On Cyclic Transitivity [1].

By

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INTRODUCTION.

Fundamental Concepts.

0.1. Let us denote by 1 a set which will serve as our space; the elements of the set 1 will be called points. We shall not assume that the set 1 is topologized in any way — that is, 1 is a wholly unconditioned set, unless a statement to the contrary is explicitly made.

0.2. Given in 1 a binary relation $\mathfrak{R}$, we shall write $a\mathfrak{R}b$ to express the fact that the points a and b of 1 are in the $\mathfrak{R}$-relation. Many important binary relations arising in algebra are reflexive, symmetric, and transitive — that is, $a\mathfrak{R}a$ for every point a; $a\mathfrak{R}b$ implies $b\mathfrak{R}a$ for every pair of points a, b; and $a\mathfrak{R}b\mathfrak{R}c$ implies $a\mathfrak{R}c$ for every triple of points a, b, c. On the other hand, the general theory of sets leads to binary relations — such as set inclusion — which are transitive, but are neither reflexive nor symmetric. Binary relations of the types just mentioned have been studied and applied extensively. Both of these types are transitive. In this paper we are concerned with binary relations which are reflexive and symmetric, but are not necessarily transitive; the requirement of transitivity is replaced by a weaker condition which we shall call *cyclic transitivity* (cf. 0.3). We were led to consider such binary relations by a study of the theory (developed by G. T. Whyburn, C. Kuratowski, R. L. Moore, and others) of the structure of various abstract spaces.

[1]) Parts of this paper were presented to the American Mathematical Society at its meeting in Chicago, 1939.

[2]) Le manuscrit de l'ouvrage de MM. T. Radó et P. Reichelderfer a été brulé par les Allemands et la plupart des feuilles déjà composées détruite. Il nous sont restées par hasard ces quelques feuilles que nous avons fait imprimer. Nous espérons que les Auteurs pourront nous fournir la suite de leur travail, qui sera imprimée alors dans le prochain volume de „Fundamenta Mathematicae".

La Rédaction

0.3. Given a binary relation $\mathfrak{R}$ in 1, we say that $\mathfrak{R}$ is *cyclicly* transitive if, for every finite cyclicly ordered set of distinct points $a_1, a_2, \dots, a_n$ satisfying $a_1\mathfrak{R}a_2\mathfrak{R}\dots\mathfrak{R}a_n\mathfrak{R}a_1$, we have $a_i\mathfrak{R}a_j$ for every choice of the subscripts i and j. Let $\mathfrak{R}$ be a reflexive and symmetric binary relation; if $\mathfrak{R}$ is transitive (cf. 0.2), then clearly $\mathfrak{R}$ is cyclicly transitive, but the converse is not true. Thus cyclic transitivity is an extension of ordinary transitivity — that is, an extension of one of the fundamental concepts arising in algebra. On the other hand, we shall see presently (cf. 0.4) that cyclic transitivity also arises in connection with certain fundamental concepts in topology.

0.4. Let us now assume that 1 is a set sufficiently topologized so that we can speak of mutually separated sets, hence of connected sets in 1. A basic concept in K. W. [3]) is that of *conjugate points*. Two (not necessarily distinct) points a and b are conjugate if, for every choice of the point x different from a and b, the points a and b are in the same component of $1-x$. Writing $a\mathfrak{K}_1b$ to mean that a and b are conjugate, we easily verify that $\mathfrak{K}_1$ is reflexive and symmetric, but is not generally transitive. However, if we have $a\mathfrak{K}_1b_1\mathfrak{K}_1c$, $a\mathfrak{K}_1b_2\mathfrak{K}_1c$ where b_1 and b_2 are distinct points, then it follows easily that $a\mathfrak{K}_1c$. A closer inspection of the properties of $\mathfrak{K}_1$ reveals that this latter property is but a special case of the more general property of cyclic transitivity (cf. 0.3) possessed by $\mathfrak{K}_1$. In W. [4]), G. T. Whyburn uses the concept of a nodular set as a basis for his work. A nodular set S is a connected set which is disconnected by none of its points — that is, $S-x$ is connected for every choice of x. A nodular set is called a maximal nodular set if it is a proper subset of no nodular set; a non-degenerate maximal nodular set is called a *nodule*. W. considers no binary relation explicity; however, we may associate a binary relation with the concepts in W. as follows: $a\mathfrak{K}_2b$ if either $a=b$ or a and b are on the same nodule. Inspection reveals that $\mathfrak{K}_2$ is a reflexive, symmetric, and cyclicly transitive binary relation which is not generally transitive. M. [5]) uses the notion of two points being separated by

[3]) In the sequel, K. W. refers to C. Kuratowski and G. T. Whyburn. *Sur les éléments cycliques et leurs applications*, Fundam. Math., **16** (1939), pp. 305-331.

[4]) In the sequel, W. refers to G. T. Whyburn, *On the structure of connected and connected im kleinen point sets*, Trans. Amer. Math. Soc. 1931.

[5]) In the sequel, M. refers to R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications, **13** (1932).

a third point as a basic concept. Two points a and b are said to be separated by a point x if $a \neq x \neq b$, and if $1-x$ is a sum of two mutually separated sets A and B such that $a \in A$, $b \in B$. Consider the following binary relation: $a\mathfrak{K}_3 b$ if and only if no point x different from a and b separates a and b in the sense of M. Obviously $\mathfrak{K}_3$ is reflexive and symmetric, but is not generally transitive. Again, a closer inspection shows that $\mathfrak{K}_3$ is cyclicly transitive.

In so far as we are aware, the cyclic transitivity property of the binary relations so intimately related to these three theories in topology has not been stated or used explicitly. Yet, once attention is called to this property, it is quite apparent that cyclic transitivity accounts for many of the fundamental results in the theories just mentioned.

0.5. Thus the concept of cyclic transitivity may be construed to have its origin both in algebra and topology (cf. 0.3, 0.4). The purpose of this paper is to study the concept of cyclic transitivity from this dual point of view. In Chapter I, two ways of generating all possible reflexive, symmetric, and cyclicly transitive binary relations are developed. Let there be given, for every point x in the space 1, a binary relation $\mathfrak{R}_x$ which is defined in $1-x$ and is reflexive, symmetric, and transitive there. Define in 1 a binary relation as follows: $a\mathfrak{R}(\mathfrak{R}_x)b$ if and only if $a\mathfrak{R}_x b$ for every choice of x different from a and b. The relation $\mathfrak{R}(\mathfrak{R}_x)$ is clearly reflexive, symmetric, and cyclicly transitive; conversely, every reflexive, symmetric, and cyclicly transitive binary relation can be generated in this way (cf. 1.24).

A second method of generation is obtained as follows. Let Γ be class of subsets of 1 possessing the following properties.

Property $\mathfrak{P}_1$. The empty set 0, the whole space 1, and every set consisting of a single point of 1 is in Γ.

Property $\mathfrak{P}_2$. If Ω be any subclass of Γ such that the product of all the sets in Ω is not empty, then the sum of all the sets in Ω is a set in Γ.

Given, now, any set S, we define a Γ-component of S to be a maximal set with respect to the property of being both a subset of S and a set in Γ. Clearly S is the sum of its Γ-components, and two distinct Γ-components of S have no point in common.

Next, we define a binary relation as follows: $a\mathfrak{R}(\Gamma)b$ if and only if, for every choice of x different from a and b, the points a and b are in the same Γ-component of $1-x$. This relation $\mathfrak{R}(\Gamma)$ is clearly reflexive, symmetric, and cyclicly transitive; conversely, every reflexive, symmetric, and cyclicly transitive binary relation can be generated in this way (cf. 1.7).

0.6. In fact, there are generally several classes Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating the same binary relation (cf. 1.10). A general example of this, important for the sequel, is the following one. Given any class Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$, define a class Γ' as follows: a set S belongs to Γ' if it is either one of the sets described in property $\mathfrak{P}_1$ (cf. 0.5) or if, for every choice of the point x in $1-S$, the set S is in one Γ-component of $1-x$. We shall call Γ' the *closure* of Γ. The class Γ' possesses properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generates the same binary relation as does Γ — that is, $\mathfrak{R}(\Gamma') = \mathfrak{R}(\Gamma)$, (cf. 1.9). A class Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ is said to be *closed* if $\Gamma = \Gamma'$ (cf. 1.9). A necessary and sufficient condition that a class Γ be closed is that Γ possesses the following additional

Property $\mathfrak{P}_3$. If Ω^* be any subclass of Γ then the product of all the sets in Ω^* is a set in Γ (cf. 1.9).

0.7. We next consider a system consisting of a reflexive, symmetric, and cyclicly transitive binary relation $\mathfrak{K}$, any on of the classes Γ generating $\mathfrak{K}$ (cf. 0.5), and its closure Γ' which also generates $\mathfrak{K}$ (cf. 0.6) — briefly, a system $(\mathfrak{K}, \Gamma, \Gamma')$. Such a system gives rise to a sequence of concepts and theorems which correspond closely to those arising in the theories referred to in 0.4. For details, the reader may consult the sections in Chapter I from 1.13 onward.

0.8. As we specified in 0.1, our space 1 is wholly untopologized in the usual sense. However, the introduction of the class of sets Γ (cf. 0.5) does topologize 1 in a fashion — the sets of Γ correspond to connected sets in the standard treatments (cf. 0.4, 2.1). This observation suggests the desirability of a full axiomatic treatment of the theory of the structure of a general space, using the notion of a „connected“ set as an undefined concept. Such a treatment is beyond the scope of this paper [6]).

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[6]) The earliest suggestion of this kind of axiomatic treatment seems to be due to F. Riesz, who proposed to use (essentially) the concept of *a pair of not mutually separated sets* as a primitive concept (*Stetigkeitsbegriff und abstrakte Mengenlehre*, Atti del IV Congresso Internazionale dei Matematici, **2** (Rome 1909), pp. 18-24). As far as we are aware, the concept of a *connected set* has not been used so far as a primitive concept.

Given a system $(\mathfrak{R}, \Gamma, \Gamma')$, (cf. 0.1), a cut point with respect to $(\mathfrak{R}, \Gamma, \Gamma')$ is defined to be any point x such that $1-x$ is not in Γ (cf. 1.14). Next, an end poind with respect to $(\mathfrak{R}, \Gamma, \Gamma')$ is defined to be any point x which is not a cut point and for which there is no point distinct from x in the $\mathfrak{R}$-relation to x (cf. 1.14). Clearly the notions of both a cut point and an end point depend essentially upon the choice of Γ. Further, this definition of a cut point is a direct generalization of that used for a Peano space in K.W., while this definition of an end point is a direct generalization of a characterizing property of the end point as defined in K.W. Generalizing the definition of a proper cyclic element found in K.W., we would have the

Definition. A *proper cyclic element* is the set of all points each of which is in the $\mathfrak{R}$-relation to some point p which is neither a cut point nor an end point.

This notion of a proper cyclic element would seem to depend upon Γ also. It is quite interesting to observe, then, that a proper cyclic element, as we shall define it (cf. 1.1), is clearly independent of the choice of Γ, and depends solely upon $\mathfrak{R}$. A proper cyclic element in our theory corresponds to a fundamental concept in algebra — that of a residue class. Given a reflexive, symmetric, and transitive binary relation $\mathfrak{R}$ (cf. 0.3), we consider in algebra classes which are maximal with respect to the property that any two elements of a class are in the $\mathfrak{R}$-relation. Our definition of a proper cyclic element, although worded for convenience in a slightly different form (cf. 1.1), is exactly equivalent to this: it is a non-degenerate set which is maximal with respect to the property that any two elements of the set are in the $\mathfrak{R}$ relation (cf. 1.23). It is thus quite interesting to note that, for the special choice of the system $(\mathfrak{R}, \Gamma, \Gamma')$ used in K.W., (cf. 2.1), our definition of a proper cyclic element is equivalent to theirs (cf. 2.22), but these two definitions of a proper cyclic element are not equivalent for a general system $(\mathfrak{R}, \Gamma, \Gamma')$ (cf. 1.23).

For divers reasons we modify most of the definitions used in the theories mentioned in 0.4 (cf. 1.1, 1.14, 1.17, 1.19). To illustrate, let us consider the fundamental concept of a cyclic chain $C(a,b)$ joining two distinct points a and b (cf. 1.19). In K.W., the cyclic chain $C(a,b)$ is defined to be the product of all sets A containing the points a and b, where a set A is defined to be a closed set which

contains every arc whose end points are in the set. Since, in Chapter I, we work with a space 1 untopologized, except in the sense mentioned at the beginning of this section, none of the concepts occurring in this definition are available. In spite of these definitional modifications, several important structure theorems remain valid — for example, the theorem concerning the structure of a cyclic chain in terms of proper cyclic elements (cf. 1.20, 2.23). Because of this fact, and because our definitions are, for the Peano space as considered in K.W., equivalent to those used by K.W., we felt justified in using their terminology, although our definitions for certain concepts differ considerably from theirs.

0.9. It is beyond the scope of this paper to discuss the implications of the abstract theory we develop here for the whole literature (cf. 0.4). To illustrate the way in which important structure theorems for special spaces are simple consequences of this general theory, we give in Chapter II a brief outline of the principal structure theorems for a Peano space appearing in K.W.

CHAPTER I.

Cyclicly Transitive Binary Relations.

1.1. Until further notice, $\mathfrak{R}$ will denote a binary relation which is reflexive, symmetric, and cyclicly transitive (cf. 0.3). Given $\mathfrak{R}$ we define various concepts which depend solely upon $\mathfrak{R}$. A set S is called *coherent* if it is non-degenerate and if every two points of S are in the $\mathfrak{R}$-relation — that is, $a \in S$, $b \in S$ imply $a\mathfrak{R}b$ (cf. 0.2). Clearly every non-degenerate subset of a coherent set is coherent. A set S is called *complete* if it is non-degenerate and if it contains every point which is in the $\mathfrak{R}$-relation to *two distinct* points of S — that is, $a\mathfrak{R}x\mathfrak{R}b$, $a \in S$, $b \in S$, $a \neq b$ imply $x \in S$. It is evident that the product of complete sets, if it be non-degenerate, is complete. A set is called a *proper cyclic element* if is both coherent and complete; the letter C is used consistently in the sequel to denote a proper cyclic element.

1.2. ***Theorem.*** *If two proper cyclic elements C_1 and C_2 have more than one point in common, they are identical.*

Proof. Suppose $a \in C_1 \cdot C_2$, $b \in C_1 \cdot C_2$, $a \neq b$. If $x \in C_1$ then $a\mathfrak{R}x\mathfrak{R}b$, since C_1 is coherent; hence $x \in C_2$, since C_2 is complete. Consequenty $C_1 \subset C_2$; similarly $C_2 \subset C_1$. Therefore $C_1 = C_2$.

1.3. **Theorem.** *Given two distinct points a and b satisfying $a\mathfrak{R}b$, there exists one and only one proper cyclic element containing them.*

Proof. Denote by S the (non-degenerate) set of points x such that $a\mathfrak{R}x\mathfrak{R}b$. We assert that S is coherent. Suppose $x \epsilon S$, $y \epsilon S$. We assume that x,y,a,b are distinct points (since otherwise the assertion is trivial). Then it follows from $x\mathfrak{R}a\mathfrak{R}y\mathfrak{R}b\mathfrak{R}x$ that $x\mathfrak{R}y$, since $\mathfrak{R}$ is cyclicly transitive (cf. 0.3). Thus S is coherent. Next, we assert that S is complete. Suppose $x\mathfrak{R}z\mathfrak{R}y$, $x \epsilon S$, $y \epsilon S$, $x \neq y$. We assume that z,x,y,a,b are distinct points (leaving the discussion of special cases to the reader). It follows from $z\mathfrak{R}x\mathfrak{R}a\mathfrak{R}b\mathfrak{R}y\mathfrak{R}z$ that $a\mathfrak{R}z\mathfrak{R}b$ (cf. 0.3), hence $z \epsilon S$. So S is complete. Consequently S is a proper cyclic element containing a and b (cf. 1.1). That it is the only one is a direct consequence of 1.2.

1.4. A finite ordered set of disting points $a_1, a_2, \ldots, a_n$ satisfying $a_1\mathfrak{R}a_2\mathfrak{R}\ldots\mathfrak{R}a_n$ is called a $\mathfrak{R}$-*chain* joining a_1 and a_n. Each of the points $a_1,\ldots,a_n$ is called a *vertex* of the $\mathfrak{R}$-chain — in particular, the points $a_2,\ldots,a_{n-1}$ are called *interior* vertices. Clearly an ordered pair of distinct points a,b is a $\mathfrak{R}$-chain joining a and b if and only if $a\mathfrak{R}b$. Similarly a *closed* $\mathfrak{R}$-chain is a finite cyclicly ordered set of distinct points $a_1,a_2,\ldots,a_n$ satisfying $a_1\mathfrak{R}a_2\mathfrak{R}\ldots\mathfrak{R}a_n\mathfrak{R}a_1$. Of course, in case of a closed $\mathfrak{R}$-chain there is no occasion to speak of inferior vertices, since every vertex plays the same role. The fact that $\mathfrak{R}$ is cyclicly transitive may now be expressed in the following equivalent form: any two vertices of a closed $\mathfrak{R}$-chain are in the $\mathfrak{R}$-relation (cf. 0.3). Consider two distinct points a and b such that there is a $\mathfrak{R}$-chain joining them [7]). Then, in the class of all $\mathfrak{R}$-chains joining a and b, there is obviously at least one *minimal* $\mathfrak{R}$-chain — that is, one with the smallest possible number of vertices. In fact, there exists exactly one minimal $\mathfrak{R}$-chain joining a and b (cf. 1.5).

Lemma. *If $\mathfrak{C}_0$ is a minimal $\mathfrak{R}$-chain joining two distinct points a and b, if $\mathfrak{C}$ is any $\mathfrak{R}$-chain joining a and b, then every vertex of $\mathfrak{C}_0$ is also a vertex of $\mathfrak{C}$.*

Proof. If $a\mathfrak{R}b$, this is obvious. If not, both $\mathfrak{C}_0$ and $\mathfrak{C}$ have interior vertices. Denote the vertices of $\mathfrak{C}_0$ by $a=x_0, x_1, \ldots, x_{m-1}, x_m=b$; those of $\mathfrak{C}$ by $a=y_0, y_1, \ldots, y_{n-1}, y_n=b$. If not all of the vertices of $\mathfrak{C}_0$ are amongst those of $\mathfrak{C}$ there is a first vertex in the sequence $x_1,\ldots,x_m$

[7]) Simple examples show that such a chain does not exist in general.

not in $\mathfrak{C}$ — denote it by x_{i_1+1}, $i_1 \geqslant 0$; then the vertex x_{i_1} of $\mathfrak{C}_0$ is also a vertex y_{j_1} of $\mathfrak{C}$. Let x_{i_2}, $i_2 > i_1+1$, be the next vertex in the sequence $x_{i_1+2},\ldots,x_m$ which $\mathfrak{C}_0$ and $\mathfrak{C}$ have in common — thus $x_{i_2}=y_{j_2}$. Either $j_1<j_2$ or $j_1>j_2$ — let us assume that $j_1>j_2$. Then the finite cyclicly ordered set $y_{j_1}=x_{i_1}, x_{i_1+1}, \ldots, x_{i_2}=y_{j_2}, y_{j_2+1}, \ldots, y_{j_1-1}$ satisfies

$$x_{i_1}\mathfrak{R}x_{i_1+1}\mathfrak{R}\ldots\mathfrak{R}x_{i_2}=y_{j_2}\mathfrak{R}y_{j_2+1}\mathfrak{R}\ldots\mathfrak{R}y_{j_1-1}\mathfrak{R}y_{j_1}=x_{i_1},$$

hence is a closed $\mathfrak{R}$-chain. Since $\mathfrak{R}$ is cyclicly transitive (cf. 1.1), $x_{i_1}\mathfrak{R}x_{i_2}, i_2>i_1+1$. But then the finite ordered set $a=x_0,\ldots,x_{i_1}, x_{i_2},\ldots,x_m=b$ satisfies $a\mathfrak{R}\ldots\mathfrak{R}x_{i_1}\mathfrak{R}x_{i_2}\mathfrak{R}\ldots\mathfrak{R}b$ — that is, is a $\mathfrak{R}$-chain joining a and b which has fewer vertices than $\mathfrak{C}_0$. This contradicts our hypothesis that $\mathfrak{C}_0$ is a minimal $\mathfrak{R}$-chain joining a and b. We conclude that every vertex of $\mathfrak{C}_0$ must also be a vertex of $\mathfrak{C}$.

1.5. In the preceding lemma we show that every vertex of the minimal $\mathfrak{R}$-chain $\mathfrak{C}_0$ is also a vertex of the $\mathfrak{R}$-chain $\mathfrak{C}$ — that is, we have $x_i=y_{j_i}$, $i=0,\ldots,m$. A somewhat closer discussion, left to the reader, would show that $j_i<j_{i+1}$, $i=0,\ldots,m-1$. This remark implies the

Theorem. *If a and b are two distinct points such that there exists a $\mathfrak{R}$-chain joining them, then there exists exactly one minimal $\mathfrak{R}$-chain joining a and b.*

1.6. With the given binary relation $\mathfrak{R}$ (cf. 1.1) we associate a class of sets, $\Lambda(\mathfrak{R})$, defined as follows: 0,1, every point in 1, is in $\Lambda(\mathfrak{R})$; a non-degenerate set E is in $\Lambda(\mathfrak{R})$ if, for every pair of distinct points a and b in E, there exists a $\mathfrak{R}$-chain joining a and b all of whose vertices are in E. Obviously $\Lambda(\mathfrak{R})$ possesses properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ (cf. 0.5). Moreover, we have the

Lemma. *$\Lambda(\mathfrak{R})$ possesses property $\mathfrak{P}_3$* (cf. 0.6).

Proof. Let Ω be any class of sets in $\Lambda(\mathfrak{R})$. Set $E=\prod_{S \epsilon \Omega} S$. If the set E is either 0,1, or a single point, then it is in $\Lambda(\mathfrak{R})$ by definition. If E is non-degenerate, let a and b be two distinct points in E. If $\mathfrak{C}(a,b)$ is a minimal $\mathfrak{R}$-chain joining a and b, it follows by 1.4 that $\mathfrak{C}(a,b) \subset S$ for every set $S \epsilon \Omega$. Thus $\mathfrak{C}(a,b) \subset E$. So E is in $\Lambda(\mathfrak{R})$ by definition, and $\Lambda(\mathfrak{R})$ possesses property $\mathfrak{P}_3$.

1.7. **Theorem.** *Given a binary relation $\mathfrak{R}^*$, there exists a class Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating $\mathfrak{R}^*$* (cf. 0.5) *if and only if $\mathfrak{R}^*$ is reflexive, symmetric, and cyclicly transitive.*

Proof. Firstly, suppose $\mathfrak{R}^*$ is a reflexive, symmetric, and cyclicly transitive binary relation (cf. 0.3). Then the class $\Lambda(\mathfrak{R}^*)$ possesses properties $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3$, (cf. 1.6). We assert that the class $\Lambda(\mathfrak{R}^*)$ generates $\mathfrak{R}^*$ (cf. 0.5). If $a\mathfrak{R}^*b$ then the set $a+b\in\Lambda(\mathfrak{R}^*)$ by definition (cf. 1.6). Hence if x be any point such that $a\neq x\neq b$, then the points a and b are in the same $\Lambda(\mathfrak{R}^*)$-component (cf. 0.5) of $1-x$-viz., the one containing the set $a+b$. Thus $a\mathfrak{R}\big(\Lambda(\mathfrak{R}^*)\big)b$. If, next, $a\mathfrak{R}\big(\Lambda(\mathfrak{R}^*)\big)b$, then for every point x distinct from a and b, the points a and b are in the same $\Lambda(\mathfrak{R}^*)$-component of $1-x$ — denote it by S_x. Now clearly $a+b=\prod_{x\in 1-(a+b)} S_x$. Thus, since $\Lambda(\mathfrak{R}^*)$ possesses property $\mathfrak{P}_3$, we have $a+b\in\Lambda(\mathfrak{R}^*)$, whence $a\mathfrak{R}^*b$. So $\Lambda(\mathfrak{R}^*)$ generates $\mathfrak{R}^*$ — that is, $\mathfrak{R}\big(\Lambda(\mathfrak{R}^*)\big)=\mathfrak{R}^*$. Secondly, suppose Γ is a class possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ such that $\mathfrak{R}(\Gamma)=\mathfrak{R}^*$. Then $\mathfrak{R}^*$ is reflexive and symmetric, since $\mathfrak{R}(\Gamma)$ is (cf. 0.5). Consider any closed $\mathfrak{R}^*$-chain $x_1\mathfrak{R}^*x_2\mathfrak{R}^*\ldots\mathfrak{R}^*x_n\mathfrak{R}^*x_1$. Let x_i and x_j, $i<j$, be any two vertices of this $\mathfrak{R}^*$-chain. We assert that $x_i\mathfrak{R}^*x_j$. For let x be any point distinct from both x_i and x_j; then x does not occur in at least one of the sets $A=\sum_{k=i}^{j}x_k$, $B=\sum_{k=j}^{m}x_k+\sum_{k=1}^{i}x_k$ — say $x\in 1-A$. Clearly the Γ-component (cf. 0.5) of $1-x$ containing x also contains every point of the set A — in particular, the point x_j. Thus it follows that $x_i\mathfrak{R}^*x_j$. So $\mathfrak{R}^*$ is cyclicly transitive (cf. 0.3).

1.8. Given a reflexive, symmetric, and cyclicly transitive binary relation $\mathfrak{K}$, how many classes do we have which possess properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generate $\mathfrak{K}$. We have exhibited one in 1.7 — viz., $\Lambda(\mathfrak{K})$. We shall show that generally there are several.

1.9. ***Lemma.*** *If Γ be any class possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating the binary relation $\mathfrak{R}(\Gamma)$* (cf. 0.5) *then its closure Γ'* (cf. 0.6) *possesses properties $\mathfrak{P}_1$, $\mathfrak{P}_2$ and $\mathfrak{P}_3$ and also generates $\mathfrak{R}(\Gamma)$ — that is, $\mathfrak{R}(\Gamma')=\mathfrak{R}(\Gamma)$.*

The proof is obvious.

A class Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ is said to be *closed* if it is identical with its closure Γ' (cf. 0.6).

Lemma. *A class Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ is closed if and only if it also possesses property $\mathfrak{P}_3$* (cf. 0.6).

Proof. That $\mathfrak{P}_3$ is a property of Γ if Γ is closed is a consequence of the preceding lemma. Conversely, suppose that Γ is a class possessing properties $\mathfrak{P}_1$, $\mathfrak{P}_2$ and $\mathfrak{P}_3$. We assert that $\Gamma=\Gamma'$. For, if E be any set in Γ' (cf. 0.6), then for every point $x\in 1-E$, E is in a Γ-component S_x of $1-x$. Now clearly $E=\prod_{x\in 1-E} S_x$. Thus $E\in\Gamma$, since Γ possesses property $\mathfrak{P}_3$. So $\Gamma'\subset\Gamma$, but we always have $\Gamma\subset\Gamma'$ (cf. 0.6). Consequently $\Gamma=\Gamma'$ — that is, Γ is closed.

1.10. Let Γ be any class possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ but not property $\mathfrak{P}_3$. Then Γ and its closure Γ' are two distinct classes generating the same reflexive, symmetric, and cyclicly transitive binary relation $\mathfrak{K}=\mathfrak{R}(\Gamma)=\mathfrak{R}(\Gamma')$ (cf. 1.9). Also $\Lambda(\mathfrak{K})$ (cf. 1.6) is a closed class generating the binary relation $\mathfrak{K}$ (cf. 1.8). Thus we see three generally distinct classes possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating the same binary relation $\mathfrak{K}$. The outstanding example where these three classes are generally distinct is furnished by the study of cyclic elements in a Peano space (cf. 1.12, 2.1). Note that the class Γ' depends solely upon the class Γ of which it is the closure (cf. 0.6); on the other hand, the class $\Lambda(\mathfrak{K})$ depends solely upon the binary relation $\mathfrak{K}$ (cf. 1.6). Moreover, the class $\Lambda(\mathfrak{K})$ is unique in the sense that it is the smallest closed class generating the binary relation $\mathfrak{K}$; this fact is a consequence of the

Lemma. *If Γ be any closed class — that is, a class possessing properties $\mathfrak{P}_1$, $\mathfrak{P}_2$ and $\mathfrak{P}_3$ — which generates the binary relation $\mathfrak{K}=\mathfrak{R}(\Gamma)$ then the class $\Lambda(\mathfrak{K})$* (cf. 1.6) *is a subclass of Γ.*

Proof. Suppose $E\in\Lambda(\mathfrak{K})$. If E is 0,1, or a single point then $E\in\Gamma$ by property $\mathfrak{P}_1$ (cf. 0.5). Otherwise let a and x be two distinct points of E; denote by $\mathfrak{C}(x)$ a $\mathfrak{K}$-chain joining a and x all of whose vertices are in E (cf. 1.6). Suppose y is any point in $1-\mathfrak{C}(x)$; then $\mathfrak{C}(x)$ is in one Γ-component S of $1-y$ (cf. 0.5). Now $\mathfrak{C}(x)=\prod_{y\in 1-\mathfrak{C}(x)} S_y$, so $\mathfrak{C}(x)\in\Gamma$ by property $\mathfrak{P}_3$ (cf. 0.6). But $E=\sum_{x\in E-a}\mathfrak{C}(x)$, hence $E\in\Gamma$ by property $\mathfrak{P}_2$ (cf. 0.5). Thus $\Lambda(\mathfrak{K})\subset\Gamma$.

1.11. Since there is a smallest closed class generating a given reflexive, symmetric, and cyclicly transitive binary relation $\mathfrak{K}$ (cf. 1.10) we naturally ask: Is there a smallest class possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating $\mathfrak{K}$. Is there a largest class possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating $\mathfrak{K}$? The following examples will show that generally the answers to both of these questions are in the negative.

Example 1. Let 1 be any set containing at least four distinct points; denote by a,b,c,d four distinct points in 1. Let $\mathfrak{R}$ be the obviously reflexive, symmetric, and cyclicly transitive binary relation: $x\mathfrak{R}y$ if and only if either $x=y$, or x and y are the points a and b. Let Γ_1 be the class comprised of the sets 0,1 every point in 1, and $a+b$; let Γ_2 be the class comprised of the sets 0,1, every point in 1 and $a+b+c$, $a+b+d$, $a+b+c+d$. Evidently both classes Γ_1 and Γ_2 possess properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generate $\mathfrak{R}$ (cf. 0.5). Suppose there is a smallest class Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating $\mathfrak{R}$; Γ is a subset of both Γ_1 and Γ_2, hence it is comprised of the sets 0,1 and every point in 1. But $x\mathfrak{R}(\Gamma)y$ if and only if $x=y$ (cf. 0.5). Hence Γ does not generate $\mathfrak{R}$ — that is, there is no smallest class possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating $\mathfrak{R}$.

Example 2. Let 1 be the set of points in a closed linear interval $a\leqslant t\leqslant b$. Let $\mathfrak{R}$ be the obviously reflexive, symmetric, and cyclicly transitive binary relation: $t_1\mathfrak{R}t_2$ if and only if $t_1=t_2$. Fix a point t in 1; let Γ_t be the class comprised of the sets 0,1, every point in 1, and $1-t$. Clearly Γ_t possesses properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generates $\mathfrak{R}$ (cf. 0.5). Assume that there is a largest class Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating $\mathfrak{R}$; then Γ contains every set in the class Γ_t for every point t in 1 — in particular, every set of the form $1-t$, $t\epsilon 1$. But clearly, for every pair of points t_1 and t_2 in 1, $t_1\mathfrak{R}(\Gamma)t_2$. Thus a contradiction to the assumption that Γ generates $\mathfrak{R}$ is reached. So there is no largest class possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generating $\mathfrak{R}$.

1.12. By way of illustration, we point out, for a Peano space 1, (cf. 2.1), four important classes which are generally distinct, possess properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generate the same reflexive, symmetric, and cyclicly transitive binary relation. Let Γ be the class comprised of all the connected sets in 1, including 0,1 and every point in 1. Let Γ' be the closure of Γ (cf. 0.6); the sets in Γ' will be called *quasi-connected* sets. Now Γ and Γ' possess properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and generate the same binary relation $\mathfrak{R}=\mathfrak{R}(\Gamma)=\mathfrak{R}(\Gamma')$ (cf. 1.9). Also $\Lambda(\mathfrak{R})$ is a class possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ (and $\mathfrak{P}_3$) and generating $\mathfrak{R}$ (cf. 1.10). But there is another important class which clearly possesses properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and which generates $\mathfrak{R}$-namely, the class Γ_0 comprised of 0,1, every point in 1, and every connected open set in 1.

1.13. Given a class of sets Γ possessing properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ (cf. 0.5); it gives rise to a closed class Γ' possessing properties $\mathfrak{P}_1$, $\mathfrak{P}_2$, and $\mathfrak{P}_3$ (cf. 0.6). We have seen that Γ and Γ' generate the same binary relation $\mathfrak{R}=\mathfrak{R}(\Gamma)=\mathfrak{R}(\Gamma')$, which is reflexive, symmetric, and cyclicly transitive (cf. 1.9). In the rest of this chapter we regard Γ as fixed, and introduce for study various concepts depending upon one or more of the entities $\mathfrak{R},\Gamma,\Gamma'$ (cf. 1.14, 1.15, 1.17, 1.19).

1.14. A point x is called a *cut point* if $1-x$ is a set not in Γ [8]). A point x is called an *end point* if it is not a cut point and there exists no point y distinct from x such that $y\mathfrak{R}x$ (cf. 0.8). A point x is said to *separate* two distinct points a and b if $a\neq x\neq b$ and a and b are in different Γ-components of $1-x$ (cf. 0.5). Denote by $K(a,b)$ the totality of points each of which separates a and b. The following statements are now obvious. Every point in $K(a,b)$ is a cut point. $K(a,b)$ is empty if and only if $a\mathfrak{R}b$ (cf. 0.5). If a and b are any two distinct points and if $a+b\subset E$ where $E\epsilon\Gamma'$, then $K(a,b)\subset E-(a+b)$. It follows that $a+K(a,b)+b\,\epsilon\,\Gamma'$. Finally, every coherent set (cf. 1.1) is in Γ'; thus every proper cyclic element (cf. 1.1) is in Γ' [9]).

1.15. A configuration consisting of $n>1$ distinct points $a_1,\dots,a_n$ and n sets $S_1,\dots,S_n$ satisfying the conditions: $S_i\epsilon\Gamma'$ for $i=1,\dots n$; $S_1\cdot S_2=a_2$, $S_2\cdot S_3=a_3$, $\dots$, $S_n\cdot S_1=a_1$; $S_i\cdot S_j=0$ for $|i-j|>1$ and $i,j=1,\dots,n$, is called a *closed polygon* [10]). Such a configuration is denoted by the symbol $(a_1,\dots,a_n;\ S_1,\dots,S_n)$. A closed polygon is a generalization of a closed $\mathfrak{R}$-chain (cf. 1.4) in the following sense: if the finite cyclicly ordered set of distinct points $a_1,\dots,a_n$ satisfies $a_1\mathfrak{R}a_2\mathfrak{R}\dots\mathfrak{R}a_n\mathfrak{R}a_1$ — that is, constitutes a closed $\mathfrak{R}$-chain — then the configuration $(a_1,a_2,\dots,a_n;\ a_1+a_2,a_2+a_3,\dots,a_n+a_1)$ is clearly a closed polygon. The points $a_1,\dots,a_n$ are called the *vertices* of the polygon; the sets $S_1,\dots,S_n$ are called the *sides* of the polygon. The two vertices on a side of a polygon are called *adjacent* vertices.

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[8]) It is understood that all the concepts considered in the sequel are relative to a fixed system $(\mathfrak{R},\Gamma,\Gamma')$.

[9]) In special cases (cf. Chapter II), it may happen that every proper cyclic element belongs to Γ itself, but this is not true for a general system $(\mathfrak{R},\Gamma,\Gamma')$.

[10]) We should speak, in fact, of a *simple* closed polygon.

1.16. ***Theorem.*** *If* $(a_1,\dots,a_n;\ S_1,\dots,S_n)$ *be a closed polygon, then the set* $\sum_{i=1}^{n} a_i$ *is coherent* (cf. 1.1).

Proof. Because $\mathfrak{R}$ is cyclicly transitive (cf. 1.13) it is sufficient to verify that adjacent vertices are in the $\mathfrak{R}$-relation (cf. 0.3). We shall show this for the adjacent vertices a_1 and a_n. Since $a_n+a_1\subset S_n$, it follows that $K(a_1,a_n)\subset S_n-(a_1+a_n)$ (cf. 1.14). Similarly, since $a_1+a_n\subset\sum_{i=1}^{n-1} S_i$ we have $K(a_1,a_n)\subset\sum_{i=1}^{n-1} S_i-(a_1+a_n)$. Consequently $K(a_1,a_n)$ is empty — that is, $a_1\mathfrak{R}a_n$ (cf. 1.14).

Corollary 1. *If the* $n>2$ *distinct points* $a_1,\dots,a_n$, *together with the set* $S\in\Gamma'$, *satisfy the relations* $a_i\mathfrak{R}a_{i+1}$, $i=1,\dots,n-1$, *and* $S\cdot\sum_{i=1}^{n} a_i=a_n+a_1$, *then the set* $\sum_{i=1}^{n} a_i$ *is coherent.*

Proof. Consider the closed polygon

$$(a_1,a_2,\dots,a_n;\ a_1+a_2,a_2+a_3,\dots,a_{n-1}+a_n,S),$$

and apply the above result.

Corollary 2. *If* S_1 *and* S_2 *are two sets satisfying* $S_1\in\Gamma'$ $S_2\in\Gamma'$, $S_1\neq 0\neq S_2$, $S_1\cdot S_2=0$, *there is at most one proper cyclic element* C *such that* $S_1\cdot C\neq 0\neq S_2\cdot C$.

Proof. Suppose C_1 and C_2 are two distinct proper cyclic elements such that $S_1\cdot C_j\neq 0\neq S_2\cdot C_j$ for $j=1,2$. Since C_1 and C_2 have at most one point in common (cf. 1.2) we may choose four points $a_{ij}\in S_i\cdot C_j$ for $i,j=1,2$ such that at least three are distinct. Now $a_{1j}\mathfrak{R}a_{2j}$ for $j=1,2$, since the proper cyclic elements C_j are coherent (cf. 1.1). Suppose all four points a_{ij} are distinct. Then the configuration $(a_{11},a_{12},a_{22},a_{21};\ S_1,\ a_{12}+a_{22},\ S_2,\ a_{21}+a_{11})$ is a closed polygon (cf. 1.15), hence the set $\sum_{i,j=1}^{2} a_{ij}$ is coherent. Since $a_{11}\mathfrak{R}a_{i2}\mathfrak{R}a_{21}$ for $i=1,2$ and $a_{12}\neq a_{22}$, it follows that $a_{i1}\in C_2$ for $i=1,2$, because C_2 is complete. Since $a_{11}\neq a_{21}$ it follows by 1.2 that $C_1=C_2$, contrary to assumption. So the theorem is established, assuming that the four points a_{ij} are distinct; we leave the treatment of the other cases for the reader.

Corollary 3. *If* x *is not a cut point* (cf. 1.14), *then there exists at most one proper cyclic element containing* x.

Proof. The sets $S_1=x$, $S_2=1-x$ satisfy the conditions in corollary 2, whence the result follows.

Definition. Let S be any non-degenerate set; denote by $\psi(S)$ the class of all proper cyclic elements C each of which has at least two distinct points in common with S — that is, for which $S\cdot C$ is non-degenerate.

Corollary 4. *Given a non-degenerate set* S *and a non-empty set* S^* *satisfying* $S\in\Gamma'$, $S^*\in\Gamma'$, $S\cdot S^*=0$, *there exists at most one proper cyclic element* $C\in\psi(S)$ *such that* $S^*\cdot C\neq 0$.

The proof follows immediately from the preceding corollary 2.

Corollary 5. *Given two distinct proper cyclic elements* C_1 *and* C_2 *in* $\psi(S)$, *where* S *is any non-degenerate set in* Γ', *either* C_1 *and* C_2 *have no points in common or else they have a single point of* S *in common, which is a cut point.*

Proof. By 1.2 the distinct proper cyclic elements C_1 and C_2 have at most one point in common. Suppose C_1 and C_2 have the point x in common. Assume that $x\in 1-S$. Set $S^*=x$; clearly S and S^* satisfy the hypotheses in corollary 4, hence $C_1=C_2$, contrary to hypothesis. Therefore $x\in S$. It follows from corollary 3 that x is a cut point.

1.17. We introduce further notions needed in the sequel. A set which is complete (cf. 1.1) and belongs to the class Γ' is called an $\mathfrak{H}$-*set*. The letter H is used consistently to denote an $\mathfrak{H}$-set By definition every $\mathfrak{H}$-set is non-degenerate. Clearly the product of $\mathfrak{H}$-sets, if it be non-degenerate, is an $\mathfrak{H}$-set (cf. 1.1, 1.13). If S be any non-degenerate set, the product of all $\mathfrak{H}$-sets containing S is evidently the smallest $\mathfrak{H}$-set containing S — denote it by $H(S)$. We now state a property of $\mathfrak{H}$-sets needed in the next chapter.

Lemma. *If* H_1 *and* H_2 *be two* $\mathfrak{H}$-*sets such that* $H_1\cdot H_2$ *consists of a single point* x, *then* H_1+H_2 *is an* $\mathfrak{H}$-*set.*

Proof. Firstly, $H_1+H_2\in\Gamma'$ by $\mathfrak{P}_2$ (cf. 0.5). Next, we assert that H_1+H_2 is complete. Let y be any point for which there are two points x_1 and x_2 in H_1+H_2 such that $x_1\mathfrak{R}y\mathfrak{R}x_2$ — suppose $x_1\in H_1$. Assume that $y\in 1-(H_1+H_2)$; then clearly we have $x_1\in H_1-H_2$, $x_2\in H_2-H_1$ (cf. 1.1). But obviously the configuration

$$(y,x_1,x,x_2;\ y+x_1H_1,H_2,x_2+y)$$

is a closed polygon (cf. 1.15), hence by 1.16, the set $y+x_1+x+x_2$ is coherent. Thus $x_1\mathfrak{R}y\mathfrak{R}x$ (cf. 1.1), and so $y\in H$, since H_1 is complete. This contradicts our assumption that $y\in 1-(H_1+H_2)$. Therefore H_1+H_2 is an $\mathfrak{H}$-set.

Extending this result, we have the

Corollary. *If $H_1,\dots,H_n$ be any finite number of $\mathfrak{H}$-sets such that every H_k for $k=2,\dots,n$ has exactly one point in common with the set $\sum_{j=1}^{k-1} H_j$, then $\sum_{j=1}^{n} H_j$ is an $\mathfrak{H}$-set.*

1.18. Regarding the structure of $H(S)$, we have the

Theorem. *If S be any non-degenerate set in Γ', then* $H(S)=S+\sum_{C\epsilon\psi(S)} C$ (cf. 1.16).

The proof of this theorem will follow easily from two facts concerning the set

$$E=S+\sum_{C\epsilon\psi(S)} C. \qquad (*)$$

Lemma 1. *The set E of $(*)$ is an $\mathfrak{H}$-set.*

Proof. Firstly, $E=\sum_{C\epsilon\psi(S)}(S+C)$, so clearly $E\epsilon\Gamma'$ by $\mathfrak{P}_2$ (cf. 0.5). Secondly, we assert that E is complete. Let x be any point for which there are two distinct points x_1 and x_2 in E such that $x_1\mathfrak{R}x\mathfrak{R}x_2$. Assume that x is not in E. Three cases arise: I) neither of the points x_1 and x_2 is in S; II) just one of the points x_1 and x_2 is in S; III) both of the points x_1 and x_2 are in S. In case I the point x_i is in a proper cyclic element $C_i\epsilon\psi(S)$ — that is, having two distinct points x_{i1} and x_{i2} in common with S — this is so for $i=1,2$. Clearly we may assume that $x_{11}\neq x_{22}$. Now $x_i\mathfrak{R}x_{ij}$ for $i,j=1,2$, since the proper cyclic element C_i is coherent (cf. 1.1). So the distinct points x_{11},x_1,x,x_2,x_{22}, together with the set S, clearly satisfy the hypotheses of corollary 1 in 1.16. Consequently $x_{11}\mathfrak{R}x\mathfrak{R}x_1$. Since $x_{11}\neq x_1$ and C_1 is complete, it follows that $x\epsilon C_1\subset E$, contrary to assumption. Thus the assertion is established for case I; the discussion of cases II and III is left to the reader.

Lemma 2. *Let S and E have meanings as above. If H be any $\mathfrak{H}$-set such that $H\cdot E$ is non-degenerate, then $H\cdot S$ is non-degenerate.*

Proof. Assume, contrariwise, that $H\cdot S$ is degenerate. Then we have two cases to consider:

(I) the sets H and S have no point in common;

(II) the sets H and S have a single point x in common.

In case (I) the sets S and $S^*=H$ satisfy the hypotheses of corollary 4 in 1.16; since $H\cdot E$ is non-degenerate, it follows that there exists a unique proper cyclic element $C\epsilon\psi(S)$ such that $C\cdot H$ is non-degenerate. Thus $C\subset H$ since C is coherent (cf. 1.1) and H is complete (cf. 1.17). Consequently $H\cdot S$ is non-degenerate because $H\cdot C$ is; this contradicts our assumption.

In case (II) there exists a proper cyclic element $C\epsilon\psi(S)$ such that $C\cdot H$ contains a point $y\neq x$, since $H\cdot E$ is non-degenerate. $C\cdot S$ contains a point $z\neq x$ since $C\cdot S$ in non-degenerate (cf. 1.16). Now clearly the configuration $(x,y,z;H,y+z,S)$ is a closed polygon (cf. 1.15), hence $x\mathfrak{R}z\mathfrak{R}y$ (cf. 1.16). Since $x\neq y$ it follows that $z\epsilon H$, because H is complete — that is, $H\cdot S$ is non-degenerate, contrary to assumption. So the assertion in lemma 2 is verified.

Obviously we have the

Corollary. *Let S and E have meanings as above. If C be any proper cyclic element such that $C\cdot E$ is non-degenerate, then $C\epsilon\psi(S)$.*

Proof of the theorem stated at the beginning of 1.18. Since E is an $\mathfrak{H}$-set containing S, surely $H(S)\subset E$ (cf. 1.17). But because $H(S)$ is complete, $E\subset H(S)$. Thus $H(S)=E=S+\sum_{C\epsilon\psi(S)} C$.

Replacing E of $(*)$ by $H(S)$ in lemma 2 and its corollary, we have the

Theorem. *Let S be any non-degenerate set in Γ'. If H be any $\mathfrak{H}$-set such that $H\cdot H(S)$ is non-degenerate, then $H\cdot S$ is non-degenerate. If C be any proper cyclic element such that $C\cdot H(S)$ is non-degenerate, then $C\epsilon\psi(S)$.*

1.19. If a and b be two distinct points, the $\mathfrak{H}$-set $H(a+b)$ is called the *cyclic chain* joining a and b (cf. 1.17). We denote this cyclic chain by $C(a,b)$.

Theorem. *For any two distinct points a and b, we have*

$$C(a,b)=H\big(a+K(a,b)+b\big).$$

Proof. By definition, $C(a,b)=H(a+b)$. Since $a+b\subset a+K(a,b)+b$ it follows that $H(a+b)\subset H\big(a+K(a,b)+b\big)$ (cf. 1.17). However, $a+K(a,b)+b\subset H(a+b)$ since $a+b\subset H(a+b)$ and $H(a+b)\epsilon\Gamma'$ (cf. 1.14); consequently $H\big(a+K(a,b)+b\big)\subset H(a+b)$. Thus

$$C(a,b)=H\big(a+K(a,b)+b\big).$$