On the decomposition of spheres.

By

Raphael M. Robinson (Princeton, N. J., U. S. A.).

§ 1. Introduction. According to a theorem of Banach and Tarski[1]), it is possible to cut a solid unit sphere into a finite number of pieces, and reassemble these to form two solid unit spheres. That is, the set S defined by $x^2+y^2+z^2 \leqslant 1$ admits three decompositions into disjoints pieces

$$S = A_1 + A_2 + \ldots + A_{k+l},$$

$$S = B_1 + \ldots + B_k, \qquad S = B_{k+1} + \ldots + B_{k+l},$$

such that

$$A_i \cong B_i \qquad \text{for } i = 1, 2, \ldots, k+l.$$

Here by $A \cong B$ is meant that A and B are congruent. We shall call two sets A and B congruent only if A can be transformed into B using translation and rotation. Thus reflection is excluded.

No estimate was given by Banach and Tarski for the number of pieces required. Von Neumann[2]) stated without proof that nine pieces are sufficient, with $k=4$ and $l=5$. Recently, Sierpiński[3]) showed that eight pieces are sufficient, with $k=3$ and $l=5$, or $k=2$ and $l=6$. In this paper, *we shall show that the smallest possible number of pieces is* 5. If we take $k=2$ and $l=3$, then A_5 may be taken to consist of a single point.

The number of pieces could not be reduced by allowing reflections. In the earlier papers, reflections were not specifically excluded, but were not actually used.

[1]) S. Banach and A. Tarski, *Sur la décomposition des ensembles de points en parties respectivement congruentes*, Fund. Math., **6** (1924), p. 244-277.

[2]) J. v. Neumann, *Zur allgemeinen Theorie des Masses*, Fund. Math., **13** (1929), pp. 73-116 (p. 77).

[3]) W. Sierpiński, *Sur le paradoxe de MM. Banach et Tarski*, Fund. Math., **33** (1945), pp. 228-234.

A similar problem for the surface S of the sphere is also considered. It is shown that four pieces are sufficient in this case. In fact, this result is obtained first, and the result for the solid sphere is then an almost immediate consequence.

The solution of the surface problem follows immediately from the following fact: *The sphere S, defined by $x^2+y^2+z^2=1$, can be divided into two pieces, each of which can be divided into two pieces congruent to itself.*

Such a decomposition of S is possible on the basis of a theorem proved in § 4, according to which S may be decomposed into pieces satisfying any system of congruences, provided that we do not demand, explicitly or implicitly, that two complementary portions of the sphere be congruent. This theorem is obtained by specializing a more general decomposition theorem proved in § 3.

The theorem of Banach and Tarski rests essentially on a special decomposition of S given by Hausdorff[4]). This decomposition has the form

$$S = A + B + C + D,$$

where the four sets are disjoint, D is denumerable, and

$$A \cong B \cong C \cong B + C.$$

Indeed, A and $B+C$ are interchanged by a rotation φ of 180^0 about one axis, and A, B, C are permuted by a rotation ψ of 120^0 about another suitably chosen axis. Disregarding the denumerable set D, we see that A is both a half and a third of S. This result was used by Hausdorff to show the non-existence of an additive measure in space which is invariant under translation and rotation.

By using the Hausdorff decomposition, and the equivalence theorem of Schröder-Bernstein, as extended by Banach[5]), Banach and Tarski were able to prove not only the result stated at the beginning of this paper, but also the more general result that any two bounded sets in space with interior points are equivalent by finite decomposition. In the special case considered in this paper, we shall avoid use of the equivalence theorem, as indeed seems essential if we are to get the minimum number of pieces.

[4]) F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914, pp. 469-472.

[5]) S. Banach, *Un théorème sur les transformations biunivoques*, Fund. Math., **6** (1924), pp. 236-239.

No use is made of results from any of the papers mentioned, except that from Hausdorff's book we take the existence of rotations φ and ψ which satisfy the relations $\varphi^2=1$ and $\psi^3=1$ (where 1 denotes the identical transformation), but no other independent identities.

In an appendix to this paper, a sharper form of the Hausdorff decomposition theorem is derived. From this and the equivalence theorem, it is easily shown that six pieces are sufficient for the problem of the solid spheres, and five pieces for the surface problem. These results were known to me before the better results stated above. Their derivation has been omitted, since the results have been superceded. The discussion of the Hausdorff decomposition has been retained, because it is believed that this decomposition is of considerable intrinsic and historical interest.

§ 2. The rotation group. By a rotation, we shall always mean a rotation of the three-dimensional space which leaves the origin fixed. If φ is a rotation, then the transform of a point u by φ will be denoted by $u\varphi$, and the transform of a point-set A by φ will be denoted by $A\varphi$.

In this section, we shall show the existence of m independent rotations, and prove some properties of the subgroup of the rotation group generated by them.

Suppose that we have two rotations φ and ψ which are independent except for possible periodicity. That is, the two rotations may satisfy one or both of the equations $\varphi^r=1$ and $\psi^s=1$, where r and s are positive integers, but no other independent identity. Suppose that $r>1$ and $s>m$. Let

$$\alpha_k=\varphi\psi^k \qquad \text{for } k=1,2,\dots,m.$$

Then the rotations α_k^2 are readily seen to be independent. Indeed, if we consider any product of factors $\alpha_k^{\pm 2}$, which cannot be simplified in terms of the α's, and then substitute the values of the α's and cancel what factors we can, the first and last factors of the form $\alpha_k^{\pm 1}$ remain untouched.

Hausdorff showed how to find two rotations φ and ψ which are independent except for the relations $\varphi^2=1$ and $\psi^3=1$. By the last paragraph, we find two completely independent rotations α_1^2 and α_2^2. If we now take these as φ and ψ, and apply the construction again, we obtain as many independent rotations as we please. A somewhat different construction has been given by Sierpiński [6]).

Let $\varphi_1,\varphi_2,\dots,\varphi_m$ be m independent rotations. Let us consider the effect of the group of rotations generated by $\varphi_1,\dots,\varphi_m$ on the points of the surface S of the unit sphere. The sphere S falls into classes of equivalent points, that is, points which may be transformed into one another by rotations of the group. A point which is fixed for some rotation of the group will be called a fixed point. We notice first that any point equivalent to a fixed point is a fixed point. Indeed, if $v\alpha=v$, then $v\beta\cdot\beta^{-1}\alpha\beta=v\beta$; that is, if v is fixed for α, then $v\beta$ is fixed for $\beta^{-1}\alpha\beta$. Thus a class of equivalent points consists entirely of fixed points, or entirely of non-fixed points.

Consider any class consisting entirely of non-fixed points. If some point u of the class is chosen, then any point of the class is representable uniquely in the form $u\beta$, where β is a rotation of the group.

Consider any class consisting entirely of fixed points. Choose a rotation θ having a fixed point in the class and which is as short as possible, that is, which is expressible as a product of the smallest possible number of factors of the form $\varphi_k^{\pm 1}$. Let v denote a point of the class such that $v\theta=v$. We shall show that if $v\alpha=v$, then $\alpha=\theta^n$, where n is an integer. If α is the identity, then the conclusion holds with $n=0$; we may exclude this case in the following discussion.

We observe first that the initial and final factors of θ cannot be inverse, since otherwise some rotation $\sigma^{-1}\theta\sigma$ would be shorter and have a fixed point in the class. Thus θ and θ^{-1} do not begin with the same factor, and do not end with the same factor.

Now if α has the same fixed point v as θ, then $\alpha\theta=\theta\alpha$. If $\theta\alpha$ does not simplify, when θ and α are written in terms of the $\varphi_k^{\pm 1}$, then $\alpha\theta$ must also not simplify. Thus α must begin with the block θ. We find inductively that α is obtained by writing the block θ n times, that is, $\alpha=\theta^n$, where n is a positive integer. In case $\theta\alpha$ does simplify, then $\theta^{-1}\alpha$ does not. Thus we may apply the same argument to the equation $\alpha\theta^{-1}=\theta^{-1}\alpha$, and find that $\alpha=\theta^{-n}$, where n is a positive integer.

Any point of the class may be written in the form $v\beta$, where β is a rotation of the group which does not start with the block θ

[6]) W. Sierpiński, *Sur le paradoxe de la sphère*, Fund. Math., **33** (1945), pp. 235-244 (p. 236).

(when written in terms of the $\varphi_k^{\pm 1}$), nor with the inverse of the last factor of θ. For the latter property may be achieved by replacing β by $\theta^n\beta$ where n is sufficiently large; and we may then simplify and remove any blocks θ remaining.

This representation is unique. For suppose that $v\beta = v\gamma$, where β and γ have the form specified. Then $v\beta\gamma^{-1} = v$, hence $\beta\gamma^{-1} = \theta^n$. If $n > 0$, this gives $\beta = \theta^n\gamma$, which is impossible, since $\theta^n\gamma$ does not simplify, and β does not begin with the block θ. If $n < 0$, we may interchange the roles of β and γ, and again reach a contradiction. Hence $n = 0$, that is, $\beta = \gamma$.

§ 3. A general decomposition theorem. We shall now prove a general theorem concerning the decomposition of the sphere S into n pieces $A_1, A_2, \ldots, A_n$. Let R be a relation whose domain and range are both identical with the set of integers $\{1, 2, \ldots, n\}$. A rotation φ is said to be compatible with R, for the subdivision of S into the pieces $A_1, \ldots, A_n$, if no point of $A_k\varphi$ lies in A_l unless kRl.

The product RR' of two relations R and R' is defined by the statement that $kRR'l$ if and only if there is an s such that kRs and $sR'l$. The converse R^{-1} of R is the relation such that $lR^{-1}k$ is equivalent to kRl. Notice that RR^{-1} is not in general the identity. We call k a fixed point for R if kRk.

Theorem. *Given m relations $R_1, \ldots, R_m$, each having $\{1, 2, \ldots, n\}$ as domain and range. Then we can decompose the sphere S into n disjoint pieces $A_1, \ldots, A_n$, and for this subdivision find m rotations $\varphi_1, \ldots, \varphi_m$ compatible with $R_1, \ldots, R_m$ respectively, if and only if every product of any number of factors of the form $R_i^{\pm 1}$ has a fixed point. If such rotations exist, they may be chosen so as to be independent.*

Proof. Necessity. Suppose that such $A_1, \ldots, A_n$ and $\varphi_1, \ldots, \varphi_m$ exist. Let some relation

$$R = R_{i_1}^{j_1} R_{i_2}^{j_2} \ldots R_{i_s}^{j_s}$$

be given, where the exponents are ± 1. Then the rotation

$$\varphi = \varphi_{i_1}^{j_1} \varphi_{i_2}^{j_2} \ldots \varphi_{i_s}^{j_s}$$

must be compatible with R. Since φ has a fixed point, so also must R.

Sufficiency. Suppose that every R generated by the given relations has a fixed point. Let $\varphi_1, \ldots, \varphi_m$ be independent rotations. We consider the group of rotations generated by $\varphi_1, \ldots, \varphi_m$. With respect to these, the points of the sphere fall into classes of equivalent points. It is clear that the distribution of points into the n sets $A_1, \ldots, A_n$ is independent for different classes. Thus we need only show how to make this distribution for any class in such a way that the rotations φ_k are compatible with the relations R_k.

Non-fixed points. Given a class consisting of non-fixed points. Choose at random a point u of the class. (We use here the axiom of choice). Any element of the class may be written uniquely in the form $u\beta$, where β is a rotation of the group. Start by assigning u to any set A_k. After $u\alpha$ has been put in some set A_k, if $\beta = \alpha\varphi_i^j$ where $j = \pm 1$ and the extra factor does not cancel with a factor of α, we put $u\beta$ in some class A_l such that $kR_i^j l$. Then all conditions are satisfied.

Fixed points. Choose a shortest rotation θ having a fixed point v in the class, as in § 2. If

$$\theta = \varphi_{i_1}^{j_1} \varphi_{i_2}^{j_2} \ldots \varphi_{i_s}^{j_s},$$

where each exponent is ± 1, then from the point v we obtain successively the points

$$v,\ v\varphi_{i_1}^{j_1},\ v\varphi_{i_1}^{j_1}\varphi_{i_2}^{j_2}, \ldots, v\varphi_{i_1}^{j_1} \ldots \varphi_{i_s}^{j_s} = v.$$

Thus we have s points forming a closed cycle.

We know that every point of the class can be written uniquely in the form $v\beta$, where β does not begin with the block θ nor with the factor $\varphi_{i_s}^{-j_s}$. Thus there are no other closed cycles. Consequently, if we can assign the s points of the cycle to the classes $A_1, \ldots, A_n$, so that the compatibility conditions are satisfied, then the remaining points may be assigned in the same manner as for the non-fixed points.

The relation corresponding to θ is

$$R = R_{i_1}^{j_1} R_{i_2}^{j_2} \ldots R_{i_s}^{j_s}.$$

Since R has a fixed point k, there must exist integers $k_0, k_1, \ldots, k_s$, chosen from $\{1, 2, \ldots, n\}$, such that

$$k_{r-1} R_{i_r}^{j_r} k_r \qquad \text{for } r = 1, 2, \ldots, s,$$

and $k_0 = k_s = k$. If we put the point v in the set A_k and the point

$$v\varphi_{i_1}^{j_1} \ldots \varphi_{i_r}^{j_r}$$

in the set A_{k_r}, for $r = 1, 2, \ldots, s-1$, then all conditions will be satisfied.

§ 4. Systems of congruences. We now ask whether it is possible to divide the spherical surface S into disjoint pieces $A_1, A_2, \ldots, A_n$, satisfying given congruences. Each congruence is to have the form

$$A_{k_1} + A_{k_2} + \ldots + A_{k_r} \cong A_{l_1} + A_{l_2} + \ldots + A_{l_s},$$

where $0 < r < n$, $0 < s < n$, and

$$1 \leqslant k_1 < k_2 < \ldots < k_r \leqslant n, \qquad 1 \leqslant l_1 < l_2 < \ldots < l_s \leqslant n.$$

We notice first that this congruence is equivalent to the existence of a rotation φ compatible for the given decomposition of S with the relation R having $\{1, 2, \ldots, n\}$ as domain and range, and for k and l in this set satisfying the condition

$$kRl \leftrightarrow (k \epsilon K \leftrightarrow l \epsilon L),$$

where

$$K = \{k_1, k_2, \ldots, k_r\}, \qquad L = \{l_1, l_2, \ldots, l_s\},$$

and the double arrow denotes logical equivalence. A relation of the above form will be called *canonical*, and will be said to correspond to the given congruence.

Since to any congruence, a relation R can be found, such that the existence of a rotation φ compatible with R is equivalent to the given congruence, it is clear that the theorem of § 3 gives an answer to the question when a given system of congruences can be satisfied. The purpose of this section is to transform this condition into a form relating directly to the congruences.

Notice that each congruence is equivalent to the complementary congruence obtained by writing on each side of the congruence the sum of the pieces $A_1, A_2, \ldots, A_n$ which are absent in the given congruence. Notice further that if two congruences have one member in common, then a congruence may be written between the remaining members. This new congruence will be said to be obtained by transitivity. With this terminology, we may state the desired condition for satisfying a system of congruences.

Theorem. *The sphere S may be decomposed into n disjoint pieces satisfying a given system of congruences, if and only if none of the given congruences and no congruence obtainable from them by taking complements or by transitivity asserts the congruence of two complementary portions of S. Furthermore, if the decomposition is possible, then each congruence may be effectuated by an independent rotation.*

Proof. The necessity of the condition is clear, since two complementary portions of S cannot be congruent. We shall derive its sufficiency from the theorem of § 3. Let R_i be the relation corresponding to the i-th congruence in the sense described above. Then the existence of a rotation φ compatible with R_i insures the truth of the given congruence. Thus by the theorem of § 3, the given congruences are solvable provided that every relation R obtained as a product of factors $R_i^{\pm 1}$ has a fixed point.

The relations $R_i^{\pm 1}$ correspond to congruences, and hence are canonical. While not every product formed from these factors need be canonical, we shall show that only those products which are canonical need be considered.

If there is an l with $1 \leqslant l \leqslant n$ such that kRl for every k with $1 \leqslant k \leqslant n$, then R includes a function defined on the domain of R and having a constant value l. We shall say briefly that R includes a constant. If R includes a constant, then every product of R by other relations having the usual domain and range evidently also includes a constant. Since a relation including a constant certainly has a fixed point, such products need not be considered.

We shall show next that a product of canonical relations is either canonical or includes a constant. It is sufficient to consider a product of two factors. Suppose that

$$kR's \leftrightarrow (k \epsilon K' \leftrightarrow s \epsilon L'),$$
$$sR''l \leftrightarrow (s \epsilon K'' \leftrightarrow l \epsilon L'').$$

Now $kR'R''l$ if and only if there is an s such that $kR's$ and $sR''l$. If K'' is either L' or its complement with respect to $\{1, 2, \ldots, n\}$, then it is easily seen that $R'R''$ is canonical. For example, if $K'' = L'$, then we have

$$kR'R''l \leftrightarrow (k \epsilon K' \leftrightarrow l \epsilon L'').$$

On the other hand, if K'' has points in common with both L' and its complement, then we find that $R'R''$ includes a constant. Indeed, $kR'R''l$ for any $l \epsilon L''$ and any k. A similar conclusion may be drawn if the complement of K'' has points in common with both L' and its complement.

Thus every product of the $R_i^{\pm 1}$ is either canonical or includes a constant. Since every product which includes a constant has a fixed point, we need only find under what conditions a product which is canonical has a fixed point.

The given relations $R_1,\dots,R_m$ correspond to the given congruences, or equally well to the complementary congruences. The converse relations $R_1^{-1},\dots,R_m^{-1}$ correspond to the same congruences with the left and right sides interchanged. The multiplication of two relations, in the cases in which the product is canonical, corresponds exactly to deducing a new congruence by transitivity from the two given congruences, if necessary replacing the second by its complementary congruence. Thus if there is any relation without a fixed point, we shall find by transitivity a congruence corresponding to it, that is, one which asserts the congruence of two complementary portions of S.

§ 5. Examples of decompositions. Suppose first that it is required to divide the sphere S into three pieces A, B, C, which are congruent. The system of congruences mentioned in the theorem of § 4 may be taken as

$$A \cong B, \qquad A \cong C.$$

By taking complements, and using transitivity, we obtain only the congruences between the three sets and their sums in pairs:

$$A \cong B \cong C, \qquad A+B \cong A+C \cong B+C.$$

Thus by the theorem, the decomposition is possible.

By taking the given system of congruences as

$$A \cong A, \qquad A \cong A, \qquad A \cong B, \qquad A \cong C,$$

we see that the pieces can be so chosen that there are two independent rotations taking A into A. Thus there are as many independent rotations as we please taking each of the sets into itself. A similar remark applies to all the other examples.

In an exactly similar way, we can decompose S into n congruent parts $A_1,\dots,A_n$, provided $n>2$. The decomposition is of course impossible when $n=2$.

The next example is of particular interest, since it is the one which we shall use in § 6. We wish to decompose S into four pieces A_1, A_2, A_3, A_4, such that

$$A_1 \cong A_2 \cong A_1+A_2, \qquad A_3 \cong A_4 \cong A_3+A_4.$$

In other words, we wish to cut the sphere into two pieces, each of which can be subdivided into two pieces congruent to itself.

The congruences obtained from the given ones by taking complements and transitivity are readily seen to be

$$A_1 \cong A_2 \cong A_1+A_2 \cong A_1+A_2+A_3 \cong A_1+A_2+A_4,$$
$$A_3 \cong A_4 \cong A_3+A_4 \cong A_1+A_3+A_4 \cong A_2+A_3+A_4.$$

Since none of these asserts the congruence of two complementary portions of the sphere, the decomposition is possible.

A slight modification of the above example will also be needed. It is desired to cut S into five pieces A_1, A_2, A_3, A_4, P, of which P consists of a single point, such that

$$A_1 \cong A_2 \cong A_1+A_2+P, \qquad A_3 \cong A_4 \cong A_3+A_4.$$

The possibility of such a decomposition will be shown by modifying the previous example. We wish to decompose S into pieces $A_1,\dots,A_4$, P, satisfying the equations

$$A_1\varphi_1 = A_1+A_2+P, \qquad A_3\varphi_3 = A_3+A_4,$$
$$A_2\varphi_2 = A_1+A_2+P, \qquad A_4\varphi_4 = A_3+A_4,$$

where $\varphi_1\, \varphi_2, \varphi_3, \varphi_4$ are independent rotations. On the basis of the previous example, we know that such a decomposition is possible with P deleted. Choose a single class of non-fixed points, and only for this class modify the assignment of points to the various sets. Take any point u from this class, and assign it to P, thus completing P. Then assign

$u\varphi_1$ to A_3 or A_4, $\qquad u\varphi_1^{-1}$ to A_1,
$u\varphi_2$ to A_3 or A_4, $\qquad u\varphi_2^{-1}$ to A_2,
$u\varphi_3$ to A_1 or A_2, $\qquad u\varphi_3^{-1}$ to A_1, A_2, or A_4,
$u\varphi_4$ to A_1 or A_2, $\qquad u\varphi_4^{-1}$ to A_1, A_2, or A_3.

The further assignments are made by exactly the same method as before. That is, we use the rule given in the proof of the theorem of § 3, where the relations R_1, R_2, R_3, R_4 hold in just the following cases:

$$1R_11,\ 1R_12,\ 2R_21,\ 2R_22,\ 3R_33,\ 3R_34,\ 4R_43,\ 4R_44.$$

We consider two additional examples. Suppose that it is required to divide S into four pieces A_1, A_2, A_3, A_4, such that

$$A_1 \cong A_2 \cong A_3+A_4, \qquad A_3 \cong A_4 \cong A_1+A_2.$$

In this case, we are led to the congruences

$$A_1 \cong A_2 \cong A_3 + A_4 \cong A_1 + A_2 + A_3 \cong A_1 + A_2 + A_4,$$
$$A_3 \cong A_4 \cong A_1 + A_2 \cong A_1 + A_3 + A_4 \cong A_2 + A_3 + A_4.$$

Thus this decomposition is also possible.

As a final example, let it be required to decompose S into five disjoint pieces $A_1, \ldots, A_5$, such that for all i, j, k, we have

$$A_i \cong A_j + A_k.$$

By taking complements and by transitivity, we obtain only those congruences which have one or two terms on each side, or else three or four terms on each side. None of these expresses the congruence of two complementary portions of the sphere. Hence we may divide S into five congruent pieces, such that the sum of any two pieces is also congruent to a single piece.

§ 6. How to make two spheres from one. As shown in § 5, we may decompose S into four disjoint pieces,

$$S = A_1 + A_2 + A_3 + A_4,$$

such that

$$A_1 \cong A_2 \cong A_1 + A_2, \qquad A_3 \cong A_4 \cong A_3 + A_4.$$

It is clear that A_1 and A_3 may be rotated in such a way as to exactly fit together to form S, and similarly for A_2 and A_4. Thus we may cut S into four pieces, and reassemble them in pairs to form two copies of S. We cannot use fewer than four pieces, since we cannot form a copy of S out of a single piece which is not all of S. Thus for the surface problem, the minimum number of pieces in which to cut S is four.

Consider now the decomposition of the solid sphere $\boldsymbol{S}$ defined by $x^2+y^2+z^2 \leqslant 1$. We wish to cut $\boldsymbol{S}$ into pieces and reassemble these to form two solid unit spheres. Choose four independent rotations $\varphi_1, \varphi_2, \varphi_3, \varphi_4$. Let $S(r)$ denote the surface $x^2+y^2+z^2=r^2$. We decompose $S(r)$ into disjoint parts,

$$S(r) = A_1(r) + A_2(r) + A_3(r) + A_4(r) \qquad \text{if } 0 < r < 1,$$
$$S(1) = A_1(1) + A_2(1) + A_3(1) + A_4(1) + P,$$

where P consists of a single point, such that

$$A_1(r)\varphi_1 = A_2(r)\varphi_2 = A_1(r) + A_2(r),$$
$$A_3(r)\varphi_3 = A_4(r)\varphi_4 = A_3(r) + A_4(r),$$

for $0 < r < 1$, and

$$A_1(1)\varphi_1 = A_2(1)\varphi_2 = A_1(1) + A_2(1) + P,$$
$$A_3(1)\varphi_3 = A_4(1)\varphi_4 = A_3(1) + A_4(1).$$

Putting

$$\boldsymbol{A}_k = \sum_{0<r\leqslant 1} A_k(r) \qquad \text{for } k = 1, 2, 3, 4,$$

we have a decomposition of $\boldsymbol{S}$ into six disjoint parts,

$$\boldsymbol{S} = \boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{A}_3 + \boldsymbol{A}_4 + O + P,$$

where O contains only the center of the sphere, and P also consists of a single point, such that

$$\boldsymbol{A}_1\varphi_1 = \boldsymbol{A}_2\varphi_2 = \boldsymbol{A}_1 + \boldsymbol{A}_2 + P,$$
$$\boldsymbol{A}_3\varphi_3 = \boldsymbol{A}_4\varphi_4 = \boldsymbol{A}_3 + \boldsymbol{A}_4.$$

Notice also that

$$(\boldsymbol{A}_3 + O)\varphi_3 = \boldsymbol{A}_3 + \boldsymbol{A}_4 + O.$$

Thus we may decompose $\boldsymbol{S}$ into five disjoint pieces, $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3 + O, \boldsymbol{A}_4$, and P, of which $\boldsymbol{A}_1$ and $\boldsymbol{A}_3 + O$ can be fitted to form one copy of $\boldsymbol{S}$, and $\boldsymbol{A}_2, \boldsymbol{A}_4$, and P can be fitted to form another copy. In the last case, a translation is used to take P into O, but in all other cases only rotations about the origin are used.

It is easily seen that we cannot use fewer than five pieces. In fact, we shall prove the following somewhat stronger result: It is impossible to decompose $\boldsymbol{S}$ into four disjoint pieces,

$$\boldsymbol{S} = \boldsymbol{B}_1 + \boldsymbol{B}_2 + \boldsymbol{B}_3 + \boldsymbol{B}_4,$$

and to find distance-preserving transformations $\psi_1, \psi_2, \psi_3, \psi_4$, such that

$$\boldsymbol{B}_1\psi_1 + \boldsymbol{B}_2\psi_2 = \boldsymbol{S}, \qquad \boldsymbol{B}_3\psi_3 + \boldsymbol{B}_4\psi_4 = \boldsymbol{S},$$

even if the terms of these sums are not required to be disjoint, and the transformations are not required to be sense-preserving.

Suppose that such a decomposition were possible. Not all of the transformations $\psi_1, \psi_2, \psi_3, \psi_4$ could leave the origin fixed, for then one copy of $\boldsymbol{S}$ would be without a center. Suppose for example that $O\psi_4 \neq O$. Notice that $\boldsymbol{S} - \boldsymbol{S}\psi_4$ includes more than a hemisphere of the surface of $\boldsymbol{S}$. Since $\boldsymbol{B}_3\psi_3$ must cover $\boldsymbol{S} - \boldsymbol{S}\psi_4$, it will also include more then a hemisphere of the surface. Now $O\psi_3 = O$, since otherwise not even $S\psi_3 + S\psi_4$ would cover S. Thus B_3 itself must include more than a hemisphere of the surface of S. Consequently, B_1 and B_2 each include less than a hemisphere of the surface, so that $B_1\psi_1 + B_2\psi_2$ cannot cover even the surface of S.

Appendix. The Hausdorff decomposition. Hausdorff showed how to decompose the surface S of the unit sphere into four disjoint sets A, B, C, D, such that

$$A \cong B \cong C \cong B + C,$$

and D is denumerable. Indeed, he started with two rotations φ and ψ, such that $\varphi^2 = 1$ and $\psi^3 = 1$, φ and ψ being otherwise independent, and constructed the sets so as to satisfy

$$A\varphi = B + C, \quad (B+C)\varphi = A, \quad D\varphi = D,$$
$$A\psi = B, \quad B\psi = C, \quad C\psi = A, \quad D\psi = D.$$

In this appendix, we shall obtain a sharper form of the Hausdorff decomposition. The arguments required are similar to those used in § 2—§ 4, and will only be sketched here.

As in § 2, we study the group of rotations generated by φ and ψ. Each rotation of the group is expressed as a product in which factors φ alternate with factors $\psi^{\pm 1}$. The surface S falls into classes of equivalent points under the group, each class consisting entirely of fixed points or entirely of non-fixed points. Each point in a class of non-fixed points can be expressed uniquely in the form $u\beta$, where u is some particular point of the class and β is a rotation of the group. Now consider a class of fixed points. Choose a rotation θ which is as short as possible when expressed as a product of factors $\varphi, \psi^{\pm 1}$, and which has a fixed point v in the class. The first and last factors of θ cannot be the same or inverse, unless there is but one factor. If $v\alpha = v$, then $\alpha = \theta^n$. Each point of the class has a unique representation in the form $v\beta$, where β does not start with the block θ, nor with the last factor of θ or its inverse.

It is clear that in the Hausdorff decomposition, the set D cannot be entirely eliminated, since then $A \cong B + C$ would represent a congruence between two complementary portions of the sphere. It follows from the preceding paragraph that each set of equivalent points is infinite. Hence D must be infinite. Thus we cannot hope to find a sharper form of the Hausdorff decomposition by reducing the number of points in D, unless some other modification is also made.

We now proceed as in § 3 to distribute the points of each class among the sets A, B, C in such a way as to be consistent with the desired equations

$$A\varphi = B + C, \quad (B+C)\varphi = A, \quad A\psi = B, \quad B\psi = C, \quad C\psi = A.$$

From each class of non-fixed points, choose a point u at random, and put it in any class. After $u\alpha$ has been put in some class, we may then assign $u\beta$ to a class, if $\beta = \alpha\varphi$ or $\beta = \alpha\psi^{\pm 1}$, where the last factor does not cancel or combine with a factor of α. Indeed under these conditions we proceed as follows:

If $x\alpha$ is in A,	put $x\alpha\varphi$ in B or C,	$x\alpha\psi$ in B,	$x\alpha\psi^{-1}$ in C.
If $x\alpha$ is in B,	put $x\alpha\varphi$ in A,	$x\alpha\psi$ in C,	$x\alpha\psi^{-1}$ in A.
If $x\alpha$ is in C,	put $x\alpha\varphi$ in A,	$x\alpha\psi$ in A,	$x\alpha\psi^{-1}$ in B.

The required conditions are satisfied. If all the fixed points were put in D, we should have exactly the Hausdorff decomposition. But we shall try to also distribute the fixed points in so far as possible.

Define relations R_1 and R_2 corresponding to φ and ψ. That is, R_1 shall hold in the cases

$$1R_12, \quad 1R_13, \quad 2R_11, \quad 3R_11,$$

and R_2 in the cases

$$1R_22, \quad 2R_23, \quad 3R_21,$$

and in no others. Express the θ for any class of fixed points as a product of factors $\varphi, \psi^{\pm 1}$. Let R be defined as the product of the corresponding factors $R_1, R_2^{\pm 1}$. If this R has a fixed point, then the points of a certain closed chain can consistently be assigned to the three sets A, B, C, and then all other points can be assigned as before.

We shall show that $R_1, R_2^{\pm 1}, R_2^{-1}R_1R_2$, and $R_2R_1R_2^{-1}$ are the only products of alternate factors $R_1, R_2^{\pm 1}$ which have no fixed points. Indeed, $R_1R_2^{\pm 1}R_1$ is found to include a constant (in the sense of § 4), so that every product including this block has a fixed point. We note also that $R_1R_2^{\pm 1}$, $R_2^{\pm 1}R_1$, $R_2R_1R_2$, and $R_2^{-1}R_1R_2^{-1}$ have fixed points. Thus only the cases listed above remain.

The corresponding rotations are $\varphi, \psi^{\pm 1}, \psi^{-1}\varphi\psi$, and $\psi\varphi\psi^{-1}$, of which only φ and ψ need be considered as values for θ. In other words, of the infinitely many classes of fixed points, all except the four determined by the fixed points of the two original rotations φ and ψ can indeed be perfectly distributed among the sets A, B, C.

Note that these four classes are distinct. For example, if $v\varphi = v$, then all transforms of v can be written uniquely in the form $v\beta$, where β does not start with φ. If $v\beta$ were fixed for ψ, then $v\beta\psi$ when simplified would give a new representation for the point $v\beta$. Thus no transform of a fixed point for φ is fixed for ψ. In a similar way, we see that the two fixed points for φ or for ψ are not transforms of each other.

For these four classes we proceed as follows. If v is one of the two fixed points for φ, write all points of the class in the form $v\beta$, where β is simplified and does not begin with φ. We then distribute the points $v\beta$ as follows:

Put v and $v\alpha\varphi$ in A, $v\alpha\psi$ in B, $v\alpha\psi^{-1}$ in C.

This is a perfect distribution, except that the point v is in A, and its transform by φ is also in A. If v is one of the two fixed points for ψ, write all points of the class in the form $v\beta$, where β is simplified and does not begin with $\psi^{\pm 1}$. We then distribute the points $v\beta$ as follows:

Put $v\alpha\varphi$ in A, $v\alpha\psi$ in B, $v\alpha\psi^{-1}$ in C, v in D.

This is a perfect distribution, except that the point $v\varphi$ is in A and its transform by φ is in D. The set D consists of the two fixed points of ψ.

If we denote by E the set consisting of the two fixed points of φ, then we see that we have a decomposition of S into four disjoint sets,

$$S = A + B + C + D,$$

such that

$$(A - E)\varphi = B + C + D, \quad (B + C + D)\varphi = A - E, \quad E\varphi = E,$$
$$A\psi = B, \quad B\psi = C, \quad C\psi = A, \quad D\psi = D.$$

Thus we have divided S into three congruent parts, with only two points left over, in such a way that one piece A, omitting two points, is congruent to the complement of A. In a similar way, each of the sets B and C, with a pair of points omitted, is congruent to its complementary set, the rotations involved being $\psi^{-1}\varphi\psi$ and $\psi\varphi\psi^{-1}$.

University of California and Princeton University.

Sur l'application de la notion d'homotopie au problème du nombre algébrique des points invariants [1]).

Par

Casimir Kuratowski (Warszawa).

1. Soit E un ensemble fermé et borné situé sur le plan euclidien $\mathcal{E}^2$. Soit f une transformation continue de E en un sous-ensemble $f(E) \subset E$. Désignons par F et I la frontière et l'intérieur de E:

$$(0) \qquad F = E \cdot \overline{\mathcal{E}^2 - E} \quad \text{et} \quad I = E - \overline{\mathcal{E}^2 - E}.$$

Nous supposons dans tout ce qui va suivre que *la fonction f n'admet aucun point invariant sur la frontière F de E*, c. à d. que

$$(1) \qquad f(x) \neq x \quad \text{quel que soit} \quad x \in F;$$

ou encore: en désignant par Z l'ensemble des points invariants de la fonction f, on a $Z \subset I$.

Dans le cas où la fonction f est *holomorphe* à l'intérieur I de E et où $p \in Z$, c. à d. où p est un zéro de la fonction

$$(2) \qquad f^*(x) = f(x) - x,$$

il est naturel de nommer *ordre du point invariant p* l'ordre du point p considéré comme zéro de la fonction f^*.

Cette notion se prête à une extension au cas où f est continue (holomorphe ou non) et où p est un point invariant *isolé* (appartenant à I) [2]). On se sert à ce but de la notion d'*indice*, qui est défini comme suit [3]).

---

[1]) Présenté à la Soc. Polon. de Math., Section de Wrocław, le 7. VI. 1946.

[2]) Voir Alexandroff-Hopf, *Topologie I*, Berlin, Springer 1935, Chap. XIV, § 2, où le cas plus général de l'espace $\mathcal{E}^n$ à n dimensions est considéré.

[3]) Voir, par exemple, mon ouvrage *Théorèmes sur l'homotopie des fonctions continues de variable complexe et leurs rapports à la Théorie des fonctions analytiques*, Fund. Math. **33** (1945), p. 320 et 351.