Théorème 9. *Admettons que la fonction $\varphi(x)$ n'est pas constante et possède partout la derivée continue. Soit $a_n\beta_n > c > 0$ pour $n=1,2,\ldots$ Dans ces hypothèses, la fonction $\Phi_\eta(x)$ définie par la formule (4) est partout dépourvue de dérivée à droite pour chaque η d'un ensemble résiduel dans (φ).*

Démonstration. Soit $\max\limits_{0\leqslant x\leqslant l}|\varphi(x)-\varphi(l)|=k>0$. Soit $\xi\in(0,l)$ une valeur de x telle que $|\varphi(\xi)-\varphi(l)|=k$. Désignons par E l'ensemble des $x\in\langle 0,l\rangle$ tels que $|\varphi'(x)|\leqslant\varrho=\dfrac{k}{4l}$.

Posons:

$$h=\begin{cases} l-x & \text{si } x\in E \text{ et } |\varphi(x)-\varphi(l)|\geqslant k/2,\\ \xi-x & \text{si } x\in E,\quad |\varphi(x)-\varphi(l)|<k/2 \text{ et } 0\leqslant x<\xi,\\ \xi-x+l & \text{si } x\in E,\quad |\varphi(x)-\varphi(l)|<k/2 \text{ et } \xi<x\leqslant l,\\ l & \text{si } x\in\langle 0,l\rangle-E.\end{cases}$$

On constate facilement que, pour $x\in E$, on a

$$\left|\frac{\varphi(x+h)-\varphi(x)}{h}-\varphi'(x)\right|\geqslant\frac{k}{4l}$$

et ailleurs

$$\left|\frac{\varphi(x+h)-\varphi(x)}{h}-\varphi'(x)\right|=|\varphi'(x)|\geqslant\frac{k}{4l}.$$

Il est ainsi démontré que, pour tout $x\in\langle 0,l\rangle$, il existe un h_x tel que

$$\left|\frac{\varphi(x+h_x)-\varphi(x)}{h_x}-\varphi'(x)\right|\geqslant\frac{k}{4l}.$$

Soit: $f_n(x)=a_n\varphi(\beta_n x)$, $a=0$, $b=2l$, $a'=0$, $b'=l$, $\lambda=ck/4l$, $\delta_n=l/\beta_n$ et $h=h_y/\beta_n$ où $y=l\beta_n\dfrac{x}{l}-l\left[\beta_n\dfrac{x}{l}\right]$. On a pour $x\in\langle 0,l\rangle$

$$\left|\frac{f_n(x+h)-f_n(x)}{h}-f_n'(x)\right|=a_n\beta_n\left|\frac{\varphi(y+h_y)-\varphi(y)}{h_y}-\varphi'(y)\right|\geqslant\frac{kc}{4l}=\lambda,$$

où $0<h\leqslant\delta_n$. Il suffit maintenant d'appliquer le théorème 7.

On Hausdorff classes

By

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This paper deals with following problem: *let $\{f_n(x)\}$ be a convergent sequence of functions of a certain class $\boldsymbol{K}$; which are necessary and sufficient conditions that the limit function should also belong to $\boldsymbol{K}$?*

The answer depends of the choice of the class $\boldsymbol{K}$. If $\boldsymbol{K}$ is the class of continuous functions it is the context of the well known theorem of Arzéla, if $\boldsymbol{K}$ is the family of functions of the class α of Baire, the answer was given by Gagaeff[1]). We give a generalization of his result for a more general class of functions.

1. Let X be an arbitrary set, $\boldsymbol{H}$ a family of subsets of X satisfying the following conditions:

(1.1) *The empty set belongs to $\boldsymbol{H}$;*

(1.2) *The common part of two sets $(\boldsymbol{H})$*[2]) *is a set $(\boldsymbol{H})$;*

(1.3) *The sum of a sequence of sets $(\boldsymbol{H})$ is a set $(\boldsymbol{H})$.*

Let Y be a separable metric space with the distance (y_1,y_2). The family $\boldsymbol{H}^*$ of all functions $f(x)$ from X to Y satisfying the condition:

(1.4) *For every open set $G\subset Y$ the set $\underset{x}{E}\{f(x)\in G\}$ belongs to $\boldsymbol{H}$;*

will be called *Hausdorff class*[3]).

Theorem 1. *The necessary and sufficient condition that $f(x)$ should belong to a Hausdorff class $\boldsymbol{H}^*$ is that, for every $\varepsilon>0$ there should exist a sequence $\{X_n\}$ of sets $(\boldsymbol{H})$ such that $X=\sum\limits_{n=1}^{\infty}X_n$, $\omega(f,X_n)<\varepsilon$*[4]).

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[1]) B. Gagaeff, *Sur les suites convergentes des fonctions mesurables B*, Fund. Math. **18** (1932), p. 182-188.

[2]) We call shortly sets (or functions) belonging to a class $\boldsymbol{H}$ *sets (or functions)* $(\boldsymbol{H})$.

[3]) These classes were introduced by F. Hausdorff: *Mengenlehre*, Leipzig 1927, p. 232-270.

[4]) $\omega(f,E)$ denotes oscillation of f on E.

Theorem 2. Let $f(x)$ be the limit of a convergent sequence $\{f_n(x)\}$ of functions of a Hausdorff class $\boldsymbol{H}^*$. Then $f(x)$ belongs to $\boldsymbol{H}^*$ if and only if to every $\varepsilon>0$ there is a sequence $\{X_n\}$ of sets $(\boldsymbol{H})$ and a sequence $\{p_n\}$ of indexes such that

$$X=\sum_{n=1}^{\infty}X_n \quad \text{and} \quad (f_{p_n}(x),f(x))<\varepsilon \quad \text{for} \quad x\in X_n,\ n=1,2,\ldots$$

The proofs of these two theorems run down in a quite analoguous manner as in the paper of Gagaeff, and then we omit them.

2. Denote now by Y the set of all real numbers, and let $\boldsymbol{H}$ be a class of subsets of X satisfying (1.1), (1.2) and (1.3). By $\boldsymbol{H}^-$ and $\boldsymbol{H}^+$ we denote respectively the family of all real functions defined in X such that for every real a the sets $\underset{x}{E}\{a<f(x)\}$ or $\underset{x}{E}\{f(x)<a\}$ respectively belong to $\boldsymbol{H}$.

Theorem 3. The necessary and sufficient condition that the limit $f(x)$ of a convergent sequence $\{f_n(x)\}$ of functions belonging to a Hausdorff class $\boldsymbol{H}^*$, should belong to $\boldsymbol{H}^-$ is that, for every $\varepsilon>0$ and m there should exist a sequence $\{X_n\}$ of sets $(\boldsymbol{H})$ and a sequence of indexes $\{p_n\}$ such that

$$X=\sum_{n=1}^{\infty}X_n,\quad m<p_n, \tag{2.1}$$

$$f_{p_n}(x)-\varepsilon<f(x) \quad \text{for} \quad x\in X_n,\ n=1,2,\ldots \tag{2.2}$$

Proof. The condition is necessary. Given an $\varepsilon>0$ and m, write $p_n=m+n$, $X_n=\underset{x}{E}\{f_{p_n}(x)-\varepsilon<f(x)\}$. Since $f_n(x)\to f(x)$, we have (2.1) and (2.2). We must show only that $X_n\in\boldsymbol{H}$. Denote by $\{r_n\}$ the sequence of all rational numbers. Then

$$X_n=\underset{x}{E}\{f_{p_n}(x)-\varepsilon<f(x)\}=\sum_{m=1}^{\infty}\underset{x}{E}\{f_{p_n}(x)-\varepsilon<r_m<f(x)\}=$$
$$=\sum_{m=1}^{\infty}\underset{x}{E}\{f_{p_n}(x)-\varepsilon<r_m\}\underset{x}{E}\{r_m<f(x)\},$$

and it follows that $X_n\in\boldsymbol{H}$.

The condition is sufficient. Let a be any real number. By hypothesis there exists for any s a sequence $\{X_k^s\}_{k=1,2,\ldots}$ of sets $(\boldsymbol{H})$ and a sequence $\{p_k^s\}_{k=1,2,\ldots}$ of indexes such that

$$X=\sum_{k=1}^{\infty}X_k^s,\quad s<p_k^s \quad \text{and} \quad f_{p_k^s}(x)-\frac{1}{s}<f(x) \quad \text{for}\ x\in X_k^s,\ k=1,2,\ldots \tag{2.3}$$

By theorem 1, it follows that for any complex (k,s,t) of indexes there is a sequence $\{Z_l^{kst}\}_{l=1,2,\ldots}$ of sets $(\boldsymbol{H})$ such that

$$X=\sum_{l=1}^{\infty}Z_l^{kst},\quad \omega(f_{p_k^s},Z_l^{kst})<\frac{1}{t} \quad \text{for}\ l=1,2,\ldots \tag{2.4}$$

Put $h_l^{kst}=\inf f_{p_k^s}[Z_l^{kst}]$ [5]. Let $(\varkappa,\sigma,\tau,\lambda)$ be all these complexes (k,s,t,l) for which $a+\frac{1}{s}<h_l^{kst}$. We shall prove that

$$X_0=\underset{x}{E}\{a<f(x)\}=\sum_{\varkappa,\sigma,\tau,\lambda}X_\varkappa^\sigma Z_\lambda^{\varkappa\sigma\tau}. \tag{2.5}$$

Let $x_0\in X_0$, then there is a N such that

$$|f_k(x_0)-f(x_0)|<\delta=\tfrac{1}{3}[f(x_0)-a] \quad \text{for}\ k>N. \tag{2.6}$$

Choose $1/\sigma<\min(\delta,1/N)$. By (2.3) there is a $\varkappa$ such that $x_0\in X_\varkappa^\sigma$. Set $\tau=\sigma$; by (2.4) we can find an index λ such that $x_0\in Z_\lambda^{\varkappa\sigma\tau}$. In order to prove that x_0 belongs to the right hand side of (2.5) we must show that $a+\frac{1}{\sigma}<h_\lambda^{\varkappa\sigma\tau}$. By (2.3), (2.6), (2.4), we have

$$h_\lambda^{\varkappa\sigma\tau}=\inf f_{p_\varkappa^\sigma}[Z_\lambda^{\varkappa\sigma\tau}]\geqslant f(x_0)-[f(x_0)-f_{p_\varkappa^\sigma}(x_0)]-\omega(f_{p_\varkappa^\sigma},Z_\lambda^{\varkappa\sigma\tau})\geqslant$$
$$\geqslant f(x_0)-\frac{1}{\sigma}-\frac{1}{\tau}\geqslant f(x_0)-2\delta=f(x_0)-\frac{2}{3}[f(x_0)-a]=a+\frac{1}{3}[f(x_0)-a]>a+\frac{1}{\sigma}.$$

Conversely, if $x_0\in X_\varkappa^\sigma Z_\lambda^{\varkappa\sigma\tau}$, then by (2.2)

$$f(x_0)=[f(x_0)-f_{p_\varkappa^\sigma}(x_0)]+f_{p_\varkappa^\sigma}(x_0)\geqslant-\frac{1}{\sigma}+h_\lambda^{\varkappa\sigma\tau}>a.$$

Since the sets $X_\varkappa^\sigma$ and $Z_\lambda^{\varkappa\sigma\tau}$ belong to $\boldsymbol{H}$, we have $X_0\in\boldsymbol{H}$ q. e. d.

A theorem analoguous to the theorem 3 can be stated also for the class $\boldsymbol{H}^+$.

3. Let X be any metric space, Y a separable metric space, $\boldsymbol{H}$ the additive class α $(0\leqslant\alpha<\Omega)$ of Borel subsets of X. Then the family $\boldsymbol{H}^*$ is identical with this of functions measurable (B) of class α. In this case we obtain from theorem 2 the necessary and sufficient condition that the limit of a convergent sequence of functions of the class α should be of the same class, given by Gagaeff [6]).

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[5]) Z being any set, $f[Z]$ denotes the set of values taken by $f(x)$ for $x\in Z$.

[6]) l. c. [1]).

If Y is the set of all real numbers, then $\boldsymbol{H}^-$ or $\boldsymbol{H}^+$ is the family of functions l^α or u^α (respectively) of W. H. Young. From theorem 3 we obtain:

Theorem 4. *The necessary and sufficient condition that the limit $f(x)$ of a convergent sequence $\{f_n(x)\}$ of functions of class α should be a function l^α is, that to every $\varepsilon>0$ and m there should exist a sequence $\{X_n\}$ of sets of additive Borel class α and a sequence $\{p_n\}$ of indexes such that $X=\sum_{n=1}^{\infty}X_n$, $m<p_n$,*

$$f_{p_n}(x)-\varepsilon<f(x) \quad \text{for} \quad x\in X_n,\ n=1,2,...$$

4. Let X be any topological (Hausdorff) space, Y a separable metric space. Let $\boldsymbol{R}$ be a class of subsets of X satisfying the following conditions:

(4.1) *If $X_1\in\boldsymbol{R}$, $X_2\subset X_1$, then $X_2\in\boldsymbol{R}$;*

(4.2) *The sum of a sequence of sets $(\boldsymbol{R})$ is a set $(\boldsymbol{R})$;*

(4.3) *Sets $(\boldsymbol{R})$ do not contain open sets.*

Let $\boldsymbol{D}$ be the family of all functions from X to Y whose points of discontinuity [7]) form a set $(\boldsymbol{R})$. Denote by $\boldsymbol{H}_1$ the family of all sets which can be written in the form $G+R$ where G is open and $R\in\boldsymbol{R}$. This family satisfies the postulates (1.1), (1.2), (1.3).

The family $\boldsymbol{D}$ is the Hausdorff class corresponding to the family $\boldsymbol{H}_1$.

In fact, let K be an open sphere in Y, $X_0=\underset{x}{E}\{f(x)\in K\}$.

Denote by R the set of points of discontinuity of f, then $X_0=X_0(X-R)+X_0R$; $f(x)$ being continuous in $X-R$, we see that the set $X_0(X-R)$ consists only of inner points and so it is open; the set $R\in\boldsymbol{R}$. It is easy to show that, conversely, $\boldsymbol{H}_1^*\subset\boldsymbol{D}$.

From theorem 2 we obtain easily:

Theorem 5. *The necessary and sufficient condition that the limit $f(x)$ of a sequence $\{f_n(x)\}$ of functions $(\boldsymbol{D})$ should be a function $(\boldsymbol{D})$ is, that to every $\varepsilon>0$ there should exist a sequence $\{G_n\}$ of open sets, a set $R\in\boldsymbol{R}$ and a sequence $\{p_n\}$ of indexes such that*

$$X=R+\sum_{n=1}^{\infty}G_n, \quad \text{and} \quad (f_{p_n}(x),f(x))<\varepsilon \quad \text{for } x\in G_n,\ n=1,2,...$$

[7]) If the axiom of enumerability is not satisfied in X, continuity may be meant in the Cauchy sense.

If Y is the set of all real numbers, then $\boldsymbol{H}_1^-$ is the family of all functions which are lower semicontinuos everywhere, excepted a set $(\boldsymbol{R})$. For this class we can formulate the theorem 4 analoguously to the theorem 5.

Examples of classes $\boldsymbol{R}$ are: sets of first category in a complete metric space, enumerable sets, sets of measure 0 in an Euclidean space. In the last case we obtain from theorem 5 the necessary and sufficient condition that the bounded limit of functions integrable (R) should be integrable (R).

5. Let X be an Euclidean space, $\boldsymbol{H}_2$ the family of sets which consist exclusively of points of density; then $\boldsymbol{H}_2^*$ is the family of all approximatively continuous functions. From theorem 2 we get

Theorem 6. *The necessary and sufficient condition that the limit $f(x)$ of a convergent sequence $\{f_n(x)\}$ of approximatively continuous functions should be approximatively continuous is that to every $\varepsilon>0$ there should exist a sequence $\{X_n\}$ of sets $(\boldsymbol{H}_2)$ and a sequence $\{p_n\}$ of indexes such that*

$$X=\sum_{n=1}^{\infty}X_n \quad \text{and} \quad (f_{p_n}(x),f(x))<\varepsilon \quad \text{for} \quad x\in X_n,\ n=1,2,...$$

If Y is the set of real numbers, we obtain from theorem 3 a generalization of the preceding result, for approximatively lower semicontinuous functions.