L'ensemble $D-(\{0\},0)$, ouvert dans D, est homéomorphe au produit cartésien de l'ensemble localement contractile C et de l'intervalle demi-ouvert $(0,1\rangle$. On en conclut que D est localement contractile à chaque point $p\neq(\{0\},0)$. Afin de prouver que l'ensemble D est localement contractile au point $(\{0\},0)$, désignons, pour tout $\varepsilon>0$, par D_ε le sous-ensemble de D composé de tous les points de la forme

$$(\{x_i\cdot y\},y) \text{ où } \{x_i\}\in C \text{ et } 0\leqslant y\leqslant\varepsilon.$$

Il est clair que D_ε est un entourage du point $(\{0\},0)$ homéomorphe à l'ensemble D, qui est contractile dans soi. En outre, pour tout entourage U de $(\{0\},0)$ dans D, il existe un nombre $\varepsilon>0$ tel que $D_\varepsilon\subset U$. Or D est localement contractile au point $(\{0\},0)$.

Il ne reste qu'à prouver que D n'est pas un rétracte absolu. Désignons par C_0 le sous-ensemble de D composé de tous les points de la forme $(\{x_i\},1)$. En posant

$$g(\{x_i\cdot y\},y)=(\{x_i\},1) \text{ pour tout } (\{x_i\cdot y\},y)\in D-(\{0\},0),$$

on obtient une fonction rétractant l'entourage $D-(\{0\},0)$ de l'ensemble C_0 en C_0. Si D était un rétracte absolu, l'ensemble C_0, en tant qu'un retracte de son entourage $D-(\{0\},0)$, serait un rétracte absolu de voisinage [7]). Or cela contredit le fait que l'ensemble C_0 est homéomorphe à l'ensemble C qui n'est pas un rétracte absolu de voisinage. La démonstration du théorème 2 est ainsi terminée.

[7]) Voir l. c. dans le renvoi 5, p. 224.

On Čech homology groups of retracts.

By

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1. Introduction.

In their axiomatic approach to the homology theory, Eilenberg and Steenrod [4] formulated the *homology sequence* (called by them the natural system of the homology theory) as an axiom which should be satisfied in all admissible homology theories. The homology sequence given in this comprehensive form is a weapon far more powerful than the classical duality theorem of Alexander. There could be many applications in attacking the homology problems, one of those is given in the present paper.

Let X denote an arbitrary topological space, X_1 a closed subset of X, and $X_0=X-X_1$ its open complement.

A *retraction* of X onto X_1 is a mapping $\theta:X\to X_1$ such that $\theta(x)=x$ for each $x\in X_1$. If such a θ exists, X_1 is called a *retract* of X. It was first proved by Borsuk [3] that the homology groups of X_1 are homomorphic images of those of X if X_1 is a retract of a compactum X. By the aid of the homology sequence we shall give a strengthened form of this theorem of Borsuk without assuming X to be a compactum.

Let $H^n(X,G)$, $H^n(X_1,G)$, $H^n(X \bmod X_1,G)$ denote the n-dimensional Čech homology groups of X, X_1, X modulo X_1 based on the finite open coverings and with coefficients in a topological abelian group G. Then our result is the following

Theorem. *For each division-closure* [5, p. 68] *group G and each $n\geqslant0$, $H^n(X,G)$ is isomorphic with the direct sum*

$$H^n(X_1,G)+H^n(X \bmod X_1,G).$$

For a discrete coefficient group G, an analogous theorem holds for Čech cohomology groups which follows from the duality between the homology and the cohomology sequences. For the sake of simplicity, only homology groups will be considered explicitly in the present work.

A brief account of the Čech homology sequence is given in § 2; and the main theorem is proved in § 3.

2. Čech homology sequence.

As in the introduction, let X denote a topological space, X_1 a closed subset of X, and $X_0=X-X_1$ its open complement. Following Lefschetz, we shall call (X_0,X_1) a *dissection* of X. A finite open covering will be simply called a covering.

Let Σ denote the set of all coverings of X, partially ordered by the statement $\alpha<\beta$ if β is a *refinement* of α [5, p. 13]. Σ is a *reflexive directed set* [5, p. 4] since any two coverings α and β have a common refinement obtained by the mutual intersection of the members of α with those of β.

Let $\alpha=\{a_1,a_2,\dots,a_r\}$ be a covering of X and denote by A the *nerve* of α [5, p. 244]. The non-void sets of $X_1\cap a_i$ $(i=1,2,\dots,r)$ form a covering α_1 of X_1 whose nerve A_1 is a closed subcomplex of A. Let $A_0=A-A_1$ denote the open complement. Then A_0, A_1 are abstract complexes in the sense of Tucker [5, p. 89] and (A_0,A_1) is a dissection of A [5, p. 112]. For an arbitrary topological coefficient group G, let us denote by

$$H^n(A,G),\quad H^n(A_0,G),\quad H^n(A_1,G)$$

the n-dimensional homology groups of A,A_0,A_1 respectively. The notations of the present paragraph are typical ones.

Suppose $\alpha<\beta$, i. e. β be a refinement of α. Let us select for each member of β a member of α containing it. This gives a simplicial mapping of the nerve B of β into the nerve A of α, which is called a projection $p_{\beta\alpha}$: $B\to A$. It is obvious that $p_{\beta\alpha}(B_1)\subset A_1$. All the possible projections $p_{\beta\alpha}$ induce unique homomorphisms of the homology groups of the dissection (B_0,B_1) into those of (A_0,A_1) as follows:

$$h^n_{\beta\alpha}:\ H^n(B,G)\to H^n(A,G);$$
$$h^n_{\beta\alpha i}:\ H^n(B_i,G)\to H^n(A_i,G),\qquad (i=0,1).$$

If $\alpha<\beta<\gamma$ and $p_{\beta\alpha},p_{\gamma\beta}$ are projections, then $p_{\beta\alpha}p_{\gamma\beta}$ is a projection $p_{\gamma\alpha}$. Hence

$$\{H^n(A,G);\ h^n_{\beta\alpha}\},$$
$$\{H^n(A_i,G);\ h^n_{\beta\alpha i}\},\qquad (i=0,1),$$

are *inverse systems* of groups [5, p. 54].

By the n-dimensional *Čech homology group* $H^n(X,G)$ of X with a topological coefficient group G we mean the limit group of the inverse system $\{H^n(A,G);\ h^n_{\beta\alpha}\}$ which is topologised in the usual way [5, p. 31]. The topologised limit group of the inverse system $\{H^n(A_0,G);\ h^n_{\beta\alpha 0}\}$ is defined to be the n-dimensional *Čech homology group* of X *modulo* X_1 with coefficient group G, denoted by the symbol $H^n(X \bmod X_1,G)$. The isomorphism

$$H^n(X_1,G)\approx\lim\{H^n(A_1,G);\ h^n_{\beta\alpha 1}\}$$

is obvious; but the groups $H^n(X \bmod X_1,G)$ and $H^n(X_0,G)$ are in general different. This justifies the notation used.

Let α be an arbitrary covering of X. Denote by

$$\eta_\alpha:\ A_1\to A,\qquad \zeta_\alpha:\ A_0\to A$$

the injections of the dissection (A_0,A_1) of the nerve A of α; and denote by

$$\pi_\alpha:\ A\to A_0,\qquad \tau_\alpha:\ A\to A_1$$

the projections of (A_0,A_1), [5, p. 113]. Following Lefschetz, we denote by F_α the boundary operator in A. The projections η_α, π_α and the operator $\partial_\alpha=\tau_\alpha F_\alpha\zeta_\alpha$ induce homomorphisms:

$$\eta^n_\alpha:\ H^n(A_1,G)\to H^n(A,G),$$
$$\pi^n_\alpha:\ H^n(A,G)\to H^n(A_0,G),$$
$$\partial^n_\alpha:\ H^n(A_0,G)\to H^{n-1}(A_1,G).$$

It can be proved that these homomorphisms determine the following homomorphisms:

$$\eta^n:\ H^n(X_1,G)\to H^n(X,G),$$
$$\pi^n:\ H^n(X,G)\to H^n(X \bmod X_1,G),$$
$$\partial^n:\ H^n(X \bmod X_1,G)\to H^{n-1}(X_1,G).$$

The theorem of Čech homology sequence of the dissection (X_0, X_1), which can be proved by the argument of Alexandroff [2], is stated as follows:

(2.1) ***Čech homology sequence.*** *If the coefficient group G is a division-closure group, the homomorphism $\eta^n, \pi^n, \partial^n$ constitute a homology sequence*

$$\ldots \xrightarrow{\partial^{n+1}} H^n(X_1,G) \xrightarrow{\eta^n} H^n(X,G) \xrightarrow{\pi^n} H^n(X \bmod X_1,G) \xrightarrow{\partial^n} H^{n-1}(X_1,G) \xrightarrow{\eta^{n-1}} \ldots$$

of the dissection (X_0, X_1) of X, which is exact in the sense that the kernel of each homomorphism coincides with the image of the preceding.

3. The main theorem.

Throughout the present section, we assume X_1 to be a retract of X with a given retraction θ: $X \to X_1$. A covering $\alpha = \{a_1, a_2, \ldots, a_r\}$ of X will be called *θ-admissible*, if (1) for each $a_i \in \alpha$ there is an $a_j \in \alpha$ with $\theta(a_i) \subset a_j$, (2) if a_i meets X_1 then $\theta(a_i) \subset a_i$.

(3.1) *The set of all θ-admissible coverings of X form a cofinal subset Σ_θ of the set Σ of all coverings of X.*

Proof. Let $\beta = \{b_1, b_2, \ldots, b_s\}$ be an arbitrary covering of X. Denote by $\gamma_1 = \{c_1, c_2, \ldots, c_p\}$ the nonvoid sets of $b_i \cap X_1$ $(i = 1, 2, \ldots, s)$, then γ_1 is a covering of X_1. Without loss of generality, we may assume that $c_i = b_i \cap X_1$ $(i = 1, 2, \ldots, p)$. Let

$$d_i = \theta^{-1}(c_i) \qquad (i = 1, 2, \ldots, p),$$

then $c_i \subset d_i$ and $\delta = \{d_1, d_2, \ldots, d_p\}$ is a covering of X. Let $a_i = b_i \cap d_i$ $(i = 1, 2, \ldots, p)$ and denote by $a_{p+1}, \ldots, a_r$ the non-void sets of the form $b_i \cap d_j \cap X_0$. It is clear that $\alpha = \{a_1, a_2, \ldots, a_r\}$ is a refinement of β. Since each a_i is contained in some d_j, $\theta(a_i) \subset c_j \subset a_j$. If a_i meets X_1 then $i \leqslant p$ and $a_i \subset d_i$, hence we have the inclusion $\theta(a_i) \subset c_i \subset a_i$. Therefore, α is θ-admissible and our assertion follows.

(3.2) *For each θ-admissible covering α of X and each $n \geqslant 0$, there is a homomorphism*

$$\varkappa_\alpha^n:\ H^n(A_0, G) \to H^n(A, G)$$

which depends only on θ and such that $\pi_\alpha^n \varkappa_\alpha^n$ is the identity on $H^n(A_0, G)$.

Proof. Let $\alpha = \{a_1, a_2, \ldots, a_r\}$ be an arbitrary θ-admissible covering of X. Let us select for each a_i of α an $a_j = t(a_i) \in \alpha$ which contains $\theta(a_i)$ under the only condition that $t(a_i) = a_i$ if a_i meets X_1. This gives a simplicial mapping t: $A \to A_1$. Since $t|A_1$ is the identity on A_1, it is called a *simplicial retraction* of A onto A_1 induced by θ. In general there will be many simplicial retractions of A onto A_1 induced by θ, corresponding to the choices of the member a_j which contains $\theta(a_i)$; but for any two such t, t' of A onto A_1, $t(\sigma)$ and $t'(\sigma)$ are faces of some simplex $\sigma_1 \in A_1$ for each simplex $\sigma \in A$.

The simplicial retraction t: $A \to A_1$ induces a simplicial chain-mapping (still denoted by t) defined as follows.

$t(\sigma^n) = t(a_{i_0} \ldots a_{i_n}) = b_{i_0} \ldots b_{i_n}$ when the $b_{i_q} = t(a_{i_q})$ $(q = 0, 1, \ldots, n)$ are distinct and $t(\sigma^n) = 0$ otherwise.

Now let $z^n = \sum g_i \sigma_i^n$ be an arbitrary n-cycle of A_0 with coefficients $g_i \in G$ and let F denote the boundary operator in A, then Fz^n is an $(n-1)$-cycle of A_1. On account of $Ft(z^n) = t(Fz^n) = Fz^n$, the chain

$$y^n = z^n - t(z^n)$$

is a cycle of A. If $z^n = Fu^{n+1} + v^n$ with $u^{n+1} \subset A_0$ and $v^n \subset A_1$, then

$$y^n = Fu^{n+1} - t(Fu^{n+1}) = F(u^{n+1} - t(u^{n+1})).$$

On the other hand, it is not difficult to see that the homology class of y^n is independent of the special choice of t. Therefore, the correspondence $z^n \to y^n$ determines a homomorphism

$$\varkappa_\alpha^n:\ H^n(A_0, G) \to H^n(A, G)$$

which depends only on θ.

Since $\pi_\alpha y^n = \pi_\alpha z^n = z^n$, it is obvious that $\pi_\alpha^n \varkappa_\alpha^n$ is the identity on $H^n(A_0, G)$. Q. E. D.

In the set Σ_θ of all θ-admissible coverings of X, let us define a new partial ordering by the statement that $\alpha <^* \beta$ if (1) β is a refinement of α and (2) there exist simplicial retractions t_α: $A \to A_1$, t_β: $B \to B_1$ induced by θ and a projection $p_{\beta\alpha}$: $B \to A$ such that

$$t_\alpha p_{\beta\alpha} = p_{\beta\alpha} t_\beta.$$

(3.3) *$\{\Sigma_\theta, <^*\}$ is a directed set and hence a cofinal subset of the directed set $\{\Sigma, <\}$ in the sense of Alexandroff* [1, p. 61].

Proof. Let $\alpha = \{a_1, a_2, \ldots, a_r\}$, $\beta = \{b_1, b_2, \ldots, b_s\}$ be two arbitrary θ-admissible coverings and let

$$t_\alpha:\ A \to A_1, \qquad t_\beta:\ B \to B_1$$

be simplicial retractions induced by θ. Let γ be the covering of X which consists of the non-void sets of the form $a_i \cap b_j$ ($i=1,2,\dots,r$; $j=1,2,\dots,s$). Since $\theta(a_i) \subset t_\alpha(a_i)$ and $\theta(b_j) \subset t_\beta(b_j)$, we have

$$\theta(a_i \cap b_j) \subset t_\alpha(a_i) \cap t_\beta(b_j).$$

Define a simplicial retraction $t_\gamma\colon C \to C_1$ by taking

$$t_\gamma(a_i \cap b_j) = t_\alpha(a_i) \cap t_\beta(b_j).$$

Define projections $p_{\gamma\alpha}\colon C \to A$ and $p_{\gamma\beta}\colon C \to B$ by taking $p_{\gamma\alpha}(a_i \cap b_j) = a_i$ and $p_{\gamma\beta}(a_i \cap b_j) = b_j$. Then we have

$$t_\alpha p_{\gamma\alpha} = p_{\gamma\alpha} t_\gamma, \quad t_\beta p_{\gamma\beta} = p_{\gamma\beta} t_\gamma.$$

Hence $\alpha, \beta <^* \gamma$ and $\{\Sigma_\theta, <^*\}$ is a directed set. The remaining part of the assertion follows from (3.1) and the definition of the ordering $<^*$. Q. E. D.

(3.4) *For each $n \geqslant 0$, there exists a homomorphism*

$$\varkappa^n\colon \quad H^n(X \bmod X_1, G) \to H^n(X, G)$$

which depends only on θ and such that $\pi^n \varkappa^n$ is the identity on $H^n(X \bmod X_1, G)$.

Proof. By (3.3), we may consider only the cofinal subset $\{\Sigma_\theta, <^*\}$ of $\{\Sigma, <\}$. For two θ-admissible coverings $\alpha <^* \beta$, it follows from the definition of the ordering $<^*$ that

$$\varkappa^n_\alpha h^n_{\beta\alpha 0} = h^n_{\beta\alpha} \varkappa^n_\beta.$$

Therefore, the homomorphisms $\{\varkappa^n_\alpha, \alpha \in \Sigma_\theta\}$ determine a homomorphism [5, p. 55]

$$\varkappa^n\colon \quad H^n(X \bmod X_1, G) \to H^n(X, G)$$

which depends only on θ.

Since $\pi^n_\alpha \varkappa^n_\alpha$ is the identity for each $\alpha \in \Sigma_\theta$, we conclude that $\pi^n \varkappa^n$ is the identity on $H^n(X \bmod X_1, G)$.

(3.5) ***Theorem.*** *If X_1 is a retract of X and G a division-closure group, then for each $n \geqslant 0$ we have*

$$H^n(X, G) \approx H^n(X_1, G) + H^n(X \bmod X_1, G).$$

Proof. Since $\pi^n \varkappa^n$ is the identity according to (3.4), it follows that $\varkappa^n$ is an isomorphism into and π^n is a homomorphism onto. Let

$$H^n_\theta(X, G) \subset H^n(X, G)$$

denote the image of $\varkappa^n$; then π^n maps $H^n_\theta(X, G)$ isomorphically onto $H^n(X \bmod X_1, G)$, i. e.

$$H^n_\theta(X, G) \approx H^n(X \bmod X_1, G).$$

Since π^{n+1} is onto, it follows from the exactness of the Čech homology sequence that $\partial^{n+1} = 0$, i. e. ∂^{n+1} maps the whole group $H^{n+1}(X \bmod X_1, G)$ into the zero element of $H^n(X_1, G)$. Another application of the exactness of the Čech homology sequence shows that η^n is an isomorphism into. Let

$$H^n_*(X, G) \subset H^n(X, G)$$

denote the image of η^n and the kernel of π^n, then it follows that

$$H^n_*(X, G) \approx H^n(X_1, G).$$

Since $H^n_*(X, G)$ is the kernel of π^n and since π^n maps $H^n_\theta(X, G)$ isomorphically, it follows from an elementary group-theoretic argument that $H^n(X, G)$ is the direct sum of the groups $H^n_*(X, G)$ and $H^n_\theta(X, G)$. Hence

$$H^n(X, G) \approx H^n(X_1, G) + H^n(X \bmod X_1, G).$$

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