An example of a simple arc in space whose projection in every plane has interior points.

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In the theory of knots one investigates simple closed curves Ω in Euclidean 3-dimensional space C_3 which are polygonal — that s, which are composed of a finite number of line segments. The projection of such a curve on a plane is again polygonal. The calculation of various invariants, by means of which important progress has been made in the basic classification problem, depends on the fact thatthe segments of the projection are distinguishable from one another. Thus, for example, to calculate the most important of these invariants – the fundamental group of $C_3 - \Omega$ – one may write down a generating symbol for each segment of a given projection and then, following certain simple rules, write down a set of generating relations which are read off at the vertices of the projection. It is possible to extend these procedures to certain types of simple closed curves Ω which are not polygonal provided that Ω has some projection which is „polygonal in character".

However these methods necessarily fail if every projection (central or parallel) of Ω into any plane is 2-dimensional. The question if such a curve Ω exists was posed by R. H. Fox, who gave me also some stimulating advice in preparing this paper. The positive answer on this question follows from following more general

Theorem. *Given any n-dimensional simplex Δ_0, lying in n-dimensional Euclidean space C_n, there exists a simple arc B_0 such that every straight line in C_n which intersects Δ_0 also intersects B_0.*

We first prove the following

Lemma. *Let Δ be an n-dimensional simplex lying in Euclidean n-dimensional space C_n and Δ' be an n-dimensional simplex lying in the interior of Δ. For each $a>0$ there exists a finite system $\Delta_1, \Delta_1', \Delta_2, \Delta_2', \ldots, \Delta_m, \Delta_m'$ of n-dimensional simplexes with the properties:*

1^0 *$\Delta_1, \Delta_2, \ldots, \Delta_m$ are disjoint,*

2^0 *The diameter of Δ_i is $<a$ for each $i=1,2,\ldots,m$,*

3^0 *Δ_i lies in the interior of Δ and Δ_i' in the interior of Δ_i, for each $i=1,2,\ldots,m$.*

4^0 *Any straight line in C_n which intersects Δ' intersects at least one of the simplexes Δ_i'.*

Proof of the lemma: Let β be a positive number so small that every point of C_n whose distance from Δ' is $\leqslant\beta$ lies in Δ. It is easy to see that there is a simplicial decomposition Σ of Δ, into simplexes of diameter $<\frac{1}{n+1}\cdot\beta$, such that the set E of vertices of Σ lying in the interior of Δ satisfies the following conditions.

(1) Any n points of E are linearly independent,

(2) If $E_1, E_2, \ldots, E_n$ are n disjoint subsets of E, and each of these systems is composed of n points, then the $(n-1)$-dimensional hyperplanes $H_1, H_2, \ldots, H_n$, determined respectively by $E_1, E_2, \ldots, E_n$ have exactly one point in common.

Let us denote by

(3) $$\Delta_1^*, \Delta_2^*, \ldots, \Delta_m^*$$

the n-dimensional simplexes of Σ which lie in the interior of Δ. It follows from (1) that every straight line L which intersects each of n disjoint simplexes $\Delta_{k_1}^*, \Delta_{k_2}^*, \ldots, \Delta_{k_n}^*$ of the system (2) along a segment of L contains an interior point of at least one of these simplexes. For otherwise there would exist in each simplex $\Delta_{k_i}^*$ $(i=1,2,\ldots,n)$ a system E_i of n vertices such that the $(n-1)$-dimensional hyperplane H_i determined by E_i contains L, which contradicts condition (2).

Let p be a point belonging to the boundary $\dot{\Delta}$ of Δ and p' a point belonging to the boundary $\dot{\Delta}'$ of Δ', and let us denote by $L(p,p')$ the straight line which contains p and p'. Since the diameter of the simplexes of Σ are $<\frac{1}{n+1}\beta$ and since the measure of $\Delta \cap L$ is $\geqslant 2\beta$ it follows easily that there exists in the system (2) at least n disjoint simplexes which intersect L in a segment. But this implies, as we have already proved, that L contains an interior point of at least one of these simplexes. Of course every straight line in C_n which intersects Δ' has points in common with both $\dot{\Delta}$ and $\dot{\Delta}'$. Thus every straight line in C_n which intersects Δ' must contain a point which is the center of a sphere of positive diameter contained in some simplex of the system (3).

Let us denote by Λ the set of lines L in C_n which intersect Δ'. For each $L \epsilon \Lambda$ denote by $\varrho(L)$ the diameter of the largest possible sphere with center lying on L and contained in any simplex of the system (3), and denote by $Q(L)$ such a sphere of diameter $\varrho(L)$.

The pair of points $p \epsilon \dot{\Delta}$ and $p' \epsilon \dot{\Delta}'$ may be considered as a point (p,p') of the Cartesian product $\dot{\Delta} \times \dot{\Delta}'$. Since $\varrho(p,p') \geqslant \beta$ the line $L(p,p')$, which may be considered as a point in the space of all lines in C_n, depends continuously on (p,p'). Since $\dot{\Delta} \times \dot{\Delta}'$ is compact it follows that Λ, which is a continuous image of $\dot{\Delta} \times \dot{\Delta}'$, is also compact. Moreover it is clear that $\varrho(L)$ is always positive and depends continuously on $L \epsilon \Lambda$ so that there must exist a positive number γ such that $\varrho(L) \geqslant \gamma$ for every $L \epsilon \Lambda$.

Evidently for every $i=1,2,\ldots,m$ there exist two simplexes Δ_i and Δ_i' such that Δ_i lies in the interior of Δ_i^* and Δ_i' lies in the interior of Δ_i and the distance of any point in Δ_i^* from Δ_i' is $<\gamma$. It is clear that $\Delta_1, \Delta_1', \Delta_2, \Delta_2', \ldots, \Delta_m, \Delta_m'$ satisfy conditions 1^0, 2^0 and 3^0. If now L is any straight line which intersects Δ' then there is an index i such that Δ_i^* contains the sphere $Q(L)$. The center $c(L)$ of $Q(L)$ is on L and since the radius $\varrho(L)$ of $Q(L)$ is $\geqslant \gamma$ it follows that $c(L) \epsilon \Delta_i'$. Thus $c(L) \epsilon L \cdot \Delta_i'$ so that condition 4^0 is also satisfied. This completes the proof of the lemma.

Proof of the theorem: It is known that, given any compact 0-dimensional subset A of Euclidean n-dimensional space C_n, there is a simple arc $B \subset C_n$ such that $A \subset B$. Therefore it will be sufficient to prove only that there exists a compact 0-dimensional subset A_0 of C_n which is intersected by every straight line L of C_n which intersects Δ_0.

The set A_0 will be the common part of a monotone decreasing sequence of polytopes $\{A_k\}$ which will be defined by recurrence as follows:

A_1 is a simplex Δ, containing Δ_0 in its interior, of diameter $\leqslant \alpha$, where α is twice the diameter of Δ_0. Evidently A_1 satisfies for $k=1$ the following condition

(V_k) A_k is the union of disjoint n-dimensional simplexes $\Delta_1^{(k)}, \Delta_2^{(k)}, \ldots, \Delta_{\mu_k}^{(k)}$, each of diameter $\leqslant \frac{\alpha}{k}$. In the interior of each $\Delta_\nu^{(k)}$ there may be found a simplex $\Delta_\nu^{(k)'}$ in such a way that any straight line L in C_n which intersects Δ_0 intersects at least one of the simplexes $\Delta_1^{(k)'}, \Delta_2^{(k)'}, \ldots, \Delta_{\mu_k}^{(k)'}$.

Suppose that a polytope A_k satisfying the condition (V_k) has been constructed. Let us denote by $\Lambda_\nu^{(k)}$ the set of all straight lines in C_n which intersect $\Delta_\nu^{(k)'}$. By (V_k) we see that

$$\Lambda^0 = \bigcup_\nu \Lambda_\nu^{(k)}, \tag{3}$$

where Λ^0 denotes the set of all straight lines in C_n which intersect Δ_0. Applying the lemma to the pair of simplexes $\Delta_\nu^{(k)}, \Delta_\nu^{(k)'}$ we find a finite system of n-dimensional simplexes

$$\Delta_{\nu,1}^{(k)}, \Delta_{\nu,1}^{(k)'}, \Delta_{\nu,2}^{(k)}, \Delta_{\nu,2}^{(k)'}, \ldots, \Delta_{\nu,m_{\nu,k}}^{(k)}, \Delta_{\nu,m_{\nu,k}}^{(k)'}$$

which satisfy the following conditions:

1_k^0 $\Delta_{\nu,1}^{(k)}, \Delta_{\nu,2}^{(k)}, \ldots, \Delta_{\nu,m_{\nu,k}}^{(k)}$ are disjoint,

2_k^0 The diameter of $\Delta_{\nu,i}^{(k)}$ is $< \frac{\alpha}{k+1}$ for each $i=1,2,\ldots,m_{\nu,k}$,

3_k^0 $\Delta_{\nu,i}^{(k)}$ lies in the interior of $\Delta_\nu^{(k)}$ and $\Delta_{\nu,i}^{(k)'}$ in the interior of $\Delta_{\nu,i}^{(k)}$ for each $i=1,2,\ldots,m_{\nu,k}$.

4^0 Each straight line $L \epsilon \Lambda_\nu^{(k)}$ intersects at least one of the simplexes $\Delta_{\nu,i}^{(k)'}$.

If we set

$$A_{k+1} = \bigcup_{\nu,i} \Delta_{\nu,i}^{(k)}$$

we obtain a polytope which satisfies the condition (V_{k+1}). From 3_k^0 it follows that $A_{k+1} \subset A_k$ and from 1_k^0 and 2_k^0 we conclude that

$$A_0 = \bigcap_k A_k$$

is a 0-dimensional compactum. By 4_k^0 and (3) we see that each straight line $L \epsilon \Lambda^0$ intersects the compact set A_k and consequently also the set A_0. This completes the proof of the theorem.

Remark. Let us suppose that the diameter of Δ is $\leqslant 1$ and that $n>2$. Evidently it is always possible to define the arc B_0 and the set $A_0^* = A_0$ satisfying our theorem in such a manner that:

(1⁰) $B_0 \subset \Delta$ and the end points of B_0 are two arbitrarily given points a, b lying on the boundary $\dot{\Delta}$ of Δ,

(2⁰) $B_0 - A_0^* - (a) - (b) = \bigcup_\nu P_\nu^0$, where $P_1^0, P_2^0, \ldots$ are disjoint open segments with the diameters <1.

Let us now define a sequence of arcs $B_0, B_1, \ldots$ by recurrence as follows:

Suppose that an arc simple B_k and a 0-dimensional compact set $A_k^* \subset B_k$ have been defined satisfying the following conditions:

(1^k) $B_k \subset \Delta$ and a, b are the end points of B_k,

(2^k) $B_k - A_k^* - (a) - (b) = \bigcup_\nu P_\nu^k$, where $P_1^k, P_2^k, \ldots$, are disjoint open segments with the diameters $<\frac{1}{k+1}$.

We define now an arc simple B_{k+1} and a compact 0-dimensional set $A_{k+1}^* \subset B_{k+1}$ as follows: Let us denote by l_k a natural number such that for $i>l$ the diameter of P_i^k is $<\frac{1}{k+2}$. It is easy to observe that for each $i=1,2,\ldots,l_k$ it exist two n-dimensional simplexes Δ_i^k and Δ_i^{0k} such that

a) The center c_i^k of the segment P_i^k belongs to the interior of Δ_i^{0k} and Δ_i^{0k} lies in the interior of Δ_i^k,

b) The diameter of Δ^k is $\leqslant \frac{1}{2^k}$,

c) The diameter of Δ_i^k is less than the quarter of the distance of the center c_i^k from the set $B_k - P_i^k$.

It follows from a) and c) that $\Delta_i^k \cap P_i^k$ is an segment with endpoints a_i^k, b_i^k lying on the boundary of Δ_i^k. If we apply our theorem and (1⁰), we infer that it exists an arc $B_i^k \subset \Delta_i^k$ with the endpoints a_i^k, b_i^k and a compact 0-dimensional set $A_i^k \subset B_i^k$ such that every straight line in C_n which intersects Δ_i^{0k} also intersects A_i^k. If we put

$$A_{k+1}^* = \Big(\bigcup_{i=1}^{l_k} A_i^k\Big) \cup A_k^*,$$

$$B_{k+1} = \Big(\bigcup_{i=1}^{l_k} B_i^k\Big) \cup \Big(B_k - \bigcup_{i=1}^{l_k} \Delta_i^k\Big),$$

we obtain a compact 0-dimensional set A_{k+1}^* and an arc B_{k+1} satisfying the conditions (1^{k+1}) and (2^{k+1}).

Let φ_k denotes a homeomorphic transformation of the interval $\langle 0,1\rangle$ onto B_k such that each open interval $\varphi_k^{-1}(P_\nu^k)$ is transformed by φ_k linearly onto open segment P_ν^k. It follows easily from our construction and from b) that a homeomorphic transformation φ_{k+1} of $\langle 0,1\rangle$ into B_{k+1} can be obtained from φ_k by a modification of φ_k in the disjoint intervals

$$\varphi_k^{-1}(B_k \cap \Delta_i^k), \quad i \leqslant l_k$$

in such a way that

$$\varrho[\varphi_k(x), \varphi_{k+1}(x)] \leqslant \frac{1}{2^k} \qquad \text{for } 0 \leqslant x \leqslant 1.$$

It follows that the sequence $\{\varphi_k\}$ is in the interval $\langle 0,1\rangle$ uniformely convergent to a continuous transformation φ. Moreover it is easy to show, applying the condition c), that φ is a 1—1 transformation. Hence φ maps the interval $\langle 0,1\rangle$ onto an arc simple

$$B = \varphi(\langle 0,1\rangle).$$

It follows from our construction that every arc simple $B' \subset B$ contains some of the sets A_i^k. Consequently every straight line in C_n which intersects the simplex Δ_i^{0k} also intersects B'. Thus we obtain the following

Theorem. *There exists in the n-dimensional Euclidean space C_n a simple arc B such that for every arc simple $B' \subset B$ it exists a n-dimensional simplex Δ' such that every straight line in C_n which intersects Δ' also intersects B'.*

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