For any natural number \( n \) we use according to Mostowski the symbol \([n]\) as an abbreviation of the conjunction \([1] \& [2] \& \ldots \& [n]\).

If \( n \) is a natural number \( > 1 \), we denote by \( p(n) \) the number \( p \) determined by two following conditions:

(i) there is a decomposition of \( n \) into a sum of primes in which \( p \)

is the smallest term;

(ii) there is no decomposition of \( n \) into a sum of primes greater than \( p \).

From theorem 1 we obtain now immediately the following

**Corollary 3.** If \( m \) and \( n \) are natural numbers such that \( m > p(n) \),

then the implication \([m] \rightarrow [n]!\) is true.

Mostowski has shown that also the converse theorem is true \(^7\). The inequality \( m > p(n) \) is not only sufficient but also necessary for the derivability of the implication \([m] \rightarrow [n]!\). Mostowski derived further from the corollary 3 a sufficient condition for the truth of the implication \([m] \rightarrow [n]!\) and proved that this condition is at the same time necessary \(^8\).

We conclude with the remark that conditions given in theorem 1 and 2 are by no means necessary for the derivability of the implication \([M] \rightarrow [n]\). For instance the implication \([3,7] \rightarrow [9]\) is true \(^9\) but neither the assumptions of theorem 1 nor that of theorem 2 are satisfied.

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\(^7\) See M. p. 162.

\(^8\) M. theorem VIII.

\(^9\) M. theorem IX.
It is difficult to predict at present whether the classes of sets and of functions dealt with in this paper will gain the same "right of citizenship" in metamathematics as the class of general recursive sets or functions. I have therefore not so much developed the theory of these classes themselves as tried rather to give some applications and to detect relations between the new classes and notions already known in this field. This explains why proofs of several known theorems are given in this paper (see 4.21, 4.43, 4.48, 5.51, 5.61). I think that owing to the use of methods familiar in the theory of projective sets I obtained not only considerable simplifications of the proofs but also some slight generalisations of the results themselves.

It seems to be possible to develop very extensively the theory of the new classes on the pattern of the theory of projective sets. From this kind of problems only one will be discussed here, to wit the analogue of Souslin's theorem\( ^{5)}\), i.e. the theorem that a recursively enumerable set whose complement is also recursively enumerable must be general recursive\(^ {6)\). The utility resulting from the analogy with projective sets is thus I think demonstrated.

§ 1. Classes \( P_ {\alpha} \) and \( Q_ {\alpha} \).

1.1. Preliminary remarks. Terminology. Metamathematical concepts occurring below (e.g. propositional function, formal proof, etc.) refer to a fixed self-consistent logical system \( S \) in which the theory of addition and of multiplication of positive integers can be built up. Hence for \( S \) may be taken e.g. the system of Principia Mathematica of course reformulated so as to render the system more exact\(^ {7)\). As the subsequent investigations are in high degree independent from the particular choice of the system \( S \) I shall give a mere sketch of its structure instead of a detailed description.

In the system \( S \) occur variables \( \varepsilon, \gamma, \ldots \) of the type of positive integers\(^ {8)\) as well as signs denoting the numbers \( 1, 2, 3, \ldots \).

\[ x = y, \quad x < y, \quad x = y + e, \quad x = y \cdot e, \quad x = y^{e} \]

with their usual meaning.

If we substitute in a propositional function, e.g. in \( \phi(x, y) \), for \( \varepsilon \) the sign denoting the number \( k \) and for \( y \) the sign denoting the number \( l \) we get a sentence which will be denoted by \( \phi(k, l) \).

The implication and the conjunction of two propositional functions \( \varphi \) and \( \psi \) will be written as \( \varphi \to \psi \) and \( \varphi \land \psi \), the negation of \( \varphi \) as \( \neg \varphi \). For quantifiers we use letters \( \forall \) and \( \exists \) with a variable written below.

We admit that the ordinary rules of inference and the ordinary arithmetical axioms are valid in \( S \). The formula \( : \varphi \) means that \( \varphi \) is a valid sentence, i.e. that there exists a formal proof of \( \varphi \).

It will be admitted that it is possible to put variables, propositional functions and formal proofs of the system \( S \) in one-to-one correspondence with positive integers\(^ {9)\}. These integers will be called the Gödel-numbers of variables or of propositional functions or of proofs. The correspondence is supposed to be not arbitrary but to fulfill some conditions which will be formulated in 3.1\(^ {10)\}.

In the simplest case \( S \) contains no other variables than those of the type of positive integers and no other propositional functions than those which can be built up from the propositional functions (1) with the help of quantifiers and logical connectives \( \to, \land, \neg \) and \( \exists \).

In this case \( S \) will be spoken of as the system of elementary arithmetic and denoted by \( \mathfrak{N} \).

The logical symbols: negation \( \neg \), implication \( \to \), equivalence \( = \), conjunction \( \land \), alternative \( \lor \) and quantifiers occur also (and more frequently) as synonyms of words not", "if... then..." etc. They are then used not as signs (primitive or defined) of the formal system \( S \) but as words of our ordinary language in which we are speaking about the system \( S \). Using Carnap's

\(^{5)\} \) Kuratowski [10], p. 251, Corollary 1.

\(^{6)\} \) After having finished the first draft of this paper I became acquainted with the paper Post [12] from which I see that this result has been obtained by E. L. Post already in 1944. From letters I understand that A. Tarski has also found the same theorem.

\(^{7)\} \) Such exact reformulations are given in Gödel [3] and Tarski [17].

\(^{8)\} \) \( S \) can contain also other types of variables.

\(^{9)\} \) Gödel [3], p. 178.

\(^{10)\} \) They represent a generalization of the three conditions of recursivity formulated in Hilbert-Bernays [5], p. 293-294.
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The proof is obvious.
According to 1.21 and 1.22 the connectives of propositional calculus as well as the "limited quantifiers" $\prod [\psi(<x) \rightarrow \ldots]$ and $\Sigma [\psi(<x) \rightarrow \ldots]$ if applied to decidable propositional functions yield again decidable functions. It will be seen later (in 4.21) that the unlimited quantifiers $\prod$ and $\Sigma$ may give undecidable propositional functions.

1.23. Let $\psi(x, n)$ and $\varphi(x, n)$ be two decidable propositional functions with $m + k$ and $l + k$ free variables. Let $\psi(x, n)$ fulfill the conditions:

\[
\prod \prod \prod [\psi(x, n) \rightarrow \varphi(x, n)] \quad (3)
\]

(3) For any $m \in R$, there is $n \in R$ such that $\neg \varphi(m, n)$. Under these assumptions the propositional function

\[
\sum \bigl[ \psi(x, n) \cdot \varphi(x, n) \bigr]
\]

is decidable.

Proof. Let us denote by $\theta(x, n)$ the propositional function (4) and assume that $m \in R$, $n \in R$. Assume that non-$\neg \theta(p, m)$ and denote by $n$ a point of $R$ such that $\neg \varphi(m, n)$. Hence it cannot be $\neg \varphi(p, n)$, since $\varphi(p, n)$ would then have $\neg \theta(p, m)$ against our assumption. Therefore $\neg \varphi(p, n)$ and hence

\[
\neg \bigl[ \varphi(m, n) \rightarrow \varphi(p, n) \bigr] \quad (5)
\]

The formula $\neg \varphi(m, n)$ yields together with (3)

\[
\neg \prod \bigl[ (\neg n) \rightarrow \varphi(m, n) \bigr] \quad (6)
\]

and hence by the ordinary rules of propositional calculus

\[
\neg \prod \bigl[ (\neg n) \rightarrow \varphi(m, n) \bigr] \rightarrow \varphi(p, n) \bigr] \quad (7)
\]

Combining this with (5) we get $\neg \prod \bigl[ \varphi(m, n) \rightarrow \varphi(p, n) \bigr]$, i.e.

\[
\neg \varphi(p, m) \bigr] \quad (8)
\]

This proves that $\theta(x, n)$ is decidable.

13) Church [1], p. 4.
14) One can avoid this duality introducing other symbols in the formal system $\mathcal{S}$ and other in the meta-system. This is done e.g. in Gödel [3].
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The analogy with the theory of projective sets needs not to be emphasised.

The class \( P^{(0)} = Q^{(0)} \) plays in our theory the same role as the class of Borel-subsets of \( k \)-dimensional space plays in the theory of projective sets.

We see from the definition that the rules of inference admitted in the system \( S \) permit to decide whether any given "point" \( n \) belongs to any given set \( A \) of the class \( P^{(0)} \) or not. Hence \( P^{(0)} \) is the class of general recursive sets mentioned in the introduction. This will be proved formally in 6.31.

From classes \( P^{(n)} \) and \( Q^{(n)} \) with \( n \geq 1 \) only one as far as I see is known in the literature. It is the class \( P^{(0)} \) which was called by Kleene the class of recursively enumerable sets \( 13 \). It will be shown later that \( A \in P^{(0)} \) if and only if \( A \) is the set of values of a general recursive function (see 6.61 and 6.23).

The whole sum \( \sum n (P^{(n)} + Q^{(n)}) \) may be characterized as the class of sets \( A \subseteq \mathbb{N} \) which are definable within the elementary arithmetics. The word "definable" is here used in the following sense: a set \( A \subseteq \mathbb{N} \) is definable within \( A \) if there is a propositional function \( \varphi(x) \) with \( k \) free variables such that\(^{14} \) for any \( n \in \mathbb{N} \) for any \( n \in \mathbb{N} \) for any \( n \in \mathbb{N} \) for any \( n \in \mathbb{N} \) for any \( n \in \mathbb{N} \)

3. Definition of classes \( P^{(n)} \) and \( Q^{(n)} \). A set \( A \subseteq \mathbb{N} \) will be said to belong to the class \( P^{(n)} \) if there is a decidable propositional function \( \varphi(x) \) with \( k \) free variables such that

\( n \in A \Rightarrow \varphi(n) \)

for any \( n \in \mathbb{N} \). We say that \( \varphi(x) \) defines \( A \). For reasons of symmetry we shall denote the class \( P^{(n)} \) also by \( Q^{(n)} \).

Let us now suppose that \( n \geq 1 \) and that classes \( P^{(n)} \) and \( Q^{(n)} \) (\( k = 1, 2, 3, \ldots \)) have already been defined. We then say that a set \( A \subseteq \mathbb{N} \) belongs to the class \( P^{(n)} \) if there is a set \( B \subseteq \mathbb{N} \) such that for any \( n \in \mathbb{N} \)

\( n \in A \Rightarrow \sum p (n, p) \in B. \)

A set \( A \subseteq \mathbb{N} \) belongs to \( Q^{(n)} \), if \( B \subseteq A \in P^{(n + 1)}. \)

\( ^{13} \) Kleene [10], theorem XI, p. 739.

\( ^{14} \) Tarski [18], p. 312.

\( ^{15} \) Tarski [18].
under the cartesian multiplication by an axis (2.14); the sum (or the common part) of an enumerable sequence of sets belonging to \( P^{n}_n \) (or to \( Q^{n}_n \)) belongs under certain assumptions to the same class (2.16). The remaining theorems are lemmas.

2.11. If \( A \in P^{n}_n \) and \( B \in P^{n}_n \), then \( A \cdot B, A \cdot B \) and \( R_{\alpha} - A \) belong to \( P^{n}_n \).

This proposition which follows immediately from 1.21 states that \( P^{n}_n \) is a field of sets.

2.12. If \( A \in P^{n}_n \), then \( R_{\alpha} - A \in Q^{n}_n \) and conversely.

This follows directly from definition.

2.13. (Change of axes). If \( \pi(1), \pi(2), \ldots, \pi(k) \) is any permutation of \( 1, 2, \ldots, k \) and if we denote for any \( A \in E_k \) the set of all \( (n_{\pi(1)}, n_{\pi(2)}, \ldots, n_{\pi(k)}) \) for which \( (n_{\pi(1)}, n_{\pi(2)}, \ldots, n_{\pi(k)}) \in A \), then \( A \in P^{n}_n \) implies \( A \in P^{n}_n \) and \( A \in Q^{n}_n \) implies \( A \in Q^{n}_n \).

The easy proof of this by induction on \( n \) will not be given here.

2.14. (Cartesian products). If \( A \in P^{n}_n \) (or \( A \in Q^{n}_n \)), then \( A \times B \in P^{m+n}_m \) (or \( A \in Q^{m+n}_n \)).

Proof. Suppose first that \( n = 0 \), and let \( \varphi(x) \) be a decidable propositional function which defines \( A \). The propositional function \( \varphi(x) \cdot (x = x) \) is of course decidable and defines \( A \times B \). Hence \( A \times B \in P^{m+n}_m \).

Suppose now that \( 2.13 \) holds for \( n < m \) and let \( A \in P^{n}_n \). By definition there is a set \( B \in Q^{m-n}_n \) such that

\[
\sum B = \{ (n, q) \in E : n \in B \}.
\]

The set \( B = \sum B \times R_{\alpha} \) arises from \( B \times R_{\alpha} \) by interchanging the two last axes and therefore \( B \in Q^{n+m}_n \) by 2.13. Since we have obviously

\[
(n, p) \in A \times B_{\alpha} \Rightarrow \sum (n, p, q) \in B_{\alpha},
\]

we infer from the definition that \( A \times B_{\alpha} \in P^{n+m}_m \).

Suppose now that \( A \in Q^{n}_n \), i.e. \( R_{\alpha} - A \in P^{n}_n \). If we repeat the above reasoning taking \( R_{\alpha} \) instead of \( A \), we obtain \( (R_{\alpha} - A) \times B_{\alpha} \in P^{n+m}_m \) or passing to complements and using 2.12

\[
R_{\alpha} - A = (R_{\alpha} - A) \times B_{\alpha} \in Q^{n+m}_n.
\]

The left side is identical with \( A \times B \), what completes the proof.

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Let us put for any \( A \in E_k \)

\[
A^* = \left\{ (n, p) \in E : (n, p) \in A \right\}.
\]

2.15. Let \( A \in P^{m+n}_n \) and \( A \in Q^{m+n}_n \).

Proof by induction on \( n \). Suppose first that \( n = 0 \) and that \( A \in P^{m+n}_n \). Let \( \varphi(t, u, v) \) be a decidable propositional function with \( t + 2 \) free variables which defines \( A \) and consider the propositional function

\[
\sum \varphi(t, u, v) \cdot z_{1}(t, u) \cdot z_{2}(v, t),
\]

where \( z_{1}(t, u) \) and \( z_{2}(v, t) \) have the meaning defined in 2.15. It is obvious that this function defines \( A^* \).

Now, suppose that \( n > 0 \) and that the theorem holds for \( n < m \). Let \( A \in P^{m+n}_m \). Hence there is a set \( B \in Q^{n+m-m}_n \) such that

\[
(n, p, q) \in A \Rightarrow \sum (n, p, q, k) \in B.
\]

The set

\[
B_{\alpha} = \sum (n, p, q, k) \in B_{\alpha}
\]

arises from \( B \) by interchanging the two last axes; consequently \( B \in Q^{n+m}_n \) by 2.13 and the inductive assumption. Now we see that

\[
(n, t, u, v) \in A \Rightarrow \sum (n, p, q, k) \in B_{\alpha} \Rightarrow \sum (n, p, q, k, h) \in B_{\alpha},
\]

which proves according to the definition 1.3 that \( A^* \in P^{n+m}_n \).

Suppose now that \( A \in P^{m+n}_m \), i.e. that there is a set \( B \in Q^{n+m}_n \) such that

\[
(n, t) \in A \Rightarrow \sum (n, p, q, k, h) \in B_{\alpha}.
\]
Let $B_1$ and $B_2$ be defined by the equivalences

\[(n, h, 1) \in B_1 = (n, 1, 1) \in B_1,
\]
\[(n, p, q, h) \in B_2 = (n, h, p, q) \in B_2.
\]

Obviously $B_2^2 = B_1$. Since $B_1 \in Q_n^{(3)}$ by 2.13, we obtain from the inductive assumption $B_2 \in Q_{n+1}^{(3)}$ and again by 2.13 $B_2 \in Q_{n+1}^{(3)}$. Now observe that

\[(n, p, q, h) \in A = \frac{1}{2} \sum_{q} (n, 2(3q + 1), h) \in B
\]
\[= \frac{1}{2} \sum_{q} (n, h, 2(3q + 1)) \in B_1 = \frac{1}{2} \sum_{q} (n, h, p, q, h) \in B_2.
\]

This equivalence proves that $A \in P_n^{(3)}$ and hence $A \in P_n^{(3)}$. Passing to complements and observing that

\[(A_1 + A) = (B_2 + A),
\]
we obtain immediately

\[A \in Q_n^{(3)} = A^* \in Q_n^{(3)}.
\]

Theorem 2.15 is thus proved completely.

2.16. (Infinite sums and products). If $n \geq 1$, then

\[A \in P_n^{(3)} \rightarrow A_x = P_n^{(3)}
\]
and

\[A \in Q_n^{(3)} \rightarrow A_p = Q_n^{(3)}.
\]

Proof. Suppose first that $A \in P_n^{(3)}$. For a suitable $B \in Q_n^{(3)}$ we have the equivalence

\[(n, p, q, h) \in A = \sum_{q} (n, p, q) \in B.
\]

Remembering the definition of the set $B^*$ we obtain

\[n \in A = \sum_{p, q} (n, q) \in A = \sum_{q} (n, p, q) \in B = \sum_{q} (n, h) \in B^*,
\]
for putting $h = 2(3q + 1)$ we have $(n, p, q) \in B = (n, h) \in B^*$. Since $B^* \in Q_n^{(3)}$, the above equivalence proves that $A \in P_n^{(3)}$. In order to obtain the second result stated in the theorem it is now sufficient to observe that

\[R_1 - A = (R_2 + A)_x.
\]
2.2. The Kuratowski-Tarski method. This well known method permits to evaluate the Borel class of the projective class of any set provided that its definition has been written down in logical symbols. The method is based on propositions of exactly the same form as our theorems 2.11, 2.15 and 2.17. Hence initiating this method we may from the mere form of the definition of any given set $A \in \mathbb{R}$ evaluate a $u$ for which $A \in \mathbb{P}_u^{(0)}$ or $A \in \mathbb{Q}_u^{(2)}$. This illustrates the importance of theorems established in 2.1.

2.3. Inclusions. Between the classes $\mathbb{P}_u^{(0)}$, $\mathbb{P}_{u+1}^{(0)}$, $\mathbb{Q}_u^{(0)}$ and $\mathbb{Q}_{u+1}^{(0)}$ hold following inclusions:

$$\mathbb{P}_u^{(0)} \subseteq \mathbb{P}_{u+1}^{(0)}, \mathbb{Q}_u^{(0)} \subseteq \mathbb{Q}_{u+1}^{(0)}.$$

Proof. Suppose that $A \in \mathbb{P}_u^{(0)}$ and put $A_1 = A \times R_1$. Evidently $u = \bigcap_{n \geq 1} \bigcap_{p < n} A_n = \bigcap_{n \geq 1} \bigcap_{p < n} R_{k+1} - A_1$; since $R_{k+1} - A_1 \subseteq R_{k+1}$ by 2.12 and 2.14, we infer direct from the definition 1.3 that $A \in \mathbb{Q}_{u+1}^{(0)}$. Hence

$$\mathbb{P}_u^{(0)} \subseteq \mathbb{Q}_{u+1}^{(0)}.$$

Passing to complements, we obtain

$$\mathbb{Q}_u^{(0)} \supseteq \mathbb{P}_{u+1}^{(0)}.$$

This gives for $n = 0$ the inclusions $\mathbb{Q}_0^{(0)} \subseteq \mathbb{Q}_1^{(0)}$ and $\mathbb{P}_0^{(0)} \subseteq \mathbb{P}_1^{(0)}$. Suppose now that $m > 0$ and that the inclusions

$$\mathbb{P}_n^{(k+1)} \subseteq \mathbb{P}_{n+1}^{(k+1)}, \mathbb{Q}_n^{(k+1)} \subseteq \mathbb{Q}_{n+1}^{(k+1)}$$

are valid for $n < m$. If $A \in \mathbb{P}_m^{(0)}$, then, for a suitable $B \in \mathbb{Q}_m^{(k+1)}$ we have $n = \bigcap_{n \geq 1} \bigcap_{p < n} A_n = \bigcap_{n \geq 1} \bigcap_{p < n} B$ which proves that $A \in \mathbb{P}_{m+1}^{(0)}$, since $B \subseteq B_{m+1}$ by the inductive assumption. Hence the first inclusion (8) is true for $n = m$ and passing to complements we obtain the same result for the second one. (8) is thus true for any $n$. The theorem results now from (6), (7) and (8).

§ 3. Existence theorems.

3.1. Condition $C_n$. It is now an appropriate place to formulate the conditions imposed upon the enumeration of variables, propositional functions and formal proofs of the system $S$. We shall consider the following sets:

$$I_k = \{ p \mid [\text{there is a propositional function } \varphi(x) \text{ with } k \text{ free
variables such that } p \text{ is the G"odel-number of } \varphi(x)\},$$

$$A = \{ q \mid [q \text{ is the G"odel-number of a formal proof of the sentence}\}$$

$$E = \{ q \mid [q \text{ is the G"odel-number of a formal proof}\},$$

$$H = \{ p \mid [p \text{ is the G"odel-number of a propositional function and } q \text{ that of its negation}\}.$$

The condition $C_n$ requires that these sets should belong to classes $\mathbb{P}_u$ with suitable upper indices:

$$(C_n) \quad \mathbb{S}_n \subseteq \mathbb{P}_{u+2}^{(2n)}, \ A \subseteq \mathbb{P}_{u+1}^{(1n)} \quad E \subseteq \mathbb{P}_{u+1}^{(2n)}, \ H \subseteq \mathbb{P}_{u+2}^{(3n)}.$$

Most systems fulfill the simplest condition $C_0$; namely all those in which a formal proof consists on performing step by step some well defined finitary rules of inference. Thus e.g. the system $P$ considered by G"odel fulfills the condition $C_0$ as is proved in details in G"odel [3] pp. 179-186. Systems with non-finitary rules of inference fulfill, in general, the condition $C_n$ for some $n > 0$ but not the condition $C_0$. For examples of such systems see e.g. Rosser [33].

3.2. Universal functions. We proceed to establish, for systems $S$ which fulfill the condition $C_n$, the existence of sets belonging to any given class $\mathbb{P}_u^{(0)}$ or $\mathbb{Q}_u^{(0)}$ but not to preceding ones. The proof is based as in the theory of Borel-sets or projective sets on the concept of universal functions $\mathbb{P}_u^{(0)}$.

Let $X$ be any class of sets. A universal function for $X$ is any function $F(h)$ defined for $h = 1, 2, 3, \ldots$ such that

$$A \subseteq X \iff \bigcap_h A = F(h).$$


$)**$ There are, however, no rules in our theory which would correspond to theorems concerning infinite sums or products of Borel (or projective)

sets.

$)^{\text{a)}}$ The assumption $E \subseteq \mathbb{P}_u^{(0)}$ is irrelevant for our present purposes but we shall need it in 5.62.

$)^{\text{b)}}$ Kuratowski [10], p. 172 and 241.
3.21. If $S$ fulfills the condition $C_0$, then there is a universal function $F_{20}^S(h)$ for the class $P_{20}^S$ such that the set

$$M_{20}^S = \{ n \in \mathbb{N} \mid \exists F_{20}^S(h) \}$$

belongs to $Q_{w2+1}^{(h)}$.

Proof. An integer $h$ is the Gödel-number of a decidable propositional function with $h$ free variables if and only if

$$\prod_{x \neq 2} \sum_{x \neq 2} \left( \frac{[\bar{l}(x), \bar{l}(m), \bar{l}(m), \bar{l}(l), \bar{l}(l), \bar{l}(h, m) \in \bar{A}] \cdot \bar{A}}{x \neq 2} \right).$$

Hence denoting with $\Theta_1$ the set of these numbers we infer by the Kuratowski-Tarski method that

$$\Theta_1 \in Q_{w2+1}^{(h)}.$$ \hspace{1cm} (9)

We put now $F_{20}^S(h) = 0$ for $h \neq 0$ and $F_{20}^S(0) = \{ \sum_{x \neq 2} \left( \frac{[\bar{l}(x), \bar{l}(m), \bar{l}(m), \bar{l}(l), \bar{l}(l), \bar{l}(h, m) \in \bar{A}] \cdot \bar{A}}{x \neq 2} \right) \}$

for $h \in \Theta_1$. The set $M_{20}^S$ belongs then to $Q_{w2+1}^{(h)}$ as we easily see from its definition

$$(n, h) \in M_{20}^S \iff h \in \Theta_1 \cdot \{ \sum_{x \neq 2} \left( \frac{[\bar{l}(x), \bar{l}(m), \bar{l}(m), \bar{l}(l), \bar{l}(l), \bar{l}(h, m) \in \bar{A}] \cdot \bar{A}}{x \neq 2} \right) \}$$

using (9), 2.31 and the Kuratowski-Tarski evaluation method.

It remains to prove that $F_{20}^S(h)$ is a universal function for the class $P_{20}^S$.

Let us suppose that $A \in P_{20}^S$ and that $\varphi(x)$ defines $A$. If $h$ is the Gödel-number of $\varphi(x)$, then $h \in \Theta_1$. Let $n \in A$ and let $p$ be the Gödel-number of $\varphi(n)$. Then $(h, p, n) \in \bar{A}$. Since $\bar{A}$ is a formal proof of $\varphi(n)$, there is a formal proof of $\varphi(n)$. Denoting with $g$ its Gödel-number, we have $(g, p) \in \bar{A}$. Now from $h \in \Theta_1$, $(h, p, n) \in \bar{A}$ and $(g, p) \in \bar{A}$ we obtain $n \in F_{20}^S(h)$. If, conversely, $n \in F_{20}^S(h)$, then there are $p, q$ such that $(h, p, n) \in \bar{A}$ and $(g, p) \in \bar{A}$. Hence $p$ is the Gödel-number of $\varphi(n)$ and $q$ is the Gödel-number of a formal proof of $\varphi(n)$. Hence there is at least one formal proof of $\varphi(n)$ which proves that $\varphi(n)$ i.e. $n \in \bar{A}$; therefore $A = F_{20}^S(h)$ which proves that

$$A \in P_{20}^S \iff \sum_{n \in A} F_{20}^S(h).$$

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Suppose, conversely, that $A = F_{20}^S(h)$. If $h$ non $\in \Theta_1$, then $A = 0$ and therefore $A = P_{20}^S$. If $h$ $\in \Theta_1$, let $\varphi(x)$ be the decidable propositional function whose Gödel-number is $h$. We prove similar as above that $n \in A = \varphi(n)$ and therefore $A \in P_{20}^S$. Hence

$$\sum_{n \in A} A = F_{20}^S(h) \rightarrow A \in P_{20}^S.$$ This completes the proof of 3.21.

3.22. If $S$ fulfills the condition $C_0$, then there are functions $E_{20}^S(h)$ and $G_{20}^S(h)$ universal for classes $P_{20}^S$ and $Q_{20}^S$ and such that the sets

$$M_{20}^S = \sum_{n \in E_{20}^S(h)} \text{ and } N_{20}^S = \sum_{n \in G_{20}^S(h)}$$

belong respectively to $P_{w2+1}$ or to $Q_{w2+1}$ if $n > 0$ and to $Q_{w2+1}$ if $n = 0$.

Proof. The theorem was proved in 3.21 for $n = 0$. Suppose that $n \geq 0$ and the theorem is true for this value of $n$ and for $k = 1, 2, \ldots$. Put

$$F_{20}^{S+1}(h) = \sum_{n \in A} F_{20}^S(h),$$

$$G_{20}^{S+1}(h) = R_h \cdot F_{20}^S(h).$$

If $h$ is any integer, then $F_{20}^{S+1}(h) = P_{20}^{S+1}$, since

$$n \in F_{20}^{S+1}(h) \iff \sum_{n \in A} \sum_{n \in B} F_{20}^S(h)$$

and $G_{20}^{S+1}(h) = G_{20}^S$ by the inductive assumption. Suppose, conversely, that $A \in P_{20}^{S+1}$, i.e.

$$n \in A \iff \sum_{n \in A} B$$

for a suitable $B \in Q_{w2+1}^{(h)}$. The function $G_{20}^{S+1}(h)$ being universal for $Q_{w2+1}^{(h)}$, there is an $h$ such that $B \in G_{20}^{S+1}(h)$ and therefore $A = F_{20}^{S+1}(h)$. Hence $F_{20}^{S+1}(h)$ is a universal function for $P_{20}^{S+1}$. Turning to complements we immediately see that $G_{20}^S(h)$ is universal for $Q_{w2+1}^{(h)}$.

It remains to consider the sets $M_{20}^{S+1}$ and $N_{20}^{S+1}$. According to their definitions we have

$$(n, h) \in M_{20}^{S+1} \iff n \in F_{20}^{S+1}(h) \iff \sum_{n \in A} F_{20}^S(h) = \sum_{n \in A} G_{20}^S(h) = \sum_{n \in A} B = \sum_{n \in A} B.$$
and this proves that $M^{(n)}_{g+1} \in P^{(n)}_{g+1, n+4}$, since $N^{(g+1)}_{a} \in Q^{(g+1)}_{a+2}$ by the inductive assumption. Further we have

\[(n, h) \in N^{(g+1)}_{a} = \pi \in G^{(g+1)}_{a+1}(h) = \pi \pi \in F^{(g+1)}_{a+1}(h) = E_{a+1} = M^{(g+1)}_{a+1},\]

and therefore $N^{(g+1)}_{a+2} \in Q^{(g+1)}_{a+2}$. The theorem is thus proved completely.

3.3. Existence-theorems. They follow now easily by the well-known Cantor's diagonal-theorem 4.

3.31. If $S$ fulfills the condition $C$, then $N^{(g)}_{a} \in P^{(g)}_{a+1}$ and $Q^{(g)}_{a} \in Q^{(g)}_{a+1}$ for any $g \geq 0$ and $a \geq 1$.

Proof. Let us suppose that $S$ fulfills the condition $C$ and that $P^{(g)}_{a} = F^{(g)}_{a}$ for some $g$ and $a$. We shall show by induction on $g$ that

\[P^{(g)}_{a} = F^{(g)}_{a} = Q^{(g)}_{a} \quad \text{for} \quad a \geq n.\]

This holds for $g = n$ since we have $Q^{(g)}_{a} \subseteq P^{(g)}_{a+1} = P^{(g)}_{a}$ which implies $P^{(g)}_{a} \subseteq Q^{(g)}_{a}$ and therefore $P^{(g)}_{a} = Q^{(g)}_{a}$. Now suppose that (10) holds for an integer $g \geq n$. Obviously $P^{(g)}_{a} \subseteq P^{(g+1)}_{a+1}$. If $A \in P^{(g+1)}_{a+1}$ then there is a set $B \in Q^{(g+1)}_{a+1}$ such that

\[\pi \in A = \sum_{(n, p) \in B} (n, p) \in B.\]

Let us write $(a_{1}, a_{2}, \ldots, a_{k})$ instead of $n$ and consider the set (see 2.15)

\[B^{*} = \sum_{(n_{1}, \ldots, n_{k-1})} (n_{1}, \ldots, n_{k-1}, a_{k}(g), s_{k}(g)) \in B.\]

Since $B^{*} \subseteq Q^{(g)}_{a}$, we have $B^{*} \subseteq P^{(g)}_{a}$ in virtue of the inductive assumption and the equivalence

\[\pi \in A = \sum_{(n, p) \in B} (n_{1}, \ldots, n_{k-1}, a_{k}(g)) \in B^{*}\]

proves that $A \in P^{(g)}_{a} = F^{(g)}_{a}$. Hence $P^{(g)}_{a+1} = P^{(g)}_{a}$. Passing to complements we obtain $Q^{(g)}_{a+1} = Q^{(g)}_{a} = P^{(g)}_{a}$. The formula (10) is thus proved.

Consider now the universal function $F^{(g)}_{a}(h)$ defined in 3.32 and put

\[A = \sum_{(n_{1}, \ldots, n_{k-1}, h) \in B} (n_{1}, \ldots, n_{k-1}, h) \in F^{(g)}_{a}(h).\]

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This set does not belong to $P^{(g)}_{a}$ because otherwise there would be an integer $h_{a}$ such that $A = F^{(g)}_{a}(h_{a})$ which is impossible since we would then have

\[(h_{a}, h_{a}, \ldots, h_{a}) \in A = (h_{a}, h_{a}, \ldots, h_{a}) \non \epsilon F^{(g)}_{a}(h_{a}) = (h_{a}, h_{a}, \ldots, h_{a}) \non \epsilon A.\]

Observe now that

\[\pi \in A = \sum_{(n_{1}, \ldots, n_{k-1}, a_{k}(g), s_{k}(g)) \in B} (n_{1}, \ldots, n_{k-1}, a_{k}(g), s_{k}(g)) \in B.\]

which proves according to 3.22 that $A \in P^{(g)}_{a+1}$ and consequently $A \in F^{(g)}_{a}$ according to (10). The assumption $P^{(g)}_{a+1} = P^{(g)}_{a}$ leads thus to a contradiction.

The inequality $Q^{(g)}_{a} \subseteq Q^{(g)}_{a+1}$ will now result if we pass to complements on both sides of $P^{(g)}_{a} \subseteq P^{(g)}_{a+1}$.

3.32. If $S$ fulfills the condition $C$, and if $a > 0$, then $P^{(g)}_{a} \subseteq Q^{(g)}_{a}$ and $Q^{(g)}_{a} \subseteq P^{(g)}_{a}$.

Proof. From $P^{(g)}_{a} \subseteq Q^{(g)}_{a}$ we obtain $Q^{(g)}_{a+1} \subseteq P^{(g)}_{a+1}$ and hence

\[P^{(g)}_{a} = Q^{(g)}_{a} \quad \text{for} \quad a \geq n.\]

Let us first suppose that $k > 1$ and $A \epsilon P^{(g)}_{a+1}$. For a suitable $B \epsilon Q^{(g)}_{a}$ we have

\[\pi \in A = \sum_{(n, p) \in B} (n, p) \in B.\]

$B$ being in $Q^{(g)}_{a}$, it is also in $P^{(g)}_{a}$ in virtue of (11) and hence $A \epsilon F^{(g)}_{a+1}$ by 2.18. We obtain thus $P^{(g+1)}_{a} \subseteq P^{(g)}_{a}$ against 3.31.

Suppose now that $k = 1$. If $A \epsilon P^{(g)}_{a}$ or $A \epsilon Q^{(g)}_{a}$, then the set $A_{1} = \sum_{(n, p) \in B} (n, p) \epsilon A$ belongs to $P^{(g)}_{a}$ or $Q^{(g)}_{a}$ since $A_{1} = A$ (comp. 2.15). It is obvious that every set of $P^{(g)}_{a}$ or $Q^{(g)}_{a}$ may, for a suitable $A \epsilon P^{(g)}_{a}$ or $A \epsilon Q^{(g)}_{a}$, be represented as $A_{1}$. Hence $P^{(g)}_{a} = Q^{(g)}_{a}$ would lead to the equality $P^{(g)}_{a} = Q^{(g)}_{a}$ which we already know to be impossible.

We have thus proved that $P^{(g)}_{a} \subseteq Q^{(g)}_{a}$ for any $k$ and $a > 0$. Passing to complements we obtain $Q^{(g)}_{a} \subseteq P^{(g)}_{a}$, q.e.d.
A. Mostowski:

We note at last the following result concerning the existence of sets not definable within arithmetic, i.e. belonging to no class $P^0_0$.

3.33. The function $F^{(0)}_{a_0}(x_2(n))$ is universal for the class $\sum_i P^0_i$ and the set

$$A_b = \sum_{\neg x_2(n)} [x_{b_1}, \ldots, x_{b_{-1}}, n] \text{ non } F^{(0)}_{a_0}(x_2(n))$$

does not belong to the sum $\sum_i P^0_i$.

Proof. The first part of the theorem results from the equivalence

$$A \in \sum_i P^0_i \Rightarrow \sum_i A \in P^0_i = \sum_i A = -P^0_0(\neg x_2(n)).$$

The second part is a particular case of the "diagonal theorem" referred to in the footnote.

§ 4. Applications to theorems of Gödel and Rosser.

4.1. $\omega$-consistency. Let us recall the following definition due to Gödel [14]: A logical system $S$ is called $\omega$-consistent if, for any propositional function $\psi(x)$ with one free variable, the following implication holds:

$$\sum_{\psi(y)} \psi(n) \Rightarrow \psi(n).$$

We could, of course, replace this implication by

$$\sum_{\neg \psi(y)} \neg \psi(n).$$

It is important to observe that the quantifier $\sum_{\psi}$ is taken meta-mathematically whereas $\sum_{\neg \psi}$ represents a sentence of the formal system $S$.

4.2. Gödel's theorem. It states

4.21. If the system $S$ is $\omega$-consistent and fulfills the condition $C_0$, then there is a sentence $\theta$ such that neither $\neg \theta$ nor $\theta$.

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Proof. According to 3.31 the class $F^{(0)}_0 - P^0_0$ is non-empty. Let $A$ be any set of this class and let $B \in P^0_0$ be a set such that $n \in A \Rightarrow \sum_{\neg x_2(n)} B$. Denoting by $\psi(x,y)$ any definable propositional function which defines $B$ we have

$$(13) \quad \sum_{\neg x_2(n)} B = \sum_{\neg x_2(n)} \neg \psi(n,p).$$

Write $\psi(x)$ instead of $\sum_{\neg x_2(n)} B$. The formula (13) gives then

$$(14) \quad \sum_{\neg x_2(n)} B = \sum_{\neg x_2(n)} \neg \psi(n).$$

This equivalence would prove that $A \in P^0_0$ if $\psi(x)$ were decidable. Since $A \in P^0_0$ $\psi(x)$ cannot be decidable, i.e. there is no integer $n_0$ such that neither $\neg \psi(n_0)$ nor $\psi(n_0)$. Denoting $\psi(n_0)$ by $\theta$ we obtain the desired result.

4.3. Remarks. 4.31. Theorem 4.21 was first established by Gödel for a concrete formal system called $P^n_0$. Rosser [20] generalized this result showing that it holds for any system $S$ in which the Gödel-numbers of valid sentences form a recursively enumerable set. This is essentially the same assumption as our condition $C_0$. Our proof of 4.21 shows that the theorem holds even under the weaker condition $C_0$. Hence the Gödel's theorem is valid for all such systems $S$ in which the set of Gödel-numbers of valid sentences is definable in elementary arithmetic.

4.32. If $S$ satisfies the stronger condition $C_0$, then as shown by Rosser [20] the assumption of $\omega$-consistency can be replaced by the (weaker) assumption of ordinary self-consistency of $S$. This is in general impossible for systems satisfying the weaker condition $C_0$ ($c > 0$) since there exists a logical system $S$ such that its valid sentences form a self-consistent and complete class whereas the set of their Gödel-number is definable within $S$ (i.e. be-

27] Gödel [3]: Satz VI.
28] Rosser [14]: Theorem 1 A, p. 89.
29] Rosser [14]: Theorem 11, p. 89.
30] I. e. for any $\theta$ either $\theta$ or $\theta'$ is valid.
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Proof. Glancing at the proof of 4.21 we see that the number \( n_0 \) which has been defined there does not belong to \( A \) since otherwise we would obtain \( \neg \psi(n_0) \) in virtue of (14). Hence (13) gives \( \prod_p \neg \psi(n_0 p) \) whereas non- \( \neg \psi(n_0) \) yields non- \( \prod_p \psi(n_0 p) \).

We introduce now the concept of an \( n \)-valid sentence (in symbols \( \neg \psi \)) as \( \neg \psi \). For \( n = 0 \) we define \( \neg \psi \) as \( \neg \psi \). Suppose now that \( n > 0 \) and that the class of \( n - 1 \)-valid sentences has already been defined. We shall write \( \neg \psi \) (read: \( \psi \) is an \( n \)-valid sentence) if \( \psi \) belongs to every class \( C \) satisfying three following conditions:

- If \( \neg \psi \), then \( \psi \) is in \( C \);
- If \( \prod \neg \psi(p) \), then \( \prod \psi(x) \) is in \( C \).

Speaking less formally, we could say that \( \neg \psi \) holds if and only if \( \psi \) can be obtained from the axioms of \( S \) with the help of rules of inference admitted in \( S \) and with the help of the rule of infinite induction, this last rule being used \( n \) times.

A propositional function \( \psi(x) \) with \( k \) free variables will be said to be \( n \)-decidable if for any \( n \in R \) either \( \neg \psi(n) \) or \( \neg \psi(n) \).

4.12. If the class of \( n \)-valid sentences is self-consistent and if \( A \in P_{n+1}^\alpha \), then there is an \( n \)-decidable propositional function \( \psi(x) \) with \( k \) free variables such that

\[
\sum_{n \in R} \psi(n) = 1
\]

for any \( n \in R \).

Proof by induction on \( n \). For \( n = 0 \) the theorem is obvious. Let us suppose that it holds for an integer \( n \geq 0 \) and for \( k - 1, 2, \ldots \)

If \( A \in P_{n+1}^\alpha \), then for a suitable \( B \in Q_{n+1}^\alpha \) we have

\[
\sum_{n \in R} \psi(n) = 1
\]

for any \( n \in R \). If a \( n + 1 \)-valid sentences form a self-consistent class, the same holds true for \( n \)-valid sentences and the inductive assumption yields the existence of a \( n \)-decidable propositional function \( \psi(x, y) \) with \( k + 1 \) free variables such that

\[
\psi(n, p) \in B = \neg \psi(n, p).
\]
§ 5. Functions of classes \( P_a^{\theta} \) and \( Q_a^{\theta} \).

5.1. Definitions. We denote by \( R_\alpha^\beta \) the class of functions mapping \( R_\alpha \) on a subset of \( R_\beta^\gamma \). A function \( f \in R_\alpha^\beta \) is said to be of class \( P_a^{\theta,0} \) or \( Q_a^{\theta,0} \) if the "curve"

\[
I_f = \int_{\infty}^{\infty} [m = f(n)]
\]

belongs to \( P_a^{\theta,0} \) or \( Q_a^{\theta,0} \).

Remark. In order to sustain the analogy with the theory of Borel functions it would be perhaps better to define \( P_a^{\theta,0} \) or \( Q_a^{\theta,0} \) as the class of functions \( f \) such that for any \( \alpha \in P_a^{\theta,0} \) the counter-image \( f^{-1}(A) \) is of class \( P_a^{\theta,0} \) or \( Q_a^{\theta,0} \). It will be proved in 5.3 that classes \( F_a^{\theta,0} \) and \( G_a^{\theta,0} \) defined above possess this property. The countable theorem seems, however, to be false. The analogy with the theory of Borel sets is here breaking down.

5.2. Images. We put for \( f \in R_\alpha^\beta \) and \( A \subset R_\alpha \)

\[
f(A) = \int_{\infty}^{\infty} [\sum (n \in A) \cdot (m = f(n))]
\]

and call \( f(A) \) the image of \( A \). Obviously

\[
m \in f(A) = \sum (n \in A) \cdot (m = f(n))
\]

from what the following theorem immediately results by the Kuratowski-Tarski evaluation method:

5.3.1. If \( A \in P_a^{\theta,0} \) and \( f \in Q_a^{\theta,0} \) (\( n \geq 0 \)), then \( f(A) \in P_a^{\theta,0} \) and if \( A \in P_a^{\theta,0} \) and \( f \in Q_a^{\theta,0} \) (\( n \geq 1 \)), then \( f(A) \in P_a^{\theta,0} \).

5.3. Counter-images. If \( f \in R_\alpha^\beta \) and \( A \subset R_\alpha \), then the counter-image of \( A \) is defined as

\[
f^{-1}(A) = \int_{\infty}^{\infty} [f(n) \in A]
\]

Evidently

\[
n \in f^{-1}(A) = \sum (m = f(n)) \cdot (m \in A) = \int \sum (m = f(n)) \cdot (m \in A).
\]

In virtue of 2.16 we obtain from these equivalences the following theorems:

5.3.1. If \( A \in P_a^{\theta,0} \) and \( f \in P_a^{\theta,0} \) (\( n \geq 1 \)), then \( f^{-1}(A) \in P_a^{\theta,0} \).

5.3.2. If \( A \in Q_a^{\theta,0} \) and \( f \in Q_a^{\theta,0} \) (\( n \geq 1 \)), then \( f^{-1}(A) \in Q_a^{\theta,0} \).

---

\( ^{w} \) Kuratowski [10], p. 190.

\( ^{w} \) Kuratowski [9], p. 177.
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The first equivalence (17) yields now (with respect to (18) and (19)) the implication:

\[ \pi \in f^{-1}(A) \rightarrow \sum_{\pi} (m \in A \cdot (m = f(n)) \rightarrow \sum_{\pi} \psi(m) \cdot \psi(n,m) \]
\[ \rightarrow \sum_{\pi} \psi(n) \cdot \psi(n,m) \rightarrow \psi(n) \]

whereas the second yields

\[ \pi \in f^{-1}(A) \rightarrow \sum_{\pi} (m non \in A \cdot (m = f(n)) \rightarrow \sum_{\pi} \psi(m) \cdot \psi(n,m) \rightarrow \sum_{\pi} \psi(n,\pi) \cdot \pi(n) \rightarrow \sum_{\pi} \psi(n) \]

i.e. with respect to (20) \[ \pi \in f^{-1}(A) \rightarrow \sum_{\pi} \psi(n) \rightarrow \sum_{\pi} \psi(n) \]

Hence \[ \pi \in f^{-1}(A) \rightarrow \psi(n) \] and therefore \[ f^{-1}(A) \in P_\pi^{(\Pi_1)} \], q.e.d.

5.4. The function \( \min \) \[ \{ (n,p) \in A \} \]. Let us suppose that \( A \) is a subset of \( R_{a+1} \) such that \( \prod_{n} \sum_{p} (n,p) \in A \) and denote by \( \mu \) the smallest integer \( p \) such that \( (n,p) \in A \):

\[ \{ p = \mu \} \mathbb{N} \]

The Kuratowski-Tarski method leads immediately to the following theorem:

5.41. If \( A \in Q^{(\Pi_1)} \), then \( \mu \in Q^{(\Pi_1)} \) and if \( A \in P_\pi^{(\Pi_1)} \) (\( \pi \geq 1 \)),

\[ \mu \in P^{(\Pi_1)} \]

For \( n = 0 \) we have the sharper evaluation:

5.42. If \( A \in P_\pi^{(\Pi_1)} \), then \( \mu \in P^{(\Pi_1)} \).

Proof. Denote by \( \psi(x) \) any definable propositional function which defines \( A \) and by \( \psi(x) \) the propositional function

\[ \psi(x) \rightarrow \psi(x,y) \]

(20) that

\[ \mu \]
5.5. Post's theorem. This theorem is an exact analogue of the well-known Souslin's theorem concerning sets which are analytically together with their complements. It can be stated as follows:

5.51. $P_0^{(i)} = Q_0^{(i)}$.

Proof. In virtue of 2.31 we only have to show that if $A \in P_0^{(i)}$ and $B_2 = A \in P_0^{(i)}$, then $A \not\in P_0^{(i)}$. Let $B_1$ and $B_2$ be two sets of $P_0^{(i+1)}$ such that

$$n \in A \iff \sum p \in B_1, \quad n \in B_2 \iff \sum p \in B_2.$$

Since $\prod (n \in A) \iff \prod (n \in B_2 \iff A)$, we have $\prod (\sum p \in B_1) \iff A$ which proves accordingly to 5.42 that $\mu_{n+2} \in P_0^{(i+1)}$, the sum $B_1 + B_2$ being of class $P_0^{(i+1)}$ by 2.17. Now define a function $f \in R_{i+1}$ putting for any $n \in B_2$:

$$f(n) = (\mu_{n+1} \sigma_{n+1}(n)).$$

We have $f \in P_0^{(i+1)}$, since

$$\{n = f(n) \iff \{n = m \iff (p = \mu_{n+1} \sigma_{n+1}(m)).$$

Evidently

$$n \in f^{-1}(B_1) \iff \sum p \in B_1, \quad \{n \in \mu_{n+1} \sigma_{n+1}(B_1) \iff \sum p \in B_1 \iff n \in A.$$

If, conversely, $n \in A$, then $(\mu_{n+2} \sigma_{n+2}(n)) \in B_2$, since otherwise we would obtain $\{\mu_{n+2} \sigma_{n+2}(n) \in B_2$ and therefore $\sum p \in B_2$ or $n \in B_2 = A$. Hence $f(n) \in B_2$ and $f^{-1}(B_2)$ proves that $A = f^{-1}(B_2)$ and the theorem 5.53 yields the desired result $A \in P_0^{(i)}$.

From 5.51 we obtain two important corollaries:

5.52. If $f \in P_0^{(i)}$ and $g \in P_0^{(i)}$, then the compounded function $f(g(m))$ is of class $P_0^{(i)}$.

Proof. We have

$$\{f(g(m)) = \sum p \in g(m), \quad \{f(g(m)) = \sum p \in f(n) = \{n = g(m) \iff (n = f(n)).$$

5.53. A set $A$ is in $P_0^{(i)}$, $Q_0^{(i)}$ if and only if its characteristic function $c_A$ is in $P_0^{(i)}, Q_0^{(i)}$.

Proof. From

$$c_A(n) = \{n \in A \iff \sum p \in B_2 \iff A \iff (p = 0)$$

we infer easily that if $A \in P_0^{(i)}, Q_0^{(i)}$, then $c_A \in P_0^{(i)}, Q_0^{(i)}$. Suppose now that $c_A \in P_0^{(i)}, Q_0^{(i)}$. From

$$\{c_A(n) = \sum p = 1 \iff (c_A(n) = p) = \prod (p = 1, \sum p = B_1 \iff A)$$

we see that if $A \subseteq B$, then $A \in P_0^{(i)}, Q_0^{(i)}$. For $n = m$ these equivalences yield $A \in P_0^{(i)}, Q_0^{(i)}$ and hence $A \in P_0^{(i)}, Q_0^{(i)}$ in virtue of 5.51.

It is interesting to observe that if $A \in P_0^{(i)}$ and $f \in P_0^{(i)}$, then the set $f(A)$ does not necessarily belong to $P_0^{(i)}$, even if $f$ is one-to-one. We see here another discrepancy between our theory and the theory of Borel-sets.

It can be shown, however, that if $f$ is an increasing function, i.e., if $n < m \iff f(n) < f(m)$, then $f(A) \in P_0^{(i)}$ (< represents here the lexicographical ordering of $k$-ads or $k$-ads of integers).

5.6. Sets of the class $P_0^{(i)}$ as values of functions $P_0^{(i)}$.

The theorem 5.51 enables us to give a simple proof of the following theorem which discloses the relationship between the concept of the class $P_0^{(i)}$ and that of recursively enumerable sets:

5.61. If $A$ fulfills the condition $G_0$, then the necessary and sufficient condition for a non-empty set $A$ be in $P_0^{(i)}$ is that there is a function $f \in P_0^{(i)}$ whose set of values is $A$.

---

84) This theorem has been proved by Kleene, see Kleene [5], theorem VII, p. 537, Rosser [14], Corollary 1, p. 45, Post [12], p. 291.
Proof. Sufficiency results at once from 5.21. Suppose now that $A \in P^{(k)}_0$ and $n_0 \in A$. Let $\varphi(x,\varepsilon)$ be a decidable propositional function with $k+1$ free variables such that

$$n \in A \iff \sum_{\varepsilon} \varphi(n,\varepsilon)$$

for any $n \in R_1$.

We shall denote by $g(n)$ the sequence of $k$ integers $s_1(s_1(n)), s_2(s_2(n)), \ldots, s_k(s_k(n))$. An easy induction on $k$ shows that for any $n \in R_2$ and $p, q \in R_1$, there is an integer $h$ such that $n = g(h)$, $p = s_q(h)$, and $q = s_h(h)$.

Let $n_0$ be the G"odel-number of $\varphi(x,\varepsilon)$.

Define now the function $f(n)$ as follows: if $g(n)$ is the G"odel-number of a formal proof of $\varphi(n,\varepsilon)$, then $f(n) = \omega(n)$; if not, then $f(n) = n_0$.

It is obvious that $f(n) \in A$ for any $n$. Conversely, if $n \in A$, then, for a suitable $p \in g(n)$, we denote by $q$ the G"odel-number of a formal proof of $\varphi(n,\varepsilon)$ and by $h$ the integer for which $g(h) = n$, $s_q(h) = q$, $s_h(h) = p$. Hence $A$ is the set of values of $f$.

It remains to consider the class of $f$. Remembering the definitions of the sets $A$, $E$ and $I^*_1$ given in 3.1 we see that

$$[m = f(n)] = ([m = g(n)] \vee [g(n) \in E] \cdot \sum_{\varepsilon} [s_q(n) \vee E] + \sum_{\varepsilon} [s_h(n) \vee E] + \sum_{\varepsilon} [s_q(n) \vee E] + \sum_{\varepsilon} [s_h(n) \vee E])$$

This proves the set $f(n)$ to be of class $P^{(k+1)}_0$. Remembering further that if $g(n) \in E$, then there is exactly one $q$ such that $s_q(n) \in E$, we can rewrite the above equivalence in the following form:

$$[m = f(n)] = [m = g(n)] \cdot [g(n) \in E] \cdot \sum_{\varepsilon} [\sum_{\varepsilon} [s_q(n) \vee E] + \sum_{\varepsilon} [s_h(n) \vee E]]$$

The set $f(n)$ is thus of class $P^{(k+1)}_0$ and hence by 5.51 it is of class $P^{(k+1)}_0$, q. e. d.

\[\sharp 6. \text{Relations with the theory of general-recursively definable functions}\]

6.1. Recursivity conditions. We shall suppose that the system $S$ fulfills the following two conditions:

\begin{enumerate}
  \item[(R_1)] Primitive recursive subsets of $R_1$ belong to $P^{(k)}_0$;
  \item[(R_2)] If $\varphi$ is any propositional function with $k$ free variables, then the relation $g(n)$ which holds between $q$ and $n$ if and only if $q$ is the G"odel-number of a formal proof of $\varphi(n)$ is primitive recursive.
\end{enumerate}

That these both conditions are fulfilled, e.g. for the system $P$ has been proved by G"odel\[40\].

6.2. Functions of class $P^{(k+1)}_0$ as general recursive functions.

6.21. If $S$ fulfills the condition $R_1$, then any general recursive function $f(n)$ is of class $P^{(k+1)}_0$.

Proof. If $f(n)$ is general recursive, then there are a primitive recursive function $h \in R_c^{(k)}$ and a primitive recursive relation $R(n,p)$ such that $\sum_{p} R(n,p)$ and

$$f(n) = h(\min R(n,p)).$$

According to (R_1) the set $A = \sum_{p} R(n,p)$ belongs to $P^{(k+1)}_0$ and the function $h$ to $P^{(k+1)}_0$. The function $\mu_4$ is of class $P^{(k+1)}_0$ by 5.42 and hence the compounded function $h(\mu_4(n))$ is of class $P^{(k+1)}_0$. This compounded function is equal to $f(n)$ since $\mu_4(n) = \min R(n,p)$ and hence $f(n)$ is of class $P^{(k+1)}_0$.

6.22. If $S$ fulfills the conditions (R_1) and (R_2) and if $f \in R_c^{(k)}$ is a function for which there is a propositional function $\varphi(x,\varepsilon)$ with $k+1$ free variables such that for any $n \in R_1$ and $n \in E$:

$$[m = f(n)] = [\neg \varphi(n,\varepsilon)],$$

then $f(n)$ is a general recursive function and hence $f \in P^{(k+1)}_0$.

\[\sharp 40\] This section we suppose the reader to be acquainted with the theory of general-recursively definable functions. See footnote 41.

\[\sharp 41\] G"odel [3], p. 156.

\[\sharp 42\] This theorem has been found by G"odel [3], p. 24. See also Remez [16], final remark, Kleene [9], theorem VIII, p. 738.
A. Mostowski:

Proof. For any \( n \) there is an integer \( q \) such that \( s_q(\bar{g})B_q(n,s_q(\bar{g})) \), hence by (B_2) the function

\[
g(n) = s_q(\min_q [s_q(\bar{g})B_q(n,s_q(\bar{g}))])
\]

is general recursive \(^{44}\). Thus it is sufficient to prove that \( f(n) = g(n) \). To show this put \( q_1 = \min_q [s_q(\bar{g})B_q(n,s_q(\bar{g}))] \). Then \( s_q(\bar{g}) \) is the Gödel-number of a formal proof of \( \phi(n,s_q(\bar{g})) \) which implies the existence of at least one formal proof of \( \phi(n,s_q(\bar{g})) \), i.e. \( \neg \phi(n,s_q(\bar{g})) \) and therefore \( f(n) = s_q(\bar{g}) \) by (2). On the other hand \( g(n) = s_q(\bar{g}) \) in virtue of the definition of \( g(n) \) and hence \( f(n) = g(n) \), q. e. d.

In order to explain the significance of 6.22 it is well to point out that in virtue of this theorem the existence of any propositional function \( \psi(x,y) \) with the property (21) implies the existence of (possibly another) decidable propositional function \( \psi(x,y) \) with the same property. A simple example will elucidate this state of affairs. Let \( \delta \) be any undecidable sentence, \( \psi(x,y) \) and \( \psi(x,y) \) the propositional functions

\[
y = 2x + (y = 2x + 1) \delta
d \text{ and } y = 2x.
\]

Then \( (n = 2n) = \neg \psi(n,n) = \neg \psi(n,n), \psi(n,x) \) is undecidable and \( \psi(x,y) \) decidable.

It is remarkable that no theorem analogous to 6.22 holds for sets. We have seen in the proof of 4.21 (formula (14)) that the equivalence

\[
x \cdot A = \neg \psi(n)
\]

may hold for any \( n \) though \( A \) does not belong to \( P^{(\infty)} \). It is to remark that \( A \) must then belong to \( P^{(\infty)} \) since

\[
x \cdot 1 = \sum_v (x \cdot B_v), n.
\]

From 6.21 and 6.22 we obtain the following corollary:

6.32. If \( S \) fulfills the condition (R_4) and (R_5), then \( P^{(\infty)} \) is the class of general recursive functions with \( k \) arguments.

6.3. Independence of classes \( P^k \) and \( Q^k \) from \( S \). Subsets of \( B \) whose characteristic functions are general recursive may be called general recursive \( k \)-adic relations. From 6.23 and 5.33 we obtain therefore:

6.31. If \( S \) fulfills the conditions (R_5) and (R_6), then \( P^k \) is the class of general recursive \( k \)-adic relations.

\(^{44}\) See e.g. Hilbert-Bernays [8], p. 402.

On definable sets of positive integers

This theorem is important because it shows that the class \( P^{(\infty)} \) though defined in 1.3 with the help of notions dependent from the logical system \( S \) taken as basis is in reality independent from \( S \), at least if we limit ourselves to consideration of systems which fulfill the recursivity conditions (R_1) and (R_4). In fact, it is known that the class of general recursive relations can be defined without any reference to formalized logical systems \(^{44}\). The independence of \( P^{(\infty)} \) from \( S \) implies of course the independence of other classes \( P^k \) and \( Q^k \) from \( S^{(\infty)} \).

We note at last the following corollary from 6.31 and 5.61:

6.32. If \( S \) fulfills the conditions \( (\mathbf{C}_0) \) (R_1) and (R_4), then \( P^{(\infty)} \) is the class of recursively enumerable sets.

Bibliography.


\(^{44}\) Hilbert-Bernays [8], p. 403-416.

\(^{44}\) It would be interesting to examine the question of independence of \( P^{(\infty)} \) from \( S \) under the assumption that \( S \) fulfills the condition \( C_0 \) but not \( C_{\infty} \).
Démonstration de l’égalité $2^m - m = 2^m$ pour les nombres cardinaux transfinis.

Par

Waclaw Sierpinski (Warszawa).

$m$ et $n$ étant deux nombres cardinaux, on dit que $m - n = p$ si $p$ est le seul nombre cardinal tel que $m = n + p$.

En 1926 M. A. Tarski a énoncé 1) ce

Théorème: On peut démontrer sans utiliser l’axiome du choix que, lorsque $m$ est un nombre cardinal transfini (c. à. d. $\geq \aleph_0$), on a

$$2^m - m = 2^m.$$  

M. Tarski n’a pas publié la démonstration de ce théorème. Il a seulement indiqué (I. c.) qu’elle s’appuie sur les lemmes 4, 6 et 8 énoncés également sans démonstration (I. c., p. 301 et p. 308).

La démonstration du théorème et les lemmes de M. Tarski m’est inconnue. Dans cette Note je vais démontrer le lemme 5 de M. Tarski et j’en déduirai son théorème (sans utiliser les lemmes 4 et 58).

Lemme 1 (de M. Tarski). On peut démontrer sans utiliser l’axiome du choix que si $A$ et $B$ sont deux ensembles tels que $A - B$, il existe des ensembles $C_1$, $C_2$, $D_1$, et $D_2$ remplissant les conditions:

$$A - B = C_1 + C_2,$$

$$B - A = D_1 + D_2,$$

$$C_1C_2 = 0 = D_1D_2,$$

$$C_1C_2 = 0 = D_1D_2.$$  


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A. Mostowski.


Note. This paper was already under press, when an interesting paper of S. C. Kleene, Recursion predicates and quantifiers (Transactions of the American Mathematical Society, vol. 53 (1943), pp. 41-43) became available in Poland.

A considerable part of the theory developed above is to be found in the Kleene’s paper. It seems me, however, that some of my results are new (e.g. remarks 4.2) and that my presentation of the theory based on analogies with the theory of projective sets may be of some interest for a mathematical reader.

Professor A. Tarski informed me that he also found already in 1942 results very similar to mine.

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