i_n}\}$ be any dyadic system satisfying (i) and (ii). We can easily define by induction [18]) a dyadic system $\{X_{i_1 i_2 \dots i_n}\}$ of sets belonging to $\boldsymbol{X}$ such that

$$A_{i_1 i_2 \dots i_n}=[X_{i_1 i_2 \dots i_n}]$$

$$X=X_0+X_1$$

$$X_{i_1 i_2 \dots i_n}=X_{i_1 i_2 \dots i_n,0}+X_{i_1 i_2 \dots i_n,1}.$$

$\boldsymbol{X}$ and $\boldsymbol{I}$ being $2^{\aleph_0}$-additive, we have by the definition of the operations „Σ" and „Π" in $\boldsymbol{X}/\boldsymbol{I}$

$$[X]=[\sum_{\{i_n\}}\prod_{n=1}^{\infty}X_{i_1 \dots i_n}]=\sum_{\{i_n\}}[\prod_{n=1}^{\infty}X_{i_1 \dots i_n}]=\sum_{\{i_n\}}\prod_{n=1}^{\infty}[X_{i_1 \dots i_n}]=\sum_{\{i_n\}}\prod_{n=1}^{\infty}A_{i_1 \dots i_n}.$$

Since $[X]\neq 0$, there exists a sequence $\{j_n\}$ such that

$$\prod_{n=1} A_{j_1 \dots j_n}\neq 0,$$

q. e. d.

§ 4. Professor A. Mostowski has posed the question whether there exists an $\aleph_{\mu+1}$-complete Boolean algebra which is isomorphic to an $\aleph_\mu$-additive field of sets but not isomorphic to any $\aleph_{\mu+1}$-additive field of sets. The answer is negative.

4.1. *Every $\aleph_\mu$-additive two-valued measure m on an $\aleph_{\mu+1}$-complete Boolean algebra $\boldsymbol{A}$ is $\aleph_{\mu+1}$-additive.*

Suppose that T_0 is a set of potency $\leqslant\aleph_{\mu+1}$ and that $A_\tau\in\boldsymbol{A}$ for every $\tau\in T_0$, $A_\tau A_{\tau'}=0$ if $\tau\neq\tau'$. We shall prove that

(i) $$m\left(\sum_{\tau\in T_0}A_\tau\right)=\sum_{\tau\in T_0}m(A_\tau).$$

The real function

$$m_0(T)=m\left(\sum_{\tau\in T}A_\tau\right)$$

defined for all subsets T of T_0 is an $\aleph_\mu$-additive two-valued measure on $\mathfrak{S}(T_0)$. Banach and Ulam have proved [19]) that every $\aleph_\mu$-additive measure defined for all subsets of a set is $\aleph_{\mu+1}$-additive. Therefore

$$m_0(T_0)=\sum_{\tau\in T_0}m((\tau))$$

[18]) Namely, if $A_i=[U_i]$ $(i=0,1)$, let $X_1=XU_1$ and $X_0=XU_0+(X-(U_0+U_1))$. Then $X=X_0+X_1$ and $A_i=[X_i]$ $(i=0,1)$. The inductive definition of $X_{i_1 \dots i_n}$ is clear.

[19]) Banach [1], p. 98 and Ulam [1], p. 141. The method of the proof of theorem 4.1 is originally due to Banach. Cf. Banach [1], p. 101.

i. e. the equality (i) is true.

4.2. *If an $\aleph_{\mu+1}$-complete Boolean algebra is isomorphic to an $\aleph_\mu$-additive field of sets, then it is isomorphic to an $\aleph_{\mu+1}$-additive field of sets.*

This theorem is an immediate consequence of 1.4 and 4.1 and it gives the answer to Mostowski's question. The following theorem can be proved in an analogous way:

4.3. *If an $\aleph_\mu$-complete Boolean algebra $\boldsymbol{A}$ is isomorphic to an $\aleph_0$-additive field of sets and if $\aleph_\mu$ is of two-valued measure zero, then $\boldsymbol{A}$ is isomorphic to an $\aleph_\mu$-additive field of sets.*

§ 5. $\mathfrak{m}$-complete Boolean algebras are constructed in practice by means of the division of $\mathfrak{m}$-additive fields of sets by $\mathfrak{m}$-additive ideals. The question arises whether every $\mathfrak{m}$-complete Boolean algebra is isomorphic to a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ is an $\mathfrak{m}$-additive field of sets and $\boldsymbol{I}$ is an $\mathfrak{m}$-additive ideal of $\boldsymbol{X}$. In the case $\mathfrak{m}=\aleph_0$ the answer is affirmative (theorem 5.3), in the case $\mathfrak{m}\geqslant 2^{\aleph_0}$ it is negative.

Let $\mathcal{S}$, $\boldsymbol{S}$, and $\mathfrak{s}$ be Stone's set, field, and isomorphism of a Boolean algebra $\boldsymbol{A}$. Admitting $\boldsymbol{S}$ as the class of neighbourhoods of elements of $\mathcal{S}$ we obtain from $\mathcal{S}$ a topological space called Stone's space of $\boldsymbol{A}$. As Stone proved the space $\mathcal{S}$ is totally disconnected and bicompact, thus normal [20]).

Let $\boldsymbol{N}$ denote in this section the class of all Borel sets $X\subset\mathcal{S}$ of first category in the space $\mathcal{S}$ and let $\boldsymbol{Z}$ denote the class of all sets $Z\in\mathfrak{B}(\mathcal{S})$ which can be represented in the form $Z=S+X-Y$ where $S\in\boldsymbol{S}$; $X\in\boldsymbol{N}$ and $Y\in\boldsymbol{N}$. Obviously $\boldsymbol{Z}$ is a field of subsets of $\mathcal{S}$ and $\boldsymbol{N}$ is a σ-ideal of $\boldsymbol{Z}$.

5.1. *The mapping $h(A)=[\mathfrak{s}(A)]$ (for $A\in\boldsymbol{A}$) is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{Z}/\boldsymbol{N}$.*

It follows immediately from the definition of $\boldsymbol{Z}$ that h maps $\boldsymbol{A}$ on $\boldsymbol{Z}/\boldsymbol{N}$. Since

$$h(A')=[\mathfrak{s}(A')]=[\mathcal{S}-\mathfrak{s}(A)]=[\mathcal{S}]-[\mathfrak{s}(A)]=(h(A))'$$

and:

$$h(A+B)=[\mathfrak{s}(A+B)]=[\mathfrak{s}(A)+\mathfrak{s}(B)]=[\mathfrak{s}(A)]+[\mathfrak{s}(B)]=h(A)+h(B),$$

[20]) Stone [3], p. 378. By definition every set $S\in\boldsymbol{S}$ is both open and closed in $\mathcal{S}$.

h is a homomorphism of $\boldsymbol{A}$ on $\boldsymbol{Z}/\boldsymbol{N}$. If $h(A)=0$, then the open set $\mathfrak{s}(A)$ is of first category. Since every set of first category in a bicompact normal space is boundary [21]), we infer that $\mathfrak{s}(A)=0$. Hence $A=0$ since $\mathfrak{s}$ is an isomorphism. This proves that h is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{Z}/\boldsymbol{N}$.

5.2. *If $\boldsymbol{A}$ is σ-complete, then $\boldsymbol{Z}$ is a σ-field.*

Let $Z_n \in \boldsymbol{Z}$, $n=1,2,\ldots$ By definition $Z_n=S_n+X_n-Y_n$ where $S_n \in \boldsymbol{S}$ and $X_n \in \boldsymbol{N}$ and $Y_n \in \boldsymbol{N}$. We obtain easily that

$$Z=\sum_{n=1}^{\infty} Z_n=\sum_{n=1}^{\infty} S_n+X-Y,$$

where $X \in \boldsymbol{N}$ and $Y \in \boldsymbol{N}$. In order to prove that $Z \in \boldsymbol{Z}$ it is sufficient to show that $\sum_{n=1} S_n \in \boldsymbol{Z}$. Let $S_n=\mathfrak{s}(A_n)$, $A_n \in \boldsymbol{A}$, and let $A=\sum_{n=1} A_n$, $S=\mathfrak{s}(A)$. Since $\mathfrak{s}$ is an isomorphism, and $A_n \subset A$, $S_n \subset S$ and finally $\sum_{n=1} S_n \subset S$. Let $U=S-\sum_{n=1} S_n$. We shall show that the set U is a boundary set. Suppose the contrary, i. e. that the interior of U is not empty. Then there would exist a non-empty neighbourhood $S_0=\mathfrak{s}(A_0)\subset U$. From the definition: $A_0 \neq 0$. We have $\sum_{n=1} S_n \subset S-S_0$, and hence $S_n \subset S-S_0$ for $n=1,2,3,\ldots$ i. e. $\mathfrak{s}(A_n)\subset\mathfrak{s}(A)-\mathfrak{s}(A_0)$. As $\mathfrak{s}$ is an isomorphism, we obtain: $A_n \subset A-A_0$ for $n=1,2,\ldots$ in contradiction to the assumption that $A=\sum_{n=1} A_n$.

Since the boundary set U is closed, it is nowhere dense, thus of the first category. Hence

$$\sum_{n=1} S_n=S-U \in \boldsymbol{Z}, \qquad \text{q. e. d.}$$

We shall say that $\boldsymbol{X}/\boldsymbol{I}$ is a *σ-quotient algebra* (of a set $\mathfrak{X}$) if $\boldsymbol{X}$ is a σ-field of subsets of $\mathfrak{X}$ and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{X}$. By 5.1 and 5.2 we obtain immediately:

5.3. *Every σ-complete Boolean algebra is isomorphic to an σ-quotient algebra.*

[21]) *In a locally compact, regular space $\mathfrak{T}$ every set of first category is boundary.* Proof: Let $\{T_n\}$ be a sequence of nowhere dense sets, $T=\sum_{n=1}^{\infty} T_n$, and let $G\neq 0$ be an arbitrary open set. Since $G-\overline{T}_1\neq 0$, there exists an open set G_1 such that $0\neq\overline{G}_1\subset G-\overline{T}_1$ and $\overline{G}_1$ is compact. By induction we define easily a sequence $\{G_n\}$ of open sets such that $0\neq\overline{G}_{n+1}\subset G_n-\overline{T}_{n+1}$ $(n=1,2,\ldots)$. $\overline{G}_1$ being compact, we infer that $U=\prod_{n=1}^{\infty}\overline{G}_n\neq 0$. By definition $U\subset G-T$. Hence $G-T\neq 0$. Since G is an arbitrary open set, T is boundary.

Similarly:

5.4. *If $\boldsymbol{A}$ is a complete Boolean algebra, then $\boldsymbol{Z}=\mathfrak{B}(\mathcal{S})$. Therefore $\boldsymbol{A}$ is isomorphic to $\mathfrak{B}(\mathcal{S})/\boldsymbol{N}$* [22]).

In order to prove the equality $\boldsymbol{Z}=\mathfrak{B}(\mathcal{S})$ it is sufficient to show that every open set $G=\sum_{\tau} S_\tau$ (where $S_\tau \in \boldsymbol{S}$) can be represented in the form $G=S-X$ where $S \in \boldsymbol{S}$ and X is boundary. The proof of this fact is analogous to the proof of 5.2. The second part of theorem 5.4 follows from 5.1.

In general, $\boldsymbol{Z}/\boldsymbol{N}$ is a subalgebra of the complete Boolean algebra $\mathfrak{B}(\mathcal{S})/\boldsymbol{N}$. It is easy to show that $\mathfrak{B}(\mathcal{S})/\boldsymbol{N}$ is the minimal extension of $\boldsymbol{A}$ in the sense defined by Mac Neille [23]).

Now let $\boldsymbol{L}$ denote the field of all measurable subsets of the interval $(0,1)$ and let $\boldsymbol{L}_0$ denote the ideal of all subsets of measure zero. It is well known that the quotient algebra $\boldsymbol{L}/\boldsymbol{L}_0$ is complete [24]) and does not satisfy the condition (M) [25]). Let $\mathfrak{m}$ be a cardinal number $\geqslant 2^{\aleph_0}$. Since the condition (M) is invariant under isomorphisms, it follows from theorem 3.1 that $\boldsymbol{L}/\boldsymbol{L}_0$ is an example of an $\mathfrak{m}$-complete Boolean algebra which is isomorphic to no quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ is an $\mathfrak{m}$-additive fields of sets and $\boldsymbol{I}$ is an $\mathfrak{m}$-additive ideal of $\boldsymbol{X}$.

References.

Banach S. [1] *Über additive Massfunktionen in abstrakten Mengen.* Fund. Math. **15** (1930), pp. 97-101.

Loomis L. H. [1] *On the representation of σ-complete Boolean algebras.* Bull. Am. Math. Soc. **53** (1947), pp. 757-760.

Mac Neille H. [1] *Partially ordered sets.* Trans. Am. Math. Soc. **42** (1937), pp. 416-460.

Marczewski E. [1] *Mesures dans les corps de Boole.* Annal. Soc. Pol. Math. **19** (1946), pp. 243-244.

Marczewski E. and Sikorski R. [1] *Measures in non-separable spaces.* Colloquium Mathematicum **1**, 2 (1948), pp. 119-125.

Sikorski R. [1] *The integral on a Boolean algebra.* Colloquium Mathematicum **1**, 4 (1948).

[22]) Theorem 2.54 is a modification of a theorem of Stone. See Stone [4], p. 261.

[23]) See Mac Neille [1], p. 437.

[24]) See Wecken [1], p. 380.

[25]) See Marczewski [1], p. 243.

— [2] *Sur l'isomorphisme des corps de Boole avec des corps d'ensembles.* Annal. Soc. Pol. Math. **19** (1946), p. 246.

— [3] *On the inducing of homomorphisms by mappings.* Fund. Math. **36** (1949).

Stone M. H. [1] *The theory of representations for Boolean algebras.* Trans. Am. Math. Soc. **40** (1936), pp. 37-111.

— [2] *The representation of Boolean algebras.* Bull. Am. Math. Soc. **44** (1938), pp. 807-816.

— [3] *Applications of the theory of Boolean rings to general topology.* Trans. Am. Math. Soc. **41** (1937), pp. 375-481.

— [4] *Algebraic characterizations of special Booleans rings.* Fund. Math. **29** (1937), pp. 223-303.

Tarski A. [1] *Zur Grundlegung der Boole'schen algebra.* Fund. Math. **24** (1935), pp. 177-198.

— [2] *Über unerreichbare Kardinalzahlen.* Fund. Math. **30** (1938), pp. 68-89.

Ulam S. [1] *Zur Masstheorie in der allgemeinen Mengenlehre.* Fund. Math. **16** (1930), pp. 140-150.

Wecken F. [1] *Abstrakte Integrale und fastperiodische Funktionen.* Math. Zeitschr. **45** (1939), pp. 377-404.

Approximation to functions by trigonometric polynomials (II).

By

A. C. Offord (London).

1. The object of this paper is to give some criteria for the convergence and strong summability of certain trigonometric polynomials introduced by Marcinkiewicz and Zygmund[1]), and defined in the following way. Suppose

$$x_i=\frac{2\pi i}{2n+1}, \qquad i=0,1,2,\dots,2n, \tag{1}$$

and that $\varphi_{2n+1}(u)$ ist a non-decreasing step function with jumps $2\pi/(2n+1)$ at the $2n+1$ equidistant points x_i. We define

$$\begin{aligned} I_{n,u}(x,f)&=\frac{1}{2n+1}\sum_{i=0}^{2n} f(x_i+u)\frac{\sin(n+\frac12)(x-x_i-u)}{\sin\frac12(x-x_i-u)} \\ &=\frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{\sin(n+\frac12)(x-t)}{\sin\frac12(x-t)}\,d\varphi_{2n+1}(t-u), \end{aligned} \tag{2}$$

so that $I_{n,u}(x,f)$ is equal to $f(x)$ at the $2n+1$ points x_i+u. If $u=0$ they become the ordinary interpolation polynomials which we denote by $I_n(x,f)$. We prove

Theorem 1. *Let $f(x)$ be periodic of period 2π and write*

$$\Delta f=\tfrac12\{f(x+t)+f(x-t)-2f(x)\}. \tag{3}$$

Then, if $r>1$, and if $f(x)$ satisfies either of the following conditions

$$\int_0^{2\pi}\frac{dt}{t^2}\int_0^{2\pi}|\Delta f|^r\,dx<\infty, \tag{4}$$

[1]) Marcinkiewicz and Zygmund, 4.