On a problem of S. Ulam concerning Cartesian products.

By

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1. In Fundamenta Mathematicae 20 (1933), p. 285, the following problem was proposed by S. Ulam:

„*Soient A et B deux espaces topologiques et A^2 et B^2 respectivement leur carrés*...[1]). *Est-il vrai que si A^2 et B^2 sont homéomorphes, A et B le sont aussi?*".

It will be shown below that the answer to this question is in the negative; moreover the topological spaces which will be considered for this purpose are not in the least pathological (as would be allowed according to the statement of the problem) but are actually combinatorial manifolds (with boundary) of dimension four.

The Ulam problem belongs to a class of problems, problems which are concerned with the „arithmetic" properties of the Cartesian product[2]). One such problem is concerned with the failure of the cancellation law. The nature of this failure, which is basic for our construction, is considered in section 5. In the final section Ulam's conjecture is shown to be true for manifolds of dimension less than three.

[1]) To shorten notations I have followed Ulam in writing X^n for the n-fold Cartesian product $X\times X\times\ldots\times X$ of the topological space X with itself. I shall also take the liberty of using the equality sign between the symbols for topological spaces to indicate the existence of a homeomorphism of one onto the other. Throughout this note the symbol E will denote the unit interval $0\leqslant x\leqslant 1$. The boundary of a manifold X will be designated by $\dot{X}$.

[2]) Of course Ulam's problem and problems of a similar nature arise in other connections; as one instance, a recent note by P. F. Kelly. (Bull. Am. Math. Soc. **51** (1945), p. 960) is concerned with Ulam's problem for the Cartesian products of metric spaces using isometry as the basis of classification.

2. The main result is the following:

(1) *There exist bounded 4-dimensional combinatorial manifolds A and B such that*[3]) $A^2=B^2$ *although* $A\neq B$.

First of all let us recall[4]) that a lens-space $L(p,q)$ is defined for any pair of relatively prime integers p and q, and that $L(p,q)$ is a closed 3-dimensional manifold. A good way to describe $L(p,q)$ is as follows[5]): Consider an anchor ring T, that is to say, the bounded 3-dimensional manifold determined by the subset $(\varrho-2)^2+z^2\leqslant 1$ (in cylindrical coordinates) of Euclidean 3-space, and denote by a and b the simple closed curves on $\dot{T}$ which correspond respectively to the circles $(\varrho-2)^2+z^2=1$, $\theta=0$ and $z=0$, $\varrho=3$. A torus knot of type (p,q) on $\dot{T}$ corresponds to the skew curve which is given by the equations $\varrho-2=\cos\frac{q\theta}{p}$, $z=\sin\frac{q\theta}{p}$ [and represents, in the sense of path-multiplication, the path-class $a^q b^p$. A lens space $L(p,q)$ may be obtained from two anchor rings T_1 and T_2 by matching their toral surfaces $\dot{T}_1$ and $\dot{T}_2$ in such a way that the torus knot $a_1^q b_1^p$ on $\dot{T}_1$ is matched with the canonical curve a_2 on $\dot{T}_2$.

From a lens-space $L(p,q)$ a bounded 3-dimensional manifold $M(p,q)$ is obtained by removing the interior of some 3-cell. If we think of this 3-cell as corresponding to the set $(\varrho-2)^2+z^2\leqslant 1$, $\varepsilon\leqslant\theta\leqslant 2\pi-\varepsilon$, then we see that $M(p,q)$ may be obtained from an anchor ring T and a cylinder C by matching the lateral surface of C with a ribbon-shaped neighborhood of the torus knot $a^q b^p$ on the toral surface $\dot{T}$.

(2) *The four dimensional manifolds A and B mentioned in* (1) *are the respective Cartesian products $M(p,q_1)\times E$ and $M(p,q_2)\times E$ where E is a line segment, p is a prime congruent to 1 modulo 4, q_1 is a quadratic residue modulo p and q_2 is a quadratic non-residue modulo p.* (To be specific one could choose $p=5$, $q_1=1$ and $q_2=2$).

The proof of (1) consists in the proofs of the two statements $A^2=B^2$ and $A\neq B$. These will be the concern of the next two sections.

[3]) Actually not only is $A^2=B^2$ but $A^2=A\times B=B^2$. This follows easily from § 4.

[4]) H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig (1934), pp. 210, 215 and 279.

[5]) This is the description of $L(p,q)$ used by J. W. Alexander in his classic paper on the lens spaces (Trans. Am. Math. Soc. **20** (1919), p. 339).

3. First we prove that $A \neq B$. Since the boundary of the 4-dimensional manifold $M(p,q) \times E$ is

$$(M \times E)^{\cdot} = \dot{M} \times E + M \times \dot{E},$$

it is easily seen to be just the double [6] (Verdoppelung) of the bounded 3-dimensional manifold M, i. e., the 3-dimensional manifold without boundary obtained from two similarly oriented copies of M by identifying their boundary 2-spheres in such a way that the orientations of these two 2-spheres agree. Let us denote the double of $M(p,q)$ by $N(p,q)$. In order to show that $A \neq B$ it is clearly sufficient to show that

(3) *if* p, q_1 *and* q_2 *are as in* (2) *then* $N(p,q_1) \neq N(p,q_2)$.

Before giving a proof of (3) let it be observed that, although $L(p,q_1)$ and $L(p,q_2)$ are distinguishable by their linking invariants [7], the manifolds $N(p,q_1)$ and $N(p,q_2)$ can not be so distinguished. In fact, the linking invariant of $N(p,q)$ is the Legendre symbol $\left(\frac{d}{p}\right)$ where d is the determinant of the matrix

$$\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$

which is always $+1$ since q^2 is always a quadratic residue.

To prove (3) we make use of the linking numbers *in relation to the fundamental group*.

Given any closed 3-dimensional manifold, consider its fundamental group G, its 1-dimensional homology group H and the natural homomorphism θ of G onto H (whose kernel is the commutator subgroup of G). As is well known, the elements of finite order in H constitute a subgroup H' called the torsion group; for every element h of H' there is defined a self-linking number [7] (Eigenverschlingungszahl) $\lambda(h)$ whose sign depends on the orientation of the given manifold and which is only determined modulo 1. Those elements g of G which are mapped by θ into the torsion group H' constitute a subgroup G' of G which is characteristic (i. e., G' is sent into itself by every automorphism of G); the concept of self-

[6]) Seifert and Threlfall, loc. cit., p. 129.

[7]) H. Seifert, *Verschlingungsinvarianten*, Sitz. Ber. Preuss. Akad. d. Wiss. Berlin (1933), p. 811.

linking may be extended to the elements g of G' by the definition $\lambda(g) = \lambda(\theta(g))$. Now the elements of finite order in G all belong to G' and form a characteristic complex G''. Thus, since G'' is sent into itself by every automorphism of G, the set Σ of self-linking numbers $\{\pm\lambda(g)\}$, g ranging over G'', is a topological invariant. The fundamental group G_i of $N(p,q_i)$, $(i=1,2)$, is the free product [8]) of two cyclic groups of order p, so that this group is defined by generators x and y satisfying the relations $x^p = y^p = 1$. The generators x and y may be so specified that $\lambda(x) \equiv \lambda(y) \equiv \pm \frac{q_i}{p} \pmod 1$. It is known [9]) that the only elements of finite order in G_i are the powers of x and the powers of y and their transforms. Hence Σ is the set $\left\{\pm \frac{r^2 q_i}{p} \pmod 1\right\}$ where r takes on the values $0,1,\ldots,\frac{p-1}{2}$. Since $p \equiv 1 \pmod 4$ the numbers $\pm r^2 q_1$ are all quadratic residues $(\text{mod } p)$ and the numbers $\pm r^2 q_2$ are all quadratic non-residues $(\text{mod } p)$. Hence $N(p,q_1)$ cannot be homeomorphic to $N(p,q_2)$. It may be of some interest to note that our condition on p,q_1,q_2 is more restrictive than the classical condition $q_1 \equiv \pm r^2 q_2 \pmod p$. For example, our argument does not apply to distinguish between $N(7,1)$ and $N(7,3)$ although the linking invariant distinguishes between $L(7,1)$ and $L(7,3)$.

4. The proof that $A^2 = B^2$ makes no use of the special properties of p, q_1, and q_2 other than that q_1 and q_2 are relatively prime to p. Before proving that $A^2 = B^2$ let us first remark that J. H. C. Whitehead has proved [10]), as a corollary to a general theory, that $M(p,q_1) \times E^6 = M(p,q_2) \times E^6$, and that this fact, together with the result of the previous section, already implies a negative solution of Ulam's problem. (In fact, the Whitehead result was my point of departure in the present investigation). For there must exist an integer $1 \leqslant k \leqslant 3$ such that

$$M(p,q_1) \times E^k \neq M(p,q_2) \times E^k$$

and

$$M(p,q_1) \times E^{2k} = M(p,q_2) \times E^{2k}.$$

[8]) Seifert and Threlfall, loc. cit., p. 177, § 52. This theorem, applied to the two copies of M, shows that the group of N is the free product of the groups of the two copies, since their intersection is a 2-sphere and therefore has a trivial group. Cf. Ibid. § 85.

[9]) O. Schreier, Hamb. Abhdl. **3**, (1924) p. 167.

[10]) Annals of Math. **41** (1940), p. 825.

Then, writing $\mathcal{A}=M(p,q_1)\times E^k$ and $\mathcal{B}=M(p,q_2)\times E^k$ (so that $\mathcal{A}$ and $\mathcal{B}$ are topologically inequivalent $(k+3)$-dimensional bounded manifolds) it follows immediately that

$$\begin{aligned}\mathcal{A}^2 &= M(p,q_1)\times(M(p,q_1)\times E^{2k})\\ &= M(p,q_1)\times(M(p,q_2)\times E^{2k})\\ &= M(p,q_2)\times(M(p,q_1)\times E^{2k})\\ &= M(p,q_2)\times(M(p,q_2)+E^{2k})\\ &= \mathcal{B}^2.\end{aligned}$$

Although this argument disposes of the Ulam problem, it is not completely satisfactory because it does not specifically determine the nature of the counterexample but merely shows that not all three pairs (A,B), $(A\times E, B\times E)$ and $(A\times E^2, B\times E^2)$ can fail to be counterexamples. For this reason, and because the general theory utilized by Whitehead is rather complicated, I prefer to give a direct proof that $A^2=B^2$.

The proof that $\mathcal{A}^2=\mathcal{B}^2$, exhibited above, with $k=1$, shows that it is sufficient for this purpose to prove that $A\times E=B\times E$, i. e., that $M(p,q_1)\times E^2=M(p,q_2)\times E^2$. To demonstrate this fact we return to the description of $M(p,q)$ as the manifold obtained from an anchor ring T and a cylinder C by matching the lateral surface of C with a ribbon-shaped neighborhood of the torus knot a^qb^p on $\dot{T}$. The manifold thus obtained is not imbeddable in 3-dimensional space. However, the part U of the manifold which lies in a neighborhood of T can be imbedded in 3-dimensional space and can be visualized as follows: Denoting by T the actual set $(\varrho-2)^2+z^2\leqslant 1$, the set U consists of T and a 3-dimensional neighborhood of a^qb^p in $\overline{R^3-T}$. Let us denote as the θ_0-cross section K_{θ_0} of U the intersection of U with the half-plane $\theta=\theta_0$. Such a cross section consists of a 2-cell W and p 2-cells $V_1,\dots,V_p$ attached to W at regular intervals about the circle $\dot{W}$. Of course the manifold U could be obtained from a cylinder standing on K as base by identifying the top with the bottom with a twist through angle $\frac{2\pi q}{p}$. The 5-dimensional manifold $U\times E^2=U\times[0,1]\times[0,1]$ has as cross section $K\times[0,1]\times[0,1]$ and this is homeomorphic to the subset

$$W\times[0,1]\times[0,1]+\sum_{i=1}^{p}V_i\times[0,\varepsilon]\times[0,\varepsilon],$$

where $0<\varepsilon<1$. Furthermore, these homeomorphisms can be applied simultaneously to all cross sections K_θ $(0\leqslant\theta<2\pi)$ and can be extended to define a homeomorphism of $M\times E^2$ into a subset of the Euclidean space of appropriate dimension. Now the intersection of $W\times[0,1]^2$ with $\sum V_i\times[0,\varepsilon]^2$ is just a 3-dimensional neighborhood of p points in the 3-sphere $(W\times E^2)^{\cdot}$; thus the intersection of $T\times E^2$ with $C\times[0,\varepsilon]^2$ is just a 4-dimensional neighborhood of the torus knot a^qb^p in the product $(T\times E^2)^{\cdot}$ of the 3-sphere $(W\times E^2)^{\cdot}$ and a circle. All such knots for which p has the same value are homotopic in the 4-dimensional manifold $(T\times E^2)^{\cdot}$ and therefore, since the dimension of this manifold is greater than three, an isotopy of $T\times E^2$ can be found which transforms the neighborhood of $a^{q_1}b^p$ into the neighborhood of $a^{q_2}b^p$. This isotopy can be extended to $C\times E^2$ and this shows that $M(p,q_1)\times E^2$ is homeomorphic to $M(p,q_2)\times E^2$.

5. The finite complexes, or rather the classes of topologically equivalent finite complexes, together with the operation $\times$ of Cartesian multiplication and the operation $+$ of union of disjoint copies, constitute an extension of the set of natural numbers. In fact, the correspondence between natural numbers and finite 0-dimensional complexes under which the natural number n corresponds to the 0-dimensional complex consisting of n points, the multiplication corresponds to Cartesian multiplication and the addition corresponds to union is faithful in every respect. Of course, one would not expect the extended system to satisfy all the rules of arithmetic. As a matter of fact the fundamental theorem of arithmetic, i. e., the uniqueness of factorization into prime factors, fails to hold, in general; the problem of Ulam may be considered as belonging to a class of problems relating to this failure.

It was shown by Borsuk[11]) that factorization into prime factors is unique provided that the prime factors are of dimensionality less than 2 and that factorization of closed manifolds into prime factors is unique when the factors are of dimensionality less than 3. On the other hand, J. H. C. Whitehead showed[10]) by examples that the cancellation law: $A\times X=B\times X\rightarrow A=B$ fails to hold even for 2-dimensional A and B and 1-dimensional X, thus

[11]) Fund. Math. **31** (1938), p. 137, and **33** (1945), p. 273. Cf. also K. Borsuk, *O rozkładzie rozmaitości na iloczyny kartezjańskie*, Tow. Nauk. Warsz. (Comptes rendus, Section III), 33-38 (1946), p. 1 and M. Wojdysławski, Mat. Sbornik **18** (60) (1946), pp. 29-40.

highlighting the sharpness of the Borsuk results. The simplest example of the failure of the cancellation law was found by the present author shortly after the appearance of the Whitehead paper but has not been hitherto published. Let A be the circular ring $2\leqslant x^2+y^2\leqslant 3$ together with the spines $-4\leqslant x\leqslant -3$, $y=0$ and $3\leqslant x\leqslant 4$, $y=0$; let B be the same circular ring together with the spines $-2\leqslant x\leqslant -1$, $y=0$ and $3\leqslant x\leqslant 4$, $y=0$ (see figure 1). Denoting, as before, by E the

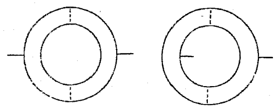

Fig. 1

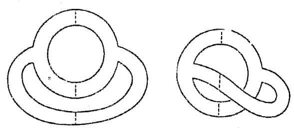

Fig. 2

unit interval, it is trivial to verify that $A\neq B$ and $A\times E=B\times E$. This example is really just a simplification of the example: $A=$sphere with three holes, $B=$ torus with one hole, given by Whitehead [10]) (see figure 2), and this example is an illustration of a theorem (trivial [12]) but not previously published).

In order that two (bounded) 2-dimensional manifolds A and B be such that $A\times E=B\times E$ it is necessary and sufficient that they have the same Euler characteristic and are either both orientable or both non-orientable.

The examples of this section also illustrate a sufficiency condition which makes it easy to construct many such examples.

Suppose that A can be decomposed into two pieces A_1 and A_2 such that $A_1\cdot A_2$ is a finite set of disjoint finite dimensional cells such that a closed neighborhood of each such cell σ^n in A is the union of cells σ_1^{n+1} in A_1 and σ_2^{n+1} in A_2 with $\sigma^n=\sigma_1^{n+1}\cdot\sigma_2^{n+1}$ and suppose that B has a decomposition B_1+B_2 with the same properties. Suppose furthermore that there exist homeomorphisms f_1 of A_1 onto B_1 and f_2 of A_2 onto B_2 such that $f_i(A_1\cdot A_2)=B_1\cdot B_2$ and $f_1|\sigma=f_2 j|\sigma$ for any cell σ of $A_1\cdot A_2$ where j is an orientation-reversing homeomorphism of σ onto itself. Then $A\times E=B\times E$.

The simple proof of this sufficiency theorem, which will be omitted, depends on the fact that if j_1 and j_2 are orientation-reversing auto-homeomorphisms of m- and n-dimensional cells then the auto-homeomorphism $j_1\times j_2$ of the product $(m+n)$-dimensional cell is isotopic to the identity.

12) This theorem is easily proved by making use of the sufficiency condition immediately below.

This detailed discussion of the failure of the cancellation law is occasioned by the fact (cf. § 4) that this failure contains the germ of the failure of Ulam's conjecture. The essential rôle played in all known examples by the boundaries suggests the following questions: (1) If A and B are closed manifolds with $A\neq B$, then is $A\times E\neq B\times E$? (2) If A and B are closed manifolds with $A\neq B$, then is $A^2\neq B^2$? (3) If A and B are topological spaces with $A\neq B$, then, denoting the circle by S^1, is $A\times S^1\neq B\times S^1$?

In every known example the complexes A and B belong to the same homotopy type. This suggests the problem of investigating the system of homotopy types of finite complexes with regard to the validity or non-validity of the cancellation law, et cetera.

6. It is natural to inquire whether four is the least dimension number possible for the construction of a counter example to Ulam's conjecture. Whether or not a counter example A,B can be found with A and B of dimension three is not known to me. However we can prove the following: *If A and B are compact manifolds, with or without boundary, of dimension $\leqslant 2$, then $A^2=B^2$ implies that $A=B$.*

That this is true for manifolds of dimension 1 or for closed manifolds of dimension 2 follows from the Borsuk result [11]), and can also be proved directly from the Künneth formula [13]).

Since $A^2=B^2$ implies that A and B have the same dimension, are either both bounded or both closed, and are either both orientable or both non-orientable, the proof reduces to showing that $A^2=B^2$ implies $A=B$ if A and B are bounded surfaces either both orientable or both non-orientable.

To demonstrate these two facts we make use of the Betti polynomial [14])

$$P_X(x)=\sum_{i=1}^{\dim X} p_i(X)x^i,$$

13) P. Alexandroff and H. Hopf, *Topologie*, Berlin (1935), p. 309.

14) S. Lefschetz, *Algebraic Topology*, New York (1942), p. 104. Betti polynomial seems a better name than the customary „Poincaré polynomial".

where $p_i(x)$ is the i^{th} Betti number of the complex X. If X is a manifold with boundary $\dot{X}$, then the boundary of X^2 is

$$(X^2)^{\cdot} = X \times \dot{X} + \dot{X} \times X,$$

where $X \times \dot{X}$ and $\dot{X} \times X$ intersect in the sub-manifold $\dot{X} \times \dot{X}$. Accordingly the Betti polynomial of $(X^2)^{\cdot}$ may be calculated from the formulas of Künneth and Mayer-Vietoris.

The Künneth formula[13]) for Betti numbers is compactly expressed as the following relationship between Betti polynomials:

$$P_{Y \times Z}(x) = P_Y(x) P_Z(x).$$

The Mayer-Vietoris formula[15]) is expressed by the formula

$$P_{Y+Z}(x) = P_Y(x) + P_Z(x) - P_{Y \cdot Z}(x) + (1+x)P^*(x),$$

where

$$P^*(x) = \sum_{i=0}^{\dim Y \cdot Z} p_i^* x^i,$$

and p_i^* is the rank of the intersection of the kernels of the two injection homomorphisms

$$H_i(Y \cdot Z) \to H_i(Y), \quad H_i(Y \cdot Z) \to H_i(Z).$$

Applied to the case under consideration, a combination of these two formulas yields

$$P_{(X^2)^{\cdot}} = (2P_X - P_{\dot{X}})P_{\dot{X}} + (1+x)P^2_{X,\dot{X}},$$

where

$$P_{X,\dot{X}}(x) = \sum_{i=0}^{\dim X} p_i(X, \dot{X}) x^i,$$

and $p_i(X, \dot{X})$ is the rank of the kernel of the injection homomorphism

$$H_i(\dot{X}) \to H_i(X).$$

Now if X is an orientable surface of genus h with $r > 0$ boundaries, then

$$P_X(x) = 1 + (2h + r - 1)x,$$
$$P_{\dot{X}}(x) = r(1+x), \quad P_{X,\dot{X}}(x) = (r-1) + x,$$

[15]) Alexandroff and Hopf, loc. cit., p. 299.

and therefore

$$P_{(X^2)^{\cdot}}(x) = 1 + (4hr + r^2 - 1)x + (4hr + r^2 - 1)x^2 + x^3,$$
$$P_{X^2}(x) = 1 + 2(2h + r - 1)x + (2h + r - 1)^2 x^2.$$

Consequently, if $A^2 = B^2$, where A is an orientable surface of genus h_A with r_A boundaries and B is an orientable surface of genus h_B with r_B boundaries, then we must have

$$(4h_A + r_A)r_A = (4h_B + r_B)r_B, \quad 2h_A + r_A = 2h_B + r_B.$$

Write θ for $2h_A + r_A = 2h_B + r_B$ and eliminate r_A and r_B. There results

$$(\theta + 2h_A)(\theta - 2h_A) = (\theta + 2h_B)(\theta - 2h_B)$$

and therefore $h_A = h_B$.

There follows immediately $r_A = r_B$ and consequently $A = B$.

On the other hand, if X is a non-orientable surface of genus k with $r > 0$ boundaries, then

$$P_X(x) = 1 + (k + r - 1)x,$$
$$P_{\dot{X}}(x) = r(1+x), \quad P_{X,\dot{X}}(x) = r - 1,$$

and therefore

$$P_{(X^2)^{\cdot}}(x) = 1 + (2kr + (r-1)^2)x + (2kr + (r-1)^2)x^2 + x^3,$$
$$P_{X^2}(x) = 1 + 2(k + r - 1)x + (k + r - 1)^2 x^2.$$

Consequently, if $A^2 = B^2$, where A is a non-orientable surface of genus k_A with r_A boundaries and B is a non-orientable surface of genus k_B with r_B boundaries, then we must have

$$2k_A r_A + (r_A - 1)^2 = 2k_B r_B + (r_B - 1)^2,$$
$$k_A + r_A - 1 = k_B + r_B - 1.$$

Write θ for $k_A + r_A - 1 = k_B + r_B - 1$ and eliminate r_A and r_B. There results

$$2k_A(\theta - k_A + 1) + (\theta - k_A)^2 = 2k_B(\theta - k_B + 1) + (\theta - k_B^2),$$
$$k_A^2 - 2k_A = k_B^2 - 2k_B, \quad (k_A - 1)^2 = (k_B - 1)^2.$$

From this follows $k_A = k_B$ or $k_A = 2 - k_B$. But $k \geqslant 1$ and therefore $k_A = k_B$. From this follows $r_A = r_B$ and hence $A = B$.