For any natural number m we use according to Mostowski the symbol $[m]!$ as an abbreviation of the conjuction $[1]\&[2]\&\ldots\&[m]$.

If n is a natural number >1, we denote by $\mu(n)$ the number p determined by two following conditions:

(i) there is a decomposition of n into a sum of primes in which p is the smallest term;

(ii) there is no decomposition of n into a sum of primes greater than p.

From theorem 1 we obtain now immediately the following

Corollary 3. *If m and n are natural numbers such that $m \geqslant \mu(n)$, then the implication $[m]! \rightarrow [n]$ is true.*

Mostowski has shown that also the converse theorem is true [7]). The inequality $m \geqslant \mu(n)$ is not only sufficient but also necessary for the derivability of the implication $[m]! \rightarrow [n]$. Mostowski derived further from the corollary 3 a sufficient condition for the truth of the implication $[m]! \rightarrow [n]!$ and proved that this condition is at the same time necessary [8]).

We conclude with the remark that conditions given in theorem 1 and 2 are by no means necessary for the derivability of the implication $[M] \rightarrow [n]$. For instance the implication $[(3,7)] \rightarrow [9]$ is true [9]) but neither the assumptions of theorem 1 nor that of theorem 2 are satisfied.

[7]) See M, p. 162.
[8]) M, theorem VIII.
[9]) M, theorem IX.

On definable sets of positive integers*).

By

Andrzej Mostowski (Warszawa).

The celebrated paper of K. Gödel on undecidable statements [1]) had (among others) the effect that several writers began to analyze the notion of functions of natural argument taking on integer values as well as related with them sets of positive integers. The chief purpose of these endeavours was to formulate an exact definition of what may be called „calculable function", i. e. such function $f(n)$ that there exists a method permitting to calculate the value of $f(n)$ for any given n in a finite number of steps. For sets we have the corresponding notion of „calculable sets" for which there is a finite method permitting to decide whether any given integer is in set or not. The solution of this problem given by Herbrand, Gödel, Church, Kleene and Turing [2]) suggested still other types of sets and of functions. So e. g. Rosser and Kleene found an interesting class of sets which they called „recursively enumerable" [3]).

The aim of this paper is to show that the two above mentioned classes of sets (and of functions) form the beginning of an infinite sequence of classes whose properties closely resemble those of projective sets [4]). For convenience of readers not acquainted with papers referred to in footnotes [2]) and [3]) I shall develop the theory without using the notion of general recursivity (the final section **6** is the only exception).

*) See note on the page 112.
[1]) Gödel [3]. Numbers in brackets refer to bibliography given at the end of this paper.
[2]) Gödel [4], [5], Church [2], Kleene [9], Turing [21]. It is now customary to call calculable functions and sets „general recursive". An excellent exposition of the theory of these functions is to be found in Hilbert-Bernays [8], Supplement II, 392-421.
[3]) Kleene [9], Rosser [14]. Further development will be found in Post [12].
[4]) I shall refer to the exposition of the theory of these sets given by Kuratowski [10].

It is difficult to predict at present whether the classes of sets and of functions dealt with in this paper will gain the same „right of citizenship" in metamathematics as the class of general recursive sets or functions. I have therefore not so much developed the theory of these classes themselves as tried rather to give some applications and to detect relations between the new classes and notions already known in this field. This explains why proofs of several known theorems are given in this paper (see 4.21, 4.41, 4.43, 5.51, 5.61). I think that owing to the use of methods familiar in the theory of projective sets I obtained not only considerable simplifications of the proofs but also some slight generalisations of the results themselves.

It seems to be possible to develop very extensively the theory of the new classes on the pattern of the theory of projective sets. From this kind of problems only one will be discussed here, to wit the analogue of Souslin's theorem [5]), i. e. the theorem that a recursively enumerable set whose complement is also recursively enumerable must be general recursive [6]). The utility resulting from the analogy with projective sets is thus I think demonstrated.

§ 1. Classes P_n and Q_n.

1.1. Preliminary remarks. Terminology. Metamathematical concepts occuring below (e. g. propositional function, formal proof etc.) refer to a fixed self-consistent logical system S in which the theory of addition and of multiplication of positive integers can be built up. Hence for S may be taken e. g. the system of *Principia Mathematica* of course reformulated so as to render the system more exact [7]). As the subsequent investigations are in high degree independent from the particular choice of the system S I shall give a mere sketch of its structure instead of a detailed description.

In the system S occur variables „x", „y",... of the type of positive integers [8]) as well as signs denoting the numbers 1,2,3,...

[5]) Kuratowski [10], p. 251, Corollaire 1.

[6]) After having finished the first draft of this paper I became acquainted with the paper Post [12] from which I see that this result has been obtained by E. L. Post already in 1944. From letters I understand that A. Tarski has also found the same theorem.

[7]) Such exact reformulations are given in Gödel [3] and Tarski [17].

[8]) S can contain also other types of variables.

Propositional functions with one, two,...,k variables of the type of positive integers will be denoted by symbols such as „$\varphi(x)$", „$\psi(x,y)$" etc. and general „$\varphi(\mathfrak{x})$", the German letter „$\mathfrak{x}$" standing for the finite sequence $x_1,x_2,...,x_k$ of k variables.

Among the propositional functions occur the arithmetical ones:

$$(1) \qquad x=y, \quad x<y, \quad x=y+z, \quad x=y\cdot z, \quad x=y^z$$

with their usual meaning.

If we substitute in a propositional function, e. g. in $\varphi(x,y)$, for „x" the sign denoting the number k and for „y" the sign denoting the number l we get a sentence which will be denoted by „$\varphi(k,l)$".

The implication and the conjunction of two propositional functions φ and ψ will be written as $\varphi\to\psi$ and $\varphi\cdot\psi$, the negation of φ as φ'. For quantifiers we use letters „Π" and „Σ" with a variable written below.

We admit that the ordinary rules of inference and the ordinary arithmetical axioms are valid in S. The formula $\vdash\varphi$ means that φ is a valid sentence, i. e. that there exists a formal proof of φ.

It will be admitted that it is possible to put variables, propositional functions and formal proofs of the system S in one-to-one correspondence with positive integers [9]). These integers will be called the Gödel-numbers of variables or of propositional functions or of proofs. The correspondence is supposed to be not arbitrary but to fulfill some conditions which will be formulated in 3.1 [10]).

In the simplest case S contains no other variables than those of the type of positive integers and no other propositional functions than those which can be built up from the propositional functions (1) with the help of quantifiers and logical connectives „$\to$", „$\cdot$" and „$'$". In this case S will be spoken of as the system of elementary arithmetic and denoted by $\mathfrak{A}$.

The logical symbols: negation „$'$", implication „$\to$", equivalence „$\equiv$", conjunction „$\cdot$", alternative „$+$" and quantifiers occur also (and more frequently) as synonyma of words „not", „if..., then..." etc. They are then used not as signs (primitive or defined) of the formal system S but as words of our ordinary language in which we are speaking about the system S. Using Carnap's

[9]) Gödel [3], p. 179.

[10]) They represent a generalization of the three conditions of recursivity formulated in Hilbert-Bernays [8], p. 393-394.

terminology we could say that we use the same symbols in object language as in syntax-language [11]). I do not think that this double meaning could cause any misunderstanding [12]).

Positive integers will be denoted by letters $m,n,h,k,\ldots$ eventually with subscripts. For any n we put

$$n=2^{s_1(n)}[2\,s_2(n)-1].$$

Ordered k-ads of positive integers will be called points of k-dimensional space R_k and denoted sometimes by a single German letter $\mathfrak{m},\mathfrak{n},\ldots$ For „$\varphi(n_1,n_2,\ldots,n_k)$" we write then shortly „$\varphi(\mathfrak{m})$".

The set-theoretical notation and terminology is that of Kuratowski [10].

1.2. Decidable functions. A propositional function $\varphi(\mathfrak{x})$ with k free variables will be called decidable, if for any $\mathfrak{n}\in R_k$ either $\vdash\varphi(\mathfrak{n})$ or $\vdash\varphi'(\mathfrak{n})$. In symbols

$$\prod_{\mathfrak{n}\in R_k}[\vdash\varphi(\mathfrak{n})+\vdash\varphi'(\mathfrak{n})].$$

Here the logical symbols except the negation-sign „$'$" are taken meta-mathematically.

E. g. the propositional functions (1) are decidable.

1.21. *The negation of a decidable propositional function and the logical product of two such functions is again a decidable propositional function.*

For negation the proposition is obvious. Suppose now that $\varphi(\mathfrak{x})$ and $\psi(\mathfrak{y})$ are decidable propositional functions with k and l free variables and let $\mathfrak{m}\in R_k$, $\mathfrak{n}\in R_l$. If $\vdash\varphi(\mathfrak{m})$ and $\vdash\psi(\mathfrak{n})$, then $\vdash\varphi(\mathfrak{m})\cdot\psi(\mathfrak{n})$. If either non $\vdash\varphi(\mathfrak{m})$ or non $\vdash\psi(\mathfrak{n})$, then $\vdash\varphi'(\mathfrak{m})$ or $\vdash\psi'(\mathfrak{n})$ since $\varphi(\mathfrak{x})$ and $\psi(\mathfrak{y})$ are both decidable and it follows by the rules of propositional calculus $\vdash[\varphi(\mathfrak{m})\cdot\psi(\mathfrak{n})]'$. Hence $\varphi(\mathfrak{x})\cdot\psi(\mathfrak{y})$ is decidable.

1.22. *If $\varphi(\mathfrak{x},y)$ is a decidable propositional function with $k+1$ free variables, then the propositional functions*

$$\prod_y[(y<x)\to\varphi(\mathfrak{x},y)] \quad \textit{and} \quad \sum_y[(y<x)\cdot\varphi(\mathfrak{x},y)]$$

are also decidable [13]).

[11]) Carnap [1], p. 4.

[12]) One could avoid this duality introducing other symbols in the formal system S and other in the meta-system. This is done e. g. in Gödel [3].

[13]) Gödel [3], Satz IV, p. 180.

The proof is obvious.

According to 1.21 and 1.22 the connectives of propositional calculus as well as the „limited quantifiers" $\prod_y[(y<x)\to(\ldots)]$ and $\sum_y[(y<x)\cdot(\ldots)]$ if applied to decidable propositional functions yield again decidable functions. It will be seen later (in 4.21) that the unlimited quantifiers $\prod_x$ and $\sum_x$ may give undecidable propositional functions.

1.23. *Let $\varphi(\mathfrak{t},\mathfrak{y})$ and $\psi(\mathfrak{x},\mathfrak{y})$ be two decidable propositional functions with $m+k$ and $l+k$ free variables. Let $\psi(\mathfrak{x},\mathfrak{y})$ fulfill the conditions:*

(2) $$\prod_{\mathfrak{x}}\prod_{\mathfrak{y}}\prod_{\mathfrak{z}}\{[\psi(\mathfrak{x},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{z})]\to(\mathfrak{y}=\mathfrak{z})\}$$ [14]);

(3) *For any $\mathfrak{m}\in R_l$ there is $\mathfrak{n}\in R_k$ such that $\vdash\psi(\mathfrak{m},\mathfrak{n})$.*

Under these assumptions the propositional function

(4) $$\sum_{\mathfrak{y}}[\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})]$$

is decidable.

Proof. Let us denote by $\vartheta(\mathfrak{t},\mathfrak{x})$ the propositional function (4) and suppose that $\mathfrak{m}\in R_l$, $\mathfrak{p}\in R_m$. Assume that non $\vdash\vartheta(\mathfrak{p},\mathfrak{m})$ and denote by $\mathfrak{n}$ a point of R_k such that $\vdash\psi(\mathfrak{m},\mathfrak{n})$. Hence it cannot be $\vdash\varphi(\mathfrak{p},\mathfrak{n})$, since we would then have $\vdash\vartheta(\mathfrak{p},\mathfrak{m})$ against our assumption. Therefore $\vdash\varphi'(\mathfrak{p},\mathfrak{n})$ and hence

(5) $$\vdash[\psi(\mathfrak{m},\mathfrak{n})\to\varphi'(\mathfrak{p},\mathfrak{n})].$$

The formula $\vdash\psi(\mathfrak{m},\mathfrak{n})$ yields together with (2)

$$\vdash\prod_{\mathfrak{y}}[(\mathfrak{y}=\mathfrak{n})'\to\psi'(\mathfrak{m},\mathfrak{y})]$$

and hence by the ordinary rules of propositional calculus

$$\vdash\prod_{\mathfrak{y}}\{(\mathfrak{y}=\mathfrak{n})'\to[\psi(\mathfrak{m},\mathfrak{y})\to\varphi'(\mathfrak{p},\mathfrak{y})]\}.$$

Combining this with (5) we get $\vdash\prod_{\mathfrak{y}}[\psi(\mathfrak{m},\mathfrak{y})\to\varphi'(\mathfrak{p},\mathfrak{y})]$, i. e. $\vdash\vartheta'(\mathfrak{p},\mathfrak{m})$. This proves that $\vartheta(\mathfrak{t},\mathfrak{x})$ is decidable.

[14]) If $\mathfrak{y}=(y_1,y_2,\ldots,y_k)$ and $\mathfrak{z}=(z_1,z_2,\ldots,z_k)$, then $\mathfrak{y}=\mathfrak{z}$ means the conjunction $(y_1=z_1)\cdot(y_2=z_2)\ldots(y_k=z_k)$.

1.24. *Under the assumptions of* 1.23

$$\vdash\{\sum_{\mathfrak{y}}[\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})]\to\prod_{\mathfrak{z}}[\psi(\mathfrak{x},\mathfrak{z})\to\varphi(\mathfrak{t},\mathfrak{z})]\}.$$

Proof. From (2) we obtain

$$\vdash\{\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{z})\to(\mathfrak{y}=\mathfrak{z})\cdot\varphi(\mathfrak{t},\mathfrak{y})\}$$

and therefore

$$\vdash\{\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{z})\to\varphi(\mathfrak{t},\mathfrak{z})\}.$$

This yields

$$\vdash\{[\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})]\to[\psi(\mathfrak{x},\mathfrak{z})\to\varphi(\mathfrak{t},\mathfrak{z})]\}.$$

If we now add the general quantifier in the second term and the particular one in the first, we obtain immediately the desired result.

1.25. *The propositional functions* $\chi(x,y,z)$, $\chi_1(x,y)$ *and* $\chi_2(x,z)$ *defined as*

$$x=2^y(2z+1),\qquad \sum_z\chi(x,y,z),\qquad \sum_y\chi(x,y,z)$$

are decidable and fulfill the formulae:

$$\vdash[\chi(x,y,z)\cdot\chi(x,y',z')\to(y=y')\cdot(z=z')],$$
$$\vdash[\chi_1(x,y)\cdot\chi_1(x,y')\to(y=y')],$$
$$\vdash[\chi_2(x,z)\cdot\chi_2(x,z')\to(z=z')].$$

1.3. Definition of classes $P_n^{(k)}$ and $Q_n^{(k)}$. A set $A\subset R_k$ will be said to belong to the class $P_0^{(k)}$ if there is a decidable propositional function $\varphi(\mathfrak{x})$ with k free variables such that

$$\mathfrak{n}\,\epsilon\,A\equiv\vdash\varphi(\mathfrak{n})$$

for any $\mathfrak{n}\,\epsilon\,R_k$. We say that $\varphi(\mathfrak{x})$ defines A. For reasons of symmetry we shall denote the class $P_0^{(k)}$ also by $Q_0^{(k)}$.

Let us now suppose that $n\geqslant 0$ and that classes $P_n^{(k)}$ and $Q_n^{(k)}$ $(k=1,2,3,\ldots)$ have already been defined. We then say that a set $A\subset R_k$ belongs to the class $P_{n+1}^{(k)}$ if there is a set $B\,\epsilon\,Q_n^{(k+1)}$ such that for any $\mathfrak{n}\,\epsilon\,R_k$

$$\mathfrak{n}\,\epsilon\,A\equiv\sum_p(\mathfrak{n},p)\,\epsilon\,B.$$

A set $A\subset R_k$ belongs to $Q_{n+1}^{(k)}$ if $R_k-A\,\epsilon\,P_{n+1}^{(k)}$.

The analogy with the theory of projective sets needs not to be emphasised.

The class $P_0^{(k)}=Q_0^{(k)}$ plays in our theory the same role as the class of Borel-subsets of k-dimensional space plays in the theory of projective sets.

We see from the definition that the rules of inference admitted in the system S permit to decide whether any given „point" $\mathfrak{n}$ belongs to any given set A of the class $P_0^{(k)}$ or not. Hence $P_0^{(k)}$ is the class of general recursive sets mentioned in the introduction. This will be proved formally in 6.31.

From classes $P_n^{(k)}$ and $Q_n^{(k)}$ with $n\geqslant 1$ only one as far as I see is known in the literature. It is the class $P_1^{(1)}$ which was called by Kleene the class of recursively enumerable sets [15]). It will be shown later that $A\,\epsilon\,P_1^{(1)}$ if and only if A is the set of values of a general recursive function (see 5.61 and 6.23).

The whole sum $\sum_{n=0}^{\infty}[P_n^{(k)}+Q_n^{(k)}]$ may be characterized as the class of sets $A\subset R_k$ which are definable within the elementary arithmetic. The word „definable" is here used in the following sense [16]): a set $A\subset R_k$ is definable within $\mathfrak{A}$ if there is in $\mathfrak{A}$ a propositional function $\varphi(\mathfrak{x})$ with k free variables such that $\mathfrak{n}\,\epsilon\,A$ if and only if $\mathfrak{n}$ fulfills $\varphi(\mathfrak{x})$. The proof of the above theorem presents no difficulty for any one who knows the notion of fulfillness [17]). As its exact definition is rather intricate, we shall omit this proof and content ourselves with the remark that the definability of sets belonging to $P_0^{(k)}$ results from theorem 6.31 given below.

The classes $P_n^{(k)}$ and $Q_n^{(k)}$ such as they were defined depend a priori from the logical system S taken as basis and should properly be denoted by symbols $P_n^{(k)}(S)$ and $Q_n^{(k)}(S)$. As a matter of fact they are independent from the system S provided that this system satisfies some very general conditions as will be shown in 6.3.

§ 2. Elementary properties of classes $P_n^{(k)}$ and $Q_n^{(k)}$.

2.1. Sums, common parts and cartesian products. The most important theorems we intend to establish in this section may be stated as follows: the classes $P_n^{(k)}$ and $Q_n^{(k)}$ are rings of sets; the property to belong to P_n (or Q_n) is invariant for any n (2.17); the property to belong to P_n (or Q_n) is invariant

[15]) Kleene [9], theorem XI, p. 739.
[16]) Tarski [18] p. 312.
[17]) Tarski [18].

under the cartesian multiplication by an axis (2.14); the sum (or the common part) of an enumerable sequence of sets belonging to $\boldsymbol{P}_n^{(k)}$ (or to $\boldsymbol{Q}_n^{(k)}$) belongs under certain assumptions to the same class (2.16). The remaining theorems are lemmas.

2.11. *If* $A \in \boldsymbol{P}_0^{(k)}$ *and* $B \in \boldsymbol{P}_0^{(k)}$, *then* $A+B$, $A\cdot B$ *and* $R_k - A$ *belong to* $\boldsymbol{P}_0^{(k)}$.

This proposition which follows immediately from 1.21 states that $\boldsymbol{P}_0^{(k)}$ is a field of sets.

2.12. *If* $A \in \boldsymbol{P}_n^{(k)}$, *then* $R_k - A \in \boldsymbol{Q}_n^{(k)}$ *and conversely.*

This follows direct from definition.

2.13. (Change of axes). *If* $\pi(1), \pi(2), \ldots, \pi(k)$ *is any permutation of* $1, 2, \ldots, k$ *and if we denote for any* $A \subset R_k$ *by* A_π *the set of all* $(n_{\pi(1)}, n_{\pi(2)}, \ldots, n_{\pi(k)})$ *for which* $(n_1, n_2, \ldots, n_k) \in A$, *then* $A \in \boldsymbol{P}_n^{(k)} \equiv A_\pi \in \boldsymbol{P}_n^{(k)}$ *and* $A \in \boldsymbol{Q}_n^{(k)} \equiv A_\pi \in \boldsymbol{Q}_n^{(k)}$.

The easy proof proceeding by induction on n will not be given here.

2.14. (Cartesian products). *If* $A \in \boldsymbol{P}_n^{(k)}$ *(or* $A \in \boldsymbol{Q}_n^{(k)}$*), then* $A \times R_1 \in \boldsymbol{P}_n^{(k+1)}$ *(or* $A \in \boldsymbol{Q}_n^{(k+1)}$*).*

Proof. Suppose first that $n=0$ and let $\varphi(\mathfrak{x})$ be a decidable propositional function which defines A. The propositional function $\varphi(\mathfrak{x})\cdot(x=x)$ is of course decidable and defines $A \times R_1$. Hence $A \times R_1 \in \boldsymbol{P}_0^{(k+1)}$.

Suppose now that 2.14 holds for $n<m$ and let $A \in \boldsymbol{P}_m^{(k)}$. By definition there is a set $B \in \boldsymbol{Q}_{m-1}^{(k+1)}$ such that

$$\mathfrak{n} \in A \equiv \sum_q (\mathfrak{n}, q) \in B.$$

The set $B_1 = \underset{(\mathfrak{n},p,q)}{E}[(\mathfrak{n}, q) \in B]$ arises from $B \times R_1$ by interchanging the two last axes and therefore $B_1 \in \boldsymbol{Q}_{m-1}^{(k+2)}$ by 2.13. Since we have obviously

$$(\mathfrak{n}, p) \in A \times R_1 \equiv \sum_q (\mathfrak{n}, p, q) \in B_1,$$

we infer from the definition that $A \times R_1 \in \boldsymbol{P}_m^{(k+1)}$.

Suppose now that $A \in \boldsymbol{Q}_m^{(k)}$, i. e. $R_k - A \in \boldsymbol{P}_m^{(k)}$. If we repeat the above reasoning taking $R_k - A$ instead of A, we obtain $(R_k - A) \times R_1 \in \boldsymbol{P}_m^{(k+1)}$ or passing to complements and using 2.12

$$R_{k+1} - (R_k - A) \times R_1 \in \boldsymbol{Q}_m^{(k+1)}.$$

The left side is identical with $A \times R_1$ what completes the proof.

Let us put for any $A \subset R_{k+2}$

$$A^* = \underset{(\mathfrak{n},p)}{E}[(\mathfrak{n}, s_1(p), s_2(p)) \in A].$$

2.15. $A \in \boldsymbol{P}_n^{(k+2)} \equiv A^* \in \boldsymbol{P}_n^{(k+1)}$ *and* $A \in \boldsymbol{Q}_n^{(k+2)} \equiv A^* \in \boldsymbol{Q}_n^{(k+1)}$.

Proof by induction on n. Suppose first that $n=0$ and that $A \in \boldsymbol{P}_0^{(k+2)}$. Let $\varphi(\mathfrak{t}, u, v)$ be a decidable propositional function with $k+2$ free variables which defines A and consider the propositional function

$$\sum_u \sum_v \varphi(\mathfrak{t}, u, v)\cdot\chi_1(z, u)\cdot\chi_2(z, v),$$

where $\chi_1(z, u)$ and $\chi_2(z, v)$ have the meaning defined in 1.25. It is obvious that this function defines A^*. It is in addition decidable because it has the form considered 1.23 with „(u, v)" instead of „$\mathfrak{y}$" and with „$\chi_1(z, u)\cdot\chi_2(z, v)$" instead of „$\psi(\mathfrak{x}, \mathfrak{y})$". The assumption (2) of 1.23 is satisfied in virtue of 1.25 whereas (3) is obvious. This proves that $A^* \in \boldsymbol{P}_0^{(k+1)}$.

Suppose now conversely that $A^* \in \boldsymbol{P}_0^{(k+1)}$ and that $\varphi(\mathfrak{t}, w)$ is a decidable propositional function which defines A^*. By 1.23 and 1.25 the propositional function

$$\sum_w \varphi(\mathfrak{t}, w)\cdot\chi(w, u, v)$$

is decidable and defines A. Therefore $A \in \boldsymbol{P}_0^{(k+2)}$.

The theorem is thus proved for $n=0$.

Suppose now that $m>0$ and that the theorem holds for $n<m$. Let $A \in \boldsymbol{P}_m^{(k+2)}$. Hence there is a set $B \in \boldsymbol{Q}_{m-1}^{(k+3)}$ such that

$$(\mathfrak{n}, p, q) \in A \equiv \sum_h (\mathfrak{n}, p, q, h) \in B.$$

The set

$$B_1 = \underset{(\mathfrak{n},l,h)}{E}[(\mathfrak{n}, s_1(l), s_2(l), h) \in B]$$

arises from B^* by interchanging the two last axes; consequently $B_1^* \in \boldsymbol{Q}_{m-1}^{(k+2)}$ by 2.13 and the inductive assumption. Now we see that

$$(\mathfrak{n}, l) \in A^* \equiv (\mathfrak{n}, s_1(l), s_2(l)) \in A \equiv \sum_h (\mathfrak{n}, s_1(l), s_2(l), h) \in B \equiv \sum_h (\mathfrak{n}, l, h) \in B_1^*$$

which proves according to the definition 1.3 that $A^* \in \boldsymbol{P}_m^{(k+1)}$. Hence $A \in \boldsymbol{P}_m^{(k+2)} \rightarrow A^* \in \boldsymbol{P}_m^{(k+1)}$.

Suppose now that $A^* \in \boldsymbol{P}_m^{(k+1)}$, i. e. that there is a set $B \in \boldsymbol{Q}_{m-1}^{(k+2)}$ such that

$$(\mathfrak{n}, l) \in A^* \equiv \sum_h (\mathfrak{n}, l, h) \in B.$$

Let B_1, B_2 and B_3 be defined by the equivalences

$$(\mathfrak{n},h,l)\,\epsilon\,B_1 \equiv (\mathfrak{n},l,h)\,\epsilon\,B,$$
$$(\mathfrak{n},h,p,q)\,\epsilon\,B_2 \equiv (\mathfrak{n},h,2^p(2q+1))\,\epsilon\,B_1,$$
$$(\mathfrak{n},p,q,h)\,\epsilon\,B_3 \equiv (\mathfrak{n},h,p,q)\,\epsilon\,B_2.$$

Obviously $B_2^* = B_1$. Since $B_1\,\epsilon\,Q_{m-1}^{(k+2)}$ by 2.13, we obtain from the inductive assumption $B_2\,\epsilon\,Q_{m-1}^{(k+3)}$ and again by 2.13 $B_3\,\epsilon\,Q_{m-1}^{(k+3)}$. Now observe that

$$(\mathfrak{n},p,q)\,\epsilon\,A \equiv (\mathfrak{n},2^p(2q+1))\,\epsilon\,A^* \equiv \sum_h (\mathfrak{n},2^p(2q+1),h)\,\epsilon\,B$$
$$\equiv \sum_h (\mathfrak{n},h,2^p(2q+1))\,\epsilon\,B_1 \equiv \sum_h (\mathfrak{n},h,p,q)\,\epsilon\,B_2 \equiv \sum_h (\mathfrak{n},p,q,h)\,\epsilon\,B_3.$$

This equivalence proves that $A\,\epsilon\,P_m^{(k+2)}$ and hence

$$A\,\epsilon\,P_m^{(k+2)} \equiv A^*\,\epsilon\,P_m^{(k+1)}.$$

Passing to complements and observing that

$$(R_{k+2}-A)^* = R_{k+1}-A^*,$$

we obtain immediately

$$A\,\epsilon\,Q_m^{(k+2)} \equiv A^*\,\epsilon\,Q_m^{(k+1)}.$$

The theorem 2.15 is thus proved completely.

We put for any $A \subset R_{k+1}$

$$A_s = \underset{\mathfrak{n}}{E}\Big[\sum_q (\mathfrak{n},q)\,\epsilon\,A\Big], \qquad A_p = \underset{\mathfrak{n}}{E}\Big[\prod_q (\mathfrak{n},q)\,\epsilon\,A\Big].$$

2.16. (Infinite sums and products). *If $n \geqslant 1$, then*

$$A\,\epsilon\,P_n^{(k+1)} \rightarrow A_s\,\epsilon\,P_n^{(k)} \quad \textit{and} \quad A\,\epsilon\,Q_n^{(k+1)} \rightarrow A_p\,\epsilon\,Q_n^{(k)}.$$

Proof. Suppose first that $A\,\epsilon\,P_n^{(k+1)}$. For a suitable $B\,\epsilon\,Q_{n-1}^{(k+2)}$ we have the equivalence

$$(\mathfrak{n},p)\,\epsilon\,A \equiv \sum_q (\mathfrak{n},p,q)\,\epsilon\,B.$$

Remembering the definition of the set B^* we obtain

$$\mathfrak{n}\,\epsilon\,A_s \equiv \sum_p (\mathfrak{n},p)\,\epsilon\,A \equiv \sum_p\sum_q (\mathfrak{n},p,q)\,\epsilon\,B \equiv \sum_h (\mathfrak{n},h)\,\epsilon\,B^*,$$

for putting $h = 2^p(2q+1)$ we have $(\mathfrak{n},p,q)\,\epsilon\,B \equiv (\mathfrak{n},h)\,\epsilon\,B^*$. Since $B^*\,\epsilon\,Q_{n-1}^{(k+1)}$, the above equivalence proves that $A_s\,\epsilon\,P_n^{(k)}$. In order to obtain the second result stated in the theorem it is now sufficient to observe that

$$R_k - A_p = (R_k - A)_s.$$

2.17. (The ring property). *If A and B belong to $P_n^{(k)}$ (or $Q_n^{(k)}$), then $A+B$ and $A\cdot B$ belong to the same class.*

Proof. In view of 2.11 we may suppose that $n \geqslant 1$ and that the theorem holds for lower values of n. From $A\,\epsilon\,P_n^{(k)}$ and $B\,\epsilon\,P_n^{(k)}$ we infer that there are sets C and D belonging both to $Q_{n-1}^{(k+1)}$ such that

$$\mathfrak{n}\,\epsilon\,A \equiv \sum_p (\mathfrak{n},p)\,\epsilon\,C, \qquad \mathfrak{n}\,\epsilon\,B \equiv \sum_p (\mathfrak{n},p)\,\epsilon\,D.$$

From these equivalences we immediately obtain

$$\mathfrak{n}\,\epsilon\,A+B \equiv \sum_p (\mathfrak{n},p)\,\epsilon\,C+D$$

and hence $A+B\,\epsilon\,P_n^{(k)}$ since $C+D\,\epsilon\,Q_{n-1}^{(k+1)}$ by the inductive assumption

Consider now the cartesian products $C \times R_1$ and $D \times R_1$ and denote by $\bar{D}$ the set arising from $D \times R_1$ by interchanging the two last axes. Putting for symmetry $\bar{C} = C \times R_1$, we have

$$(\mathfrak{n},p,q)\,\epsilon\,\bar{C} \equiv (\mathfrak{n},p)\,\epsilon\,C,$$
$$(\mathfrak{n},p,q)\,\epsilon\,\bar{D} \equiv (\mathfrak{n},q)\,\epsilon\,D$$

and $\bar{C},\bar{D}\,\epsilon\,Q_{n-1}^{(k+2)}$ in virtue of 2.13 and 2.14. By the inductive assumption we infer that $\bar{C}\cdot\bar{D}\,\epsilon\,Q_{n-1}^{(k+2)}$ and since

$$\mathfrak{n}\,\epsilon\,A\cdot B \equiv \Big[\sum_p (\mathfrak{n},p)\,\epsilon\,C\Big]\cdot\Big[\sum_q (\mathfrak{n},q)\,\epsilon\,D\Big] \equiv$$
$$\equiv \sum_p\sum_q [(\mathfrak{n},p)\,\epsilon\,C]\cdot[(\mathfrak{n},q)\,\epsilon\,D] \equiv \sum_p\sum_q (\mathfrak{n},p,q)\,\epsilon\,\bar{C}\cdot\bar{D},$$

we infer from 2.16 that $A\cdot B\,\epsilon\,P_n^{(k)}$.

Passing to complements and applying 2.12 we obtain the further result that if A and B are in $Q_n^{(k)}$, then $A+B$ and $A\cdot B$ are both in $Q_n^{(k)}$. The theorem 2.17 is thus proved.

It will be proved in 3.32 that neither $P_n^{(k)}$ nor $Q_n^{(k)}$ is a field of sets for $n \geqslant 1$, i. e., that the difference of two sets of the class $P_n^{(k)}$ (or $Q_n^{(k)}$) does not, in general, belong to this class. From 2.17 and 2.12 we obtain however

2.18. *The common part $P_n^{(k)}\cdot Q_n^{(k)}$ is a field of sets for any $n \geqslant 0$.*

The sense of this proposition is that the class $P_n^{(k)}\cdot Q_n^{(k)}$ is closed under the three operations $A+B$, $A\cdot B$ and $A-B$.

2.2. The Kuratowski-Tarski method. This well known method permits to evaluate the Borel class or the projective class of any set provided that its definition has been written down in logical symbols [18]). The method is based on propositions of exactly the same form as our theorems 2.11-2.15 and 2.17 [19]). Hence imitating this method we may from the mere form of the definition of any given set $A \subset R_k$ evaluate a n for which $A \in \boldsymbol{P}_n^{(k)}$ or $A \in \boldsymbol{Q}_n^{(k)}$. This illustrates the importance of theorems established in **2.1.**

2.3. Inclusions. Between the classes $\boldsymbol{P}_n^{(k)}$, $\boldsymbol{P}_{n+1}^{(k)}$, $\boldsymbol{Q}_n^{(k)}$ and $\boldsymbol{Q}_{n+1}^{(k)}$ hold following inclusions:

2.31. $\boldsymbol{P}_n^{(k)} \subset \boldsymbol{P}_{n+1}^{(k)} \cdot \boldsymbol{Q}_{n+1}^{(k)}$ *and* $\boldsymbol{Q}_n^{(k)} \subset \boldsymbol{P}_{n+1}^{(k)} \cdot \boldsymbol{Q}_{n+1}^{(k)}$.

Proof. Suppose that $A \in \boldsymbol{P}_n^{(k)}$ and put $A_1 = A \times R_1$. Evidently $\mathfrak{n} \in A \equiv \prod_p (\mathfrak{n}, p) \in A_1 \equiv [\sum_p (\mathfrak{n}, p) \in R_{k+1} - A_1]'$; since $R_{k+1} - A_1 \in \boldsymbol{Q}_n^{(k+1)}$ by 2.12 and 2.14, we infer direct from the definition **1.3** that $A \in \boldsymbol{Q}_{n+1}^{(k)}$. Hence

$$\boldsymbol{P}_n^{(k)} \subset \boldsymbol{Q}_{n+1}^{(k)}. \tag{6}$$

Passing to complements, we obtain

$$\boldsymbol{Q}_n^{(k)} \subset \boldsymbol{P}_{n+1}^{(k)}. \tag{7}$$

This gives for $n=0$ the inclusions $\boldsymbol{Q}_0^{(k)} \subset \boldsymbol{Q}_1^{(k)}$ and $\boldsymbol{P}_0^{(k)} \subset \boldsymbol{P}_1^{(k)}$. Suppose now that $m > 0$ and that the inclusions

$$\boldsymbol{P}_n^{(k)} \subset \boldsymbol{P}_{n+1}^{(k)}, \qquad \boldsymbol{Q}_n^{(k)} \subset \boldsymbol{Q}_{n+1}^{(k)} \qquad (k=1,2,\ldots) \tag{8}$$

are valid for $n < m$. If $A \in \boldsymbol{P}_m^{(k)}$, then, for a suitable $B \in \boldsymbol{Q}_{m-1}^{(k+1)}$, we have $\mathfrak{n} \in A \equiv \sum_p (\mathfrak{n}, p) \in B$ which proves that $A \in \boldsymbol{P}_{m+1}^{(k)}$ since $B \in \boldsymbol{Q}_m^{(k+1)}$ by the inductive assumption. Hence the first inclusion (8) is true for $n=m$ and passing to complements we obtain the same result for the second one. (8) is thus true for any n. The theorem results now from (6), (7) and (8).

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[18]) Kuratowski-Tarski [11], p. 242, Kuratowski [10], p. 168 and 243.

[19]) There are, however, no rules in our theory which would correspond to theorems concerning infinite sums or products of Borel (or projective) sets.

§ 3. Existence theorems.

3.1. Condition C_s. It is now an appropriate place to formulate the conditions imposed upon the enumeration of variables, propositional functions and formal proofs of the system S. We shall consider the following sets:

$\Gamma_k = \underset{(p,q,\mathfrak{n})}{E}$ [*there is a propositional function* $\varphi(\mathfrak{x})$ *with* k *free variables such that* p *is the Gödel-number of* $\varphi(\mathfrak{x})$ *and* q *that of* $\varphi(\mathfrak{n})$],

$\Delta = \underset{(q,n)}{E}$ [q *is the Gödel-number of a formal proof of the sentence whose Gödel-number is* n],

$E = \underset{q}{E}$ [q *is the Gödel-number of a formal proof*],

$H = \underset{(p,q)}{E}$ [p *is the Gödel-number of a propositional function and* q *that of its negation*].

The condition C_s requires that these sets should belong to classes $\boldsymbol{P}_s$ with suitable upper indices:

(C_s) $\qquad \Gamma_k \in \boldsymbol{P}_s^{(k+2)}, \quad \Delta \in \boldsymbol{P}_s^{(2)}, \quad E \in \boldsymbol{P}_s^{(1)}, \quad H \in \boldsymbol{P}_s^{(2)}$ [20]).

Most systems fulfill the simplest condition C_0; namely all those in which a formal proof consists on performing step by step some well defined finitary rules of inference. Thus e. g. the system P considered by Gödel fulfills the condition C_0 as is proved in details in Gödel [3] pp. 179-186. Systems with non-finitary rules of inference fulfill, in general, the condition C_s for some $s > 0$ but not the condition C_0. For examples of such systems see e. g. Rosser [13].

3.2. Universal functions. We proceed to establish, for systems S which fulfill the condition C_s, the existence of sets belonging to any given class $\boldsymbol{P}_n^{(k)}$ or $\boldsymbol{Q}_n^{(k)}$ but not to preceding ones. The proof is based as in the theory of Borel-sets or projective sets on the concept of universal functions [21]).

Let $\boldsymbol{X}$ be any class of sets. A universal function for $\boldsymbol{X}$ is any function $\boldsymbol{F}(h)$ defined for $h=1,2,3,\ldots$ such that

$$A \in \boldsymbol{X} \equiv \sum_h A = \boldsymbol{F}(h).$$

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[20]) The assumption $E \in \boldsymbol{P}_s^{(1)}$ is irrelevant for our present purposes but we shall need it in 5.61.

[21]) Kuratowski [10], p. 172 and 241.

3.21. *If S fulfills the condition C_s, then there is a universal function $\boldsymbol{F}_0^{(k)}(h)$ for the class $\boldsymbol{P}_0^{(k)}$ such that the set*

$$M_0^{(k)} = \underset{(\mathfrak{n},h)}{E}[\mathfrak{n} \in \boldsymbol{F}_0^{(k)}(h)]$$

belongs to $\boldsymbol{Q}_{s+2}^{(k+1)}$.

Proof.[22]). An integer h is the Gödel-number of a decidable propositional function with k free variables if and only if

$$\prod_{\mathfrak{n}} \sum_{q} \sum_{l} \sum_{m} \{(h,l,\mathfrak{n}) \in \Gamma_k \cdot (l,m) \in H \cdot [(q,l) \in \Delta + (q,m) \in \Delta]\}.$$

Hence denoting with Θ_k the set of these numbers we infer by the Kuratowski-Tarski method that

(9) $$\Theta_k \in \boldsymbol{Q}_{s+2}^{(1)}\ ^{23}).$$

We put now $\boldsymbol{F}_0^{(k)}(h) = 0$ for h non $\in \Theta_k$ and

$$\boldsymbol{F}_0^{(k)}(h) = \underset{\mathfrak{n}}{E}[\sum_{p} \sum_{q} (h,p,\mathfrak{n}) \in \Gamma_k \cdot (q,p) \in \Delta]$$

for $h \in \Theta_k$. The set $M_0^{(k)}$ belongs then to $\boldsymbol{Q}_{s+2}^{(k+1)}$ as we easily see from its definition

$$(\mathfrak{n},h) \in M_0^{(k)} \equiv (h \in \Theta_k) \cdot \sum_{p} \sum_{q} [(h,p,\mathfrak{n}) \in \Gamma_k \cdot (q,p) \in \Delta]$$

using (9), 2.31 and the Kuratowski-Tarski evaluation method.

It remains to prove that $\boldsymbol{F}_0^{(k)}(h)$ is a universal function for the class $\boldsymbol{P}_0^{(k)}$.

Let us suppose that $A \in \boldsymbol{P}_0^{(k)}$ and that $\varphi(\mathfrak{x})$ defines A. If h is the Gödel-number of $\varphi(\mathfrak{x})$, then $h \in \Theta_k$. Let $\mathfrak{n} \in A$ and let p be the Gödel-number of $\varphi(\mathfrak{n})$. Then $(h,p,\mathfrak{n}) \in \Gamma_k$. Since $\vdash \varphi(\mathfrak{n})$, there is a formal proof of $\varphi(\mathfrak{n})$. Denoting with q its Gödel-number, we have $(q,p) \in \Delta$. Now from $h \in \Theta_k$, $(h,p,\mathfrak{n}) \in \Gamma_k$ and $(q,p) \in \Delta$ we obtain $\mathfrak{n} \in \boldsymbol{F}_0^{(k)}(h)$. If, conversely, $\mathfrak{n} \in \boldsymbol{F}_0^{(k)}(h)$, then there are p, q such that $(h,p,\mathfrak{n}) \in \Gamma_k$ and $(q,p) \in \Delta$. Hence p is the Gödel-number of $\varphi(\mathfrak{n})$ and q is the Gödel-number of a formal proof of $\varphi(\mathfrak{n})$. Hence there is at least one formal proof of $\varphi(\mathfrak{n})$ which proves that $\vdash \varphi(\mathfrak{n})$ i. e. $\mathfrak{n} \in A$; therefore $A = \boldsymbol{F}_0^{(k)}(h)$ which proves that

$$A \in \boldsymbol{P}_0^{(k)} \to \sum_{h} A = \boldsymbol{F}_0^{(k)}(h).$$

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[22]) This proof is essentially due to Kleene [9], theorem IV, p. 736.

[23]) More exactly: for $s=0$ $\Theta_k \in \boldsymbol{Q}_2^{(1)}$ and for $s>0$ $\Theta_k \in \boldsymbol{Q}_{s+1}^{(1)}$. The formula (9) includes both cases.

Suppose, conversely, that $A = \boldsymbol{F}_0^{(k)}(h)$. If h non $\in \Theta_k$, then $A = 0$ and therefore $A \in \boldsymbol{P}_0^{(k)}$. If $h \in \Theta_k$, let $\varphi(\mathfrak{x})$ be the decidable propositional function whose Gödel-number is h. We prove similar as above that $\mathfrak{n} \in A \equiv \vdash \varphi(\mathfrak{n})$ and therefore $A \in \boldsymbol{P}_0^{(k)}$. Hence

$$\sum_{h} A = \boldsymbol{F}_0^{(k)}(h) \to A \in \boldsymbol{P}_0^{(k)}.$$

This completes the proof of 3.21.

3.22 *If S fulfills the condition C_s, then there are functions $\boldsymbol{F}_n^{(k)}(h)$ and $\boldsymbol{G}_n^{(k)}(h)$ universal for classes $\boldsymbol{P}_n^{(k)}$ and $\boldsymbol{Q}_n^{(k)}$ and such that the sets*

$$M_n^{(k)} = \underset{(\mathfrak{n},h)}{E}[\mathfrak{n} \in \boldsymbol{F}_n^{(k)}(h)] \quad \textit{and} \quad N_n^{(k)} = \underset{(\mathfrak{n},h)}{E}[\mathfrak{n} \in \boldsymbol{G}_n^{(k)}(h)]$$

belong respectively to $\boldsymbol{P}_{s+n+2}^{(k+1)}$ or to $\boldsymbol{Q}_{s+n+2}^{(k+1)}$ if $n>0$ and to $\boldsymbol{Q}_{s+2}^{(k+1)}$ if $n=0$.

Proof. The theorem was proved in 3.21 for $n=0$. Suppose that $n \geqslant 0$ and the theorem is true for this value of n and for $k=1,2,\ldots$ Put

$$\boldsymbol{F}_{n+1}^{(k)}(h) = \underset{\mathfrak{n}}{E}[\sum_{p} (\mathfrak{n},p) \in \boldsymbol{G}_n^{(k+1)}(h)],$$

$$\boldsymbol{G}_{n+1}^{(k)}(h) = R_k - \boldsymbol{F}_{n+1}^{(k)}(h).$$

If h is any integer, then $\boldsymbol{F}_{n+1}^{(k)}(h) \in \boldsymbol{P}_{n+1}^{(k)}$, since

$$\mathfrak{n} \in \boldsymbol{F}_{n+1}^{(k)}(h) \equiv \sum_{p} [(\mathfrak{n},p) \in \boldsymbol{G}_n^{(k+1)}(h)]$$

and $\boldsymbol{G}_n^{(k+1)}(h) \in \boldsymbol{Q}_n^{(k+1)}$ by the inductive assumption. Suppose, conversely, that $A \in \boldsymbol{P}_{n+1}^{(k)}$, i. e.

$$\mathfrak{n} \in A \equiv \sum_{p} (\mathfrak{n},p) \in B$$

for a suitable $B \in \boldsymbol{Q}_n^{(k+1)}$. The function $\boldsymbol{G}_n^{(k+1)}(h)$ being universal for $\boldsymbol{Q}_n^{(k+1)}$, there is an h such that $B \in \boldsymbol{G}_n^{(k+1)}(h)$ and therefore $A = \boldsymbol{F}_n^{(k+1)}(h)$. Hence $\boldsymbol{F}_{n+1}^{(k)}(h)$ is a universal function for $\boldsymbol{P}_{n+1}^{(k)}$. Passing to complements we immediately see that $\boldsymbol{G}_{n+1}^{(k)}(h)$ is universal for $\boldsymbol{Q}_{n+1}^{(k)}$.

It remains to consider the sets $M_{n+1}^{(k)}$ and $N_{n+1}^{(k)}$. According to their definitions we have

$$(\mathfrak{n},h) \in M_{n+1}^{(k)} \equiv \mathfrak{n} \in \boldsymbol{F}_{n+1}^{(k)}(h) \equiv \sum_{p} (\mathfrak{n},p) \in \boldsymbol{G}_n^{(k+1)}(h) \equiv \sum_{p} (\mathfrak{n},p,h) \in N_n^{(k+1)}$$

and this proves that $M_{n+1}^{(k)} \epsilon P_{s+n+3}^{(k+1)}$, since $N_n^{(k+1)} \epsilon Q_{s+n+2}^{(k+2)}$ by the inductive assumption. Further we have

$$(\mathfrak{n},h) \epsilon N_{n+1}^{(k)} \equiv \mathfrak{n} \epsilon G_{n+1}^{(k)}(h) \equiv \mathfrak{n} \text{ non } \epsilon F_{n+1}^{(k)}(h) \equiv (\mathfrak{n},h) \epsilon R_{k+1} - M_{n+1}^{(k)}$$

and therefore $N_{n+1}^{(k)} \epsilon Q_{s+n+3}^{(k+1)}$. The theorem is thus proved completely.

3.3. Existence-theorems. They follow now easily by the well-known Cantor's diagonal-theorem [24]).

3.31. *If S fulfills the condition C_s, then $P_n^{(k)} \neq P_{n+1}^{(k)}$ and $Q_n^{(k)} \neq Q_{n+1}^{(k)}$ for any $n \geqslant 0$ and $k \geqslant 1$.*

Proof. Let us suppose that S fulfills the condition C_s and that $P_n^{(k)} = P_{n+1}^{(k)}$ for some k and n. We shall show by induction on m that then

(10) $$P_n^{(k)} = P_m^{(k)} = Q_m^{(k)} \quad \text{for} \quad m \geqslant n.$$

This holds for $m=n$ since we have $Q_n^{(k)} \subset P_{n+1}^{(k)} = P_n^{(k)}$ which implies $P_n^{(k)} \subset Q_n^{(k)}$ and therefore $P_n^{(k)} = Q_n^{(k)}$. Now suppose that (10) holds for an integer $m \geqslant n$. Obviously $P_n^{(k)} \subset P_{m+1}^{(k)}$. If $A \epsilon P_{m+1}^{(k)}$, then there is a set $B \epsilon Q_m^{(k+1)}$ such that

$$\mathfrak{n} \epsilon A \equiv \sum_p (\mathfrak{n},p) \epsilon B.$$

Let us write $(n_1,n_2,\ldots,n_k)$ instead of $\mathfrak{n}$ and consider the set (see 2.15)

$$B^* = \underset{(n_1,\ldots,n_{k-1},q)}{E} [(n_1,\ldots,n_{k-1},s_1(q),s_2(q)) \epsilon B].$$

Since $B^* \epsilon Q_m^{(k)}$, we have $B^* \epsilon P_n^{(k)}$ in virtue of the inductive assumption and the equivalence

$$\mathfrak{n} \epsilon A \equiv \sum_q (s_1(q)=n_k) \cdot [(n_1,\ldots,n_{k-1},q) \epsilon B^*]$$

proves that $A \epsilon P_{n+1}^{(k)} = P_n^{(k)}$. Hence $P_{m+1}^{(k)} = P_n^{(k)}$. Passing to complements we obtain $Q_{m+1}^{(k)} = Q_n^{(k)} = P_n^{(k)}$. The formula (10) is thus proved.

Consider now the universal function $F_n^{(k)}(h)$ defined in 3.22 and put

$$A = \underset{(n_1,\ldots,n_{k-1},h)}{E} [(n_1,\ldots,n_{k-1},h) \text{ non } \epsilon F_n^{(k)}(h)].$$

[24]) Kuratowski [10], p. 175.

This set does not belong to $P_n^{(k)}$ because otherwise there would be an integer h_0 such that $A = F_n^{(k)}(h_0)$ which is impossible since we would then have

$$(h_0,h_0,\ldots,h_0) \epsilon A \equiv (h_0,h_0,\ldots,h_0) \text{ non } \epsilon F_n^{(k)}(h_0) \equiv (h_0,h_0,\ldots,h_0) \text{ non } \epsilon A.$$

Observe now that

$$(n_1,\ldots,n_{k-1},h) \epsilon A \equiv \sum_q (q=h) \cdot [(n_1,\ldots,n_{n-1},q) \text{ non } \epsilon F_n^{(k)}(h)] \equiv$$
$$\equiv \sum_q (q=h) \cdot [(n_1,\ldots,n_{k-1},q,h) \epsilon R_{k+1} - M_n^{(k)}]$$

which proves according to 3.22 that $A \epsilon P_{s+n+3}^{(k)}$ and consequently $A \epsilon P_n^{(k)}$ according to (10). The assumption $P_n^{(k)} = P_{n+1}^{(k)}$ leads thus to a contradiction.

The inequality $Q_n^{(k)} \neq Q_{n+1}^{(k)}$ will now result if we pass to complements on both sides of $P_n^{(k)} \neq P_{n+1}^{(k)}$.

3.32. *If S fulfills the condition C_s and if $n>0$, then $P_n^{(k)}$ non $\subset Q_n^{(k)}$ and $Q_n^{(k)}$ non $\subset P_n^{(k)}$.*

Proof. From $P_n^{(k)} \subset Q_n^{(k)}$ we obtain $Q_n^{(k)} \subset P_n^{(k)}$ and hence

(11) $$P_n^{(k)} = Q_n^{(k)}.$$

Let us first suppose that $k>1$ and $A \epsilon P_{n+1}^{(k-1)}$. For a suitable $B \epsilon Q_n^{(k)}$ we have

$$\mathfrak{n} \epsilon A \equiv \sum_p (\mathfrak{n},p) \epsilon B.$$

B being in $Q_n^{(k)}$, it is also in $P_n^{(k)}$ in virtue of (11) and hence $A \epsilon P_n^{(k-1)}$ by 2.16. We obtain thus $P_{n+1}^{(k-1)} \subset P_n^{(k-1)}$ against 3.31.

Suppose now that $k=1$. If $A \epsilon P_n^{(1)}$ or $A \epsilon Q_n^{(1)}$, then the set $A_1 = \underset{(p,q)}{E} [2^p(2q+1) \epsilon A]$ belongs to $P_n^{(2)}$ or $Q_n^{(2)}$ since $A_1^* = A$ (comp. 2.15). It is obvious that every set of $P_n^{(2)}$ or $Q_n^{(2)}$ may, for a suitable $A \epsilon P_n^{(1)}$ or $A \epsilon Q_n^{(1)}$, be represented as A_1. Hence $P_n^{(1)} = Q_n^{(1)}$ would lead to the equality $P_n^{(2)} = Q_n^{(2)}$ which we already know to be impossible.

We have thus proved that $P_n^{(k)}$ non $\subset Q_n^{(k)}$ for any k and $n>0$. Passing to complements we obtain $Q_n^{(k)}$ non $\subset P_n^{(k)}$, q. e. d.

We note at last the following result concerning the existence of sets not definable within arithmetic i. e. belonging to no class $\boldsymbol{P}_n^{(k)}$ [25]).

3.33. *The function* $\boldsymbol{F}^{(k)}_{s_1(n)}(s_2(n))$ *is universal for the class* $\sum\limits_{l=0}^{\infty}\boldsymbol{P}_l^{(k)}$ *and the set*

$$A_0 = \underset{(n_1,\dots,n_{k-1},n)}{E}[(n_1,\dots,n_{k-1},n) \text{ non} \in \boldsymbol{F}^{(k)}_{s_1(n)}(s_2(n))]$$

does not belong to the sum $\sum\limits_{l=0}^{\infty}\boldsymbol{P}_l^{(k)}$.

Proof. The first part of the theorem results from the equivalence

$$A \in \sum_{l=0}^{\infty}\boldsymbol{P}_l^{(k)} \equiv \sum_l A \in \boldsymbol{P}_l^{(k)} \equiv \sum_l\sum_h A = \boldsymbol{F}_l^{(k)}(h) \equiv \sum_n A = \boldsymbol{F}^{(k)}_{s_1(n)}(s_2(n)).$$

The second part is a particular case of the „diagonal theorem" referred to in the foot-note [24]).

§ 4. Applications to theorems of Gödel and Rosser.

4.1. ω-consistency. Let us recall the following definition due to Gödel [26]): A logical system S is called ω-consistent if, for any propositional function $\varphi(x)$ with one free variable, the following implication holds:

$$\prod_n \vdash \varphi(n) \rightarrow \text{non} \vdash \sum_y \varphi'(y). \tag{12}$$

We could, of course, replace this implication by

$$\vdash \sum_y \varphi(y) \rightarrow \sum_n \vdash \varphi(n).$$

It is important to observe that the quantifier „$\sum\limits_n$" is taken meta-mathematically whereas „$\sum\limits_y \varphi(y)$" represents a sentence of the formal system S.

4.2. Gödel's theorem. It states

4.21. *If the system S is ω-consistent and fulfills the condiiton C_s, then there is a sentence ϑ such that neither $\vdash\vartheta$ nor $\vdash\vartheta'$.*

25) The existence of not definable sets has been stated by Tarski [19], p. 221. See also Carnap [1], p. 89.

26) Gödel [3], p. 187.

Proof. According to 3.31 the class $\boldsymbol{P}_1^{(1)} - \boldsymbol{P}_0^{(1)}$ is non-empty. Let A by any set of this class and let $B \in \boldsymbol{P}_0^{(2)}$ be a set such that $n \in A \equiv \sum\limits_p (n,p) \in B$. Denoting by $\varphi(x,y)$ any decidable propositional function which defines B we have

$$n \in A \equiv \sum_p \vdash \varphi(n,p). \tag{13}$$

Write $\psi(x)$ instead of $\sum\limits_y \varphi(x,y)$, The formula (13) gives then

$$n \in A \rightarrow \vdash \psi(n)$$

since $\vdash \varphi(n,p) \rightarrow \vdash \sum\limits_y \varphi(n,y)$. If $n \text{ non} \in A$, then $\prod\limits_p \text{non} \vdash \varphi(n,p)$ and therefore $\prod\limits_p \vdash \varphi'(n,p)$ because of the decidability of $\varphi(x,y)$. Using (12) we obtain $\text{non} \vdash \sum\limits_y \varphi(n,y)$, i.e. $\text{non} \vdash \psi(n)$. Hence $n \text{ non} \in A \rightarrow \text{non} \vdash \psi(n)$ and we see that

$$n \in A \equiv \vdash \psi(n). \tag{14}$$

This equivalence would prove that $A \in \boldsymbol{P}_0^{(1)}$ if $\psi(x)$ were decidable. Since $A \text{ non} \in \boldsymbol{P}_0^{(1)}$, $\psi(x)$ cannot be decidable, i. e. there is an integer n_0 such that neither $\vdash \psi(n_0)$ nor $\vdash \psi'(n_0)$. Denoting $\psi(n_0)$ by ϑ we obtain the desired result.

4.3. Remarks. 4.31. Theorem 4.21 was first established by Gödel for a concrete formal system called P [27]). Rosser [28]) generalised this result showing that it holds for any system S in which the Gödel-numbers of valid sentences form a recursively enumerable set. This is essentially the same assumption as our condition C_0. Our proof of 4.21 shows that the theorem holds even under the weaker condition C_s. Hence the Gödel's theorem is valid for all such systems S in which the set of Gödel-numbers of valid sentences is definable in elementary arithmetic.

4.32. If S satisfies the stronger condition C_0, then as shown by Rosser [29]) the assumption of ω-consistency can by replaced by the (weaker) assumption of ordinary self-consistency of S. This is in general impossible for systems satisfying the weaker condition C_s $(s>0)$ since there exists a logical system S such that its valid sentences form a self-consistent and complete [30]) class whereas the set of theirs Gödel-number is definable within $\mathfrak{A}$ (i. e. be-

27) Gödel [3], Satz VI.

28) Rosser [14], Theorem I A, p. 89.

29) Rosser [14], Theorem II, p. 89.

30) I. e. for any ϑ either ϑ or ϑ' is valid.

longs to one of classes $P_s^{(1)}$). Hence S fulfills the condition C_s for an $s>0$ and there are no undecidable propositions in S. In order to get such a system it is sufficient to apply to system P the procedure with the help of which Lindenbaum has shown that there are complete and self-consistent enlargements of any self-consistent class of sentences [31]).

4.33. It follows from 4.21 that any complete and self-consistent class C of sentences (e. g. of the system P or, more general, of any system which fulfills our condition C_s for an $s\geqslant 0$) must be ω-inconsistent if the set of Gödel-numbers of sentences of this class is definable within the elementary arithmetic $\mathfrak{A}$ (i. e. belongs to $\sum_{n=0}^{\infty} P_n^{(1)}$). It has been stated already by Tarski [32]) that under these conditions C must contain false statements. The ω-inconsistency of C seems to be a new result [32a]).

4.34. The proof of 4.21 remains still valid if we modify the definition 4.1 restricting (12) to decidable propositional functions. The question remains open whether there is an ω-inconsistent system S satisfying this modified definition of ω-consistency.

4.35. Remark 4.31 makes desirable examples of formal systems satisfying C_s for some $s>0$ but not C_0. One such example will be treated in **4.4** and **4.5** in connection with the so called rule of infinite induction. Another example may be suggested here: Suppose that S contains variables $X, Y, Z, \ldots$ of the type of classes of positive integers and enlarge the system adding to its rules the following one: if $\varphi(A)$ is valid in S for any set A definable within S, then $\prod_X \varphi(X)$ is valid in the enlarged system. The enlarged system is probably self-consistent and fulfills C_s for some $s>0$ but not for $s=0$.

4.4. Rule of infinite induction [33]). This rule states that $\prod_x \varphi(x)$ may be admitted as proved if $\prod_p \vdash \varphi(p)$.

4.41. *Under the assumptions of* 4.21 *the system S is not closed under the rule of infinite induction* [34]).

[31]) See Tarski [16], Satz I. 56, p. 394. The same result is to be found in Gödel [6], pp. 20-21. The theorem in question is also known in the theory of Boolean algebras as the „fundamental theorem of ideal-theory". See e. g. my paper in the previous volume of these Fundamenta, pp. 7-8, footnote [5]).

[32]) Tarski [18], p. 378.

[32a]) This result has been also found independently by A. Tarski in 1942.

[33]) This rule has been introduced by Hilbert [7] which ascribed it a finitary character. The rule was further studied by Tarski [18], pp. 383-387, Carnap [1], p. 26 and Rosser [13], pp. 129-133. This last author calls it „Carnap rule".

[34]) Tarski says: S is ω-incomplete. See [17], p. 105. The theorem 4.41 has been proved by Gödel [3], p. 190 and generalised afterward by Rosser [14], p. 89. Comp. remark 4.31.

Proof. Glancing at the proof of 4.21 we see that the number n_0 which has been defined there does not belong to A since otherwise we would obtain $\vdash \psi(n_0)$ in virtue of (14). Hence (13) gives $\prod_p \vdash \varphi(n_0, p)$ whereas non $\vdash \psi'(n_0)$ yields non $\vdash \prod_x \varphi(n_0, x)$.

We introduce now the concept of an n-valid sentence (in symbols $\vdash_n \varphi$) [35]). For $n=0$ we define $\vdash_0 \varphi$ as $\vdash \varphi$. Suppose now that $n>0$ and that the class of $n-1$-valid sentences has already been defined. We shall write $\vdash_n \varphi$ (read: φ is an n-valid sentence) if φ belongs to every class C satisfying three following conditions:

If $\vdash_{n-1} \psi$, then ψ is in C;
C is closed under the rules of inference of S;
If $\prod_p \vdash_{n-1} \psi(p)$, then $\prod_x \psi(x)$ is in C.

Speaking less formally, we could say that $\vdash_n \varphi$ holds if and only if φ can be obtained from the axioms of S with the help of rules of inference admitted in S and with the help of the rule of infinite induction, this last rule being used but n times.

A propositional function $\varphi(\mathfrak{x})$ with k free variables will be said to be n-decidable if for any $\mathfrak{n} \in R_k$ either $\vdash_n \varphi(\mathfrak{n})$ or $\vdash_n \varphi'(\mathfrak{n})$.

4.42. *If the class of n-valid sentences is self-consistent and if $A \in P_n^{(k)} + Q_n^{(k)}$, then there is an n-decidable propositional function $\varphi(\mathfrak{x})$ with k free variables such that*

$$\mathfrak{n} \in A \equiv \vdash_n \varphi(\mathfrak{n})$$

for any $\mathfrak{n} \in R_k$.

Proof by induction on n. For $n=0$ the theorem is obvious. Let us suppose that it holds for an integer $n\geqslant 0$ and for $k=1,2,\ldots$ If $A \in P_{n+1}^{(k)}$, then for a suitable $B \in Q_n^{(k+1)}$ we have

$$\mathfrak{n} \in A \equiv \sum_p (\mathfrak{n}, p) \in B$$

for any $\mathfrak{n} \in R_k$. If $n+1$-valid sentences form a self-consistent class, the same holds true for n-valid sentences and the inductive assumption yields the existence of an n-decidable propositional function $\varphi(\mathfrak{x}, y)$ with $k+1$ free variables such that

$$(\mathfrak{n}, p) \in B \equiv \vdash_n \varphi(\mathfrak{n}, p).$$

[35]) Rosser [13], pp. 129-130.

Let $\psi(\mathfrak{x})$ denote the propositional function $\sum_y \varphi(\mathfrak{x},y)$. From two last equivalences we obtain immediately $\mathfrak{n} \epsilon A \to \vdash_n \psi(\mathfrak{n})$ and hence

(15) $$\mathfrak{n} \epsilon A \to \vdash_{n+1} \psi(\mathfrak{n}).$$

If $\mathfrak{n}$ non ϵA, then $\prod_p$ non $\vdash_n \varphi(\mathfrak{n},p)$ and hence $\prod_p \vdash_n \varphi'(\mathfrak{n},p)$ which proves accordingly to the definition of $n+1$-valid sentences that $\vdash_{n+1} \psi'(\mathfrak{n})$. Therefore

(16) $$\mathfrak{n} \text{ non } \epsilon A \to \vdash_{n+1} \psi'(\mathfrak{n}).$$

Assuming that the $n+1$-valid sentences form a self-consistent class, we obtain now $\mathfrak{n}$ non $\epsilon A \to$ non $\vdash_{n+1} \psi(\mathfrak{n})$ and finally in virtue of (15)

$$\mathfrak{n} \epsilon A \equiv \vdash_{n+1} \psi(\mathfrak{n}).$$

It remains to prove that $\psi(\mathfrak{x})$ is $n+1$-decidable. We have in fact for any $\mathfrak{n} \epsilon R_k$

$$\text{either } \mathfrak{n} \epsilon A \quad \text{or} \quad \mathfrak{n} \text{ non } \epsilon A,$$

i. e. with respect to (15) and (16)

$$\text{either } \vdash_{n+1} \psi(\mathfrak{n}) \quad \text{or} \quad \vdash_{n+1} \psi'(\mathfrak{n}).$$

The theorem is thus proved for $A \epsilon \boldsymbol{P}_{n+1}^{(k)}$. In order to prove it for $A \epsilon \boldsymbol{Q}_{n+1}^{(k)}$ it is sufficient to observe that (15) and (16) yield the equivalence

$$\mathfrak{n} \epsilon R_k - A \equiv \vdash_{n+1} \psi'(\mathfrak{n})$$

and that the propositional function $\psi'(\mathfrak{x})$ is $n+1$-decidable.

4.43. (Rosser's theorems). *If S fulfills the condition C_s (for an $s \geqslant 0$) and if the n-valid sentences form an ω-consistent class, then: 1^0 this class is not closed under the rule of infinite induction; 2^0 there is a ϑ such that neither $\vdash_n \vartheta$ nor $\vdash_n \vartheta'$* [36]).

To obtain the proof of 2^0 we repeat the proof of 4.21 taking as A any set from $\boldsymbol{P}_{n+1}^{(k)} - \boldsymbol{P}_n^{(k)}$ and replacing „$\vdash$“ throughout by „$\vdash_n$“. 1^0 follows then as in 4.41.

[36]) Rosser [14], theorem VI, p. 132. Similar remarks as in 4.31 apply here.

§ 5. Functions of classes $\boldsymbol{P}_n^{k,l}$ and $\boldsymbol{Q}_n^{k,l}$.

5.1. Definitions. We denote by $R_l^{R_k}$ the class of functions mapping R_k on a subset of R_l [37]). A function $f \epsilon R_l^{R_k}$ is said to be of class $\boldsymbol{P}_n^{(k,l)}$ or $\boldsymbol{Q}_n^{(k,l)}$ if the „curve“

$$I_f = \underset{(\mathfrak{n},\mathfrak{m})}{E} [\mathfrak{m} = f(\mathfrak{n})]$$

belongs to $\boldsymbol{P}_n^{(k+l)}$ or $\boldsymbol{Q}_n^{(k+l)}$.

Remark. In order to maintain the analogy with the theory of Borel-functions it would be perhaps better to define $\boldsymbol{P}_n^{(k,l)}$ or $\boldsymbol{Q}_n^{(k,l)}$ as the class of functions f such that for any $A \epsilon \boldsymbol{P}_0^{(l)}$ the counter-image $f^{-1}(A)$ is of class $\boldsymbol{P}_n^{(k)}$ or $\boldsymbol{Q}_n^{(k)}$ [38]). It will be proved in **5.3** that classes $\boldsymbol{P}_n^{(k,l)}$ and $\boldsymbol{Q}_n^{(k,l)}$ defined above possess this property. The converse theorem seems, however, to be false. The analogy with the theory of Borel-sets is here breaking down.

5.2. Images. We put for $f \epsilon R_l^{R_k}$ and $A \subset R_k$

$$f(A) = \underset{\mathfrak{m}}{E} [\sum_{\mathfrak{n}} (\mathfrak{n} \epsilon A) \cdot (\mathfrak{m} = f(\mathfrak{n}))]$$

and call $f(A)$ the image of A. Obviously

$$\mathfrak{m} \epsilon f(A) \equiv \sum_{\mathfrak{n}} (\mathfrak{n} \epsilon A) \cdot (\mathfrak{m} = f(\mathfrak{n}))$$

from what the following theorem immediately results by the Kuratowski-Tarski evaluation method:

5.21. *If $A \epsilon \boldsymbol{P}_n^{(k)}$ and $f \epsilon \boldsymbol{Q}_n^{(k,l)}$ $(n \geqslant 0)$, then $f(A) \epsilon \boldsymbol{P}_{n+1}^{(l)}$ and if $A \epsilon \boldsymbol{P}_n^{(k)}$ and $f \epsilon \boldsymbol{P}_n^{(k,l)}$ $(n \geqslant 1)$, then $f(A) \epsilon \boldsymbol{P}_n^{(l)}$.*

5.3. Counter-images. If $f \epsilon R_l^{R_k}$ and $A \subset R_l$, then the counter-image of A is defined as

$$f^{-1}(A) = \underset{\mathfrak{n}}{E} [f(\mathfrak{n}) \epsilon A].$$

Evidently

(17) $$\mathfrak{n} \epsilon f^{-1}(A) \equiv \sum_{\mathfrak{m}} [(\mathfrak{m} = f(\mathfrak{n})) \cdot (\mathfrak{m} \epsilon A)] \equiv \prod_{\mathfrak{m}} [(\mathfrak{m} = f(\mathfrak{n})) \to (\mathfrak{m} \epsilon A)].$$

In virtue of 2.16 we obtain from these equivalences the following theorems:

5.31. *If $A \epsilon \boldsymbol{P}_n^{(l)}$ and $f \epsilon \boldsymbol{P}_n^{(k,l)}$ $(n \geqslant 1)$, then $f^{-1}(A) \epsilon \boldsymbol{P}_n^{(k)}$.*

5.32. *If $A \epsilon \boldsymbol{Q}_n^{(l)}$ and $f \epsilon \boldsymbol{P}_n^{(k,l)}$ $(n \geqslant 1)$, then $f^{-1}(A) \epsilon \boldsymbol{Q}_n^{(k)}$.*

[37]) Kuratowski [10], p. 199.
[38]) Kuratowski [10], p. 177.

The evaluation in case $n=0$ is given in the following theorem:

5.33. *If $A \in P_0^{(l)}$ and $f \in P_0^{(k,l)}$, then $f^{-1}(A) \in P_0^{(k)}$.*

Proof. The assumptions $A \in P_0^{(l)}$ and $f \in P_0^{(k,l)}$ secure the existence of two decidable propositional functions $\varphi(\mathfrak{y})$ and $\psi(\mathfrak{x},\mathfrak{y})$ with l and $k+l$ free variables such that

(18) $$\mathfrak{m} \in A \equiv \vdash \varphi(\mathfrak{m}) \quad \text{and} \quad [\mathfrak{m}=f(\mathfrak{n})] \equiv \vdash \psi(\mathfrak{n},\mathfrak{m}).$$

Here $\mathfrak{y}$ symbolizes a sequence of l variables $y_1, y_2, \ldots, y_l$. Let us denote by $\mathfrak{y}<\overline{\mathfrak{y}}$ the following propositional function

$$(y_1<\bar{y}_1)+(y_1=\bar{y}_1)\cdot(y_2<\bar{y}_2)+\ldots+(y_1=\bar{y}_1)\ldots(y_{l-1}=\bar{y}_{l-1})\cdot(y_l<\bar{y}_l).$$

$\mathfrak{m}<\overline{\mathfrak{m}}$ says that $\mathfrak{m}$ preceeds $\overline{\mathfrak{m}}$ in the lexicographical ordering of R_l. Consider now the propositional function $\psi^*(\mathfrak{x},\mathfrak{y})$ defined as

$$\psi(\mathfrak{x},\mathfrak{y}) \cdot \prod_{\overline{\mathfrak{y}}} [(\overline{\mathfrak{y}}<\mathfrak{y}) \to \psi'(\mathfrak{x},\overline{\mathfrak{y}})].$$

$\psi^*(\mathfrak{n},\mathfrak{m})$ says that $\mathfrak{m}$ is the first point of R_l (with respect to lexicographical ordering) such that $\psi(\mathfrak{n},\mathfrak{m})$. If $\vdash\psi^*(\mathfrak{n},\mathfrak{m})$, then $\mathfrak{m}=f(\mathfrak{n})$ in virtue of (18). If, conversely, $\mathfrak{m}=f(\mathfrak{n})$, then $\mathfrak{p}\neq f(\mathfrak{n})$ for any $\mathfrak{p}$ which preceeds $\mathfrak{m}$ in the lexicographical ordering of R_l and we obtain easily $\vdash\psi^*(\mathfrak{n},\mathfrak{m})$. Hence

(19) $$[\mathfrak{m}=f(\mathfrak{n})] \equiv \vdash \psi^*(\mathfrak{n},\mathfrak{m}).$$

Further it is plain that

$$\vdash \prod_{\mathfrak{x}} \prod_{\mathfrak{y}} \prod_{\mathfrak{z}} [\psi^*(\mathfrak{x},\mathfrak{y}) \cdot \psi^*(\mathfrak{x},\mathfrak{z}) \to (\mathfrak{y}=\mathfrak{z})].$$

Using 1.22 we see that the propositional function $\psi^*(\mathfrak{x},\mathfrak{y})$ is decidable. From $\vdash \psi(\mathfrak{n},f(\mathfrak{n}))$ we infer at last that for any $\mathfrak{n} \in R_k$ there is an $\mathfrak{m} \in R_l$ such that $\vdash\psi^*(\mathfrak{n},\mathfrak{m})$. Hence all assumptions of 1.23 are fulfilled and we obtain the result that the propositional function $\vartheta(\mathfrak{x})$ defined as

$$\sum_{\mathfrak{y}} \varphi(\mathfrak{y}) \cdot \psi^*(\mathfrak{x},\mathfrak{y})$$

is decidable. Denoting by $\zeta(\mathfrak{x})$ the propositional function

$$\prod_{\mathfrak{y}} [\psi^*(\mathfrak{x},\mathfrak{y}) \to \varphi(\mathfrak{y})],$$

we infer from 1.24 that

(20) $$\vdash \vartheta(\mathfrak{x}) \to \zeta(\mathfrak{x}).$$

The first equivalence (17) yields now (with respect to (18) and (19)) the implication:

$$\mathfrak{n} \in f^{-1}(A) \to \sum_{\mathfrak{m}} (\mathfrak{m} \in A) \cdot (\mathfrak{m}=f(\mathfrak{n})) \to \sum_{\mathfrak{m}} \vdash \varphi(\mathfrak{m}) \cdot \psi^*(\mathfrak{n},\mathfrak{m})$$
$$\to \vdash \sum_{\mathfrak{y}} \varphi(\mathfrak{y}) \cdot \psi^*(\mathfrak{n},\mathfrak{y}) \to \vdash \vartheta(\mathfrak{n})$$

whereas the second yields

$$\mathfrak{n} \text{ non} \in f^{-1}(A) \to \sum_{\mathfrak{m}} (\mathfrak{m} \text{ non} \in A) \cdot (\mathfrak{m}=f(\mathfrak{n})) \to$$
$$\to \sum_{\mathfrak{m}} \vdash \varphi'(\mathfrak{m}) \cdot \psi^*(\mathfrak{n},\mathfrak{m}) \to \vdash \sum_{\mathfrak{y}} \psi^*(\mathfrak{n},\mathfrak{y}) \cdot \varphi'(\mathfrak{y}) \to \vdash \zeta'(\mathfrak{n}),$$

i. e. with respect to (20) $\mathfrak{n} \text{ non} \in f^{-1}(A) \to \vdash \vartheta'(\mathfrak{n}) \to \text{non} \vdash \vartheta(\mathfrak{n})$. Hence $\mathfrak{n} \in f^{-1}(A) \equiv \vdash \vartheta(\mathfrak{n})$ and therefore $f^{-1}(A) \in P_0^{(k)}$, q. e. d.

5.4. The function $\min_p [(\mathfrak{n},p) \in A]$. Let us suppose that A is a subset of R_{k+1} such that $\prod_{\mathfrak{n}} \sum_p (\mathfrak{n},p) \in A$ and denote by $\mu_A(\mathfrak{n})$ the smallest integer p such that $(\mathfrak{n},p) \in A$:

$$[p=\mu_A(\mathfrak{n})] \equiv \{(\mathfrak{n},p) \in A \cdot \prod_q [(q \geqslant p)+(\mathfrak{n},q) \text{ non} \in A]\}.$$

The Kuratowski-Tarski method leads immediately to the following theorem:

5.41. *If $A \in Q_n^{(k+1)}$, then $\mu_A \in Q_{n+1}^{(k,1)}$ and if $A \in P_n^{(k+1)}$ $(n \geqslant 1)$, then $\mu_A \in P_{n+1}^{(k,1)} \cdot Q_{n+1}^{(k,1)}$.*

For $n=0$ we have the sharper evaluation:

5.42. *If $A \in P_0^{(k+1)}$, then $\mu_A \in P_0^{(k,1)}$.*

Proof. Denote by $\varphi(\mathfrak{x},x)$ any decidable propositional function which defines A and by $\psi(\mathfrak{x},x)$ the propositional function

$$\varphi(\mathfrak{x},x) \cdot \prod_y [y<x \to \varphi'(\mathfrak{x},y)].$$

$\psi(\mathfrak{x},x)$ is decidable by 1.22 and it is obvious that it defines the set $\underset{(\mathfrak{n},p)}{E}[p=\mu_A(\mathfrak{n})]$. Hence $\mu_A \in P_0^{(k,1)}$, q. e. d.

5.5. Post's theorem. This theorem is an exact analogue of the well-known Souslin's theorem concerning sets which are analytical together with their complements. It can be stated as follows:

5.51. $\boldsymbol{P}_1^{(k)} \cdot \boldsymbol{Q}_1^{(k)} = \boldsymbol{P}_0^{(k)}$ [39].

Proof. In virtue of 2.31 we have only to show that if $A \in \boldsymbol{P}_1^{(k)}$ and $R_k - A \in \boldsymbol{P}_1^{(k)}$, then $A \in \boldsymbol{P}_0^{(k)}$. Let B_1 and B_2 be two sets of $\boldsymbol{P}_0^{(k+1)}$ such that

$$\mathfrak{n} \in A \equiv \sum_p (\mathfrak{n}, p) \in B_1, \qquad \mathfrak{n} \in R_k - A \equiv \sum_p (\mathfrak{n}, p) \in B_2.$$

Since $\prod_{\mathfrak{n}} [(\mathfrak{n} \in A) + (\mathfrak{n} \in R_k - A)]$, we have $\prod_{\mathfrak{n}} \sum_p [(\mathfrak{n}, p) \in B_1 + B_2]$ which proves accordingly to 5.42 that $\mu_{B_1+B_2} \in \boldsymbol{P}_0^{(k+1,1)}$, the sum $B_1 + B_2$ being of class $\boldsymbol{P}_0^{(k+1)}$ by 2.17. Now define a function $f \in R_{k+1}^{R_k}$ putting for any $\mathfrak{n} \in R_k$:

$$f(\mathfrak{n}) = (\mathfrak{n}, \mu_{B_1+B_2}(\mathfrak{n})).$$

We have $f \in \boldsymbol{P}_0^{(k,k+1)}$, since

$$[(\mathfrak{m}, p) = f(\mathfrak{n})] \equiv [(\mathfrak{m} = \mathfrak{n}) \cdot (p = \mu_{B_1+B_2}(\mathfrak{n}))].$$

Evidently

$$\mathfrak{n} \in f^{-1}(B_1) \equiv \sum_{(\mathfrak{m}, p)} [(\mathfrak{m}, p) = f(\mathfrak{n})] \cdot [(\mathfrak{m}, p) \in B_1] \rightarrow$$
$$\rightarrow \sum_{(\mathfrak{m}, p)} (\mathfrak{m} = \mathfrak{n}) \cdot [(\mathfrak{m}, p) \in B_1] \rightarrow \sum_p (\mathfrak{n}, p) \in B_1 \rightarrow \mathfrak{n} \in A.$$

If, conversely, $\mathfrak{n} \in A$, then $(\mathfrak{n}, \mu_{B_1+B_2}(\mathfrak{n})) \in B_1$, since otherwise we would obtain $(\mathfrak{n}, \mu_{B_1+B_2}(\mathfrak{n})) \in B_2$ and therefore $\sum_p (\mathfrak{n}, p) \in B_2$ or $\mathfrak{n} \in R_k - A$. Hence $f(\mathfrak{n}) \in B_1$ and $\mathfrak{n} \in f^{-1}(B_1)$. This proves that $A = f^{-1}(B_1)$ and the theorem 5.33 yields the desired result $A \in \boldsymbol{P}_0^{(k)}$.

From 5.51 we obtain two important corollaries:

5.52. *If $f \in \boldsymbol{P}_0^{(k,l)}$ and $g \in \boldsymbol{P}_0^{(m,k)}$, then the compounded function $f(g(\mathfrak{m}))$ is of class $\boldsymbol{P}_0^{(m,l)}$.*

Proof. We have

$$[\mathfrak{l} = f(g(\mathfrak{m}))] \equiv \sum_{\mathfrak{n}} (\mathfrak{n} = g(\mathfrak{m})) \cdot (\mathfrak{l} = f(\mathfrak{n})) \equiv \prod_{\mathfrak{n}} [(\mathfrak{n} = g(\mathfrak{m})) \rightarrow (\mathfrak{l} = f(\mathfrak{n}))].$$

[39]) Post [12], p. 290. See footnote [6]) on page 82.

The first equivalence shows that the set $\underset{(\mathfrak{l}, \mathfrak{m})}{E} [\mathfrak{l} = f(g(\mathfrak{m}))]$ is of class $\boldsymbol{P}_1^{(m+k)}$ and the second that it is of class $\boldsymbol{Q}_1^{(m+k)}$. Hence by 5.51 it is of class $\boldsymbol{P}_0^{(m+k)}$.

5.53. *A set $A \subset R_k$ is in $\boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$ if and only if its characteristic function c_A is in $\boldsymbol{P}_n^{(k,1)} \cdot \boldsymbol{Q}_n^{(k,1)}$.*

Proof. From

$$[c_A(\mathfrak{n}) = p] \equiv [(\mathfrak{n} \in A) \cdot (p = 1) + (\mathfrak{n} \in R_k - A) \cdot (p = 0)]$$

we infer easily that if $A \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$, then $c_A \in \boldsymbol{P}_n^{(k,1)} \cdot \boldsymbol{Q}_n^{(k,1)}$. Suppose now that $c_A \in \boldsymbol{P}_n^{(k,1)} \cdot \boldsymbol{Q}_n^{(k,1)}$. From

$$(\mathfrak{n} \in A) \equiv \sum_p [(p = 1) \cdot (c_A(\mathfrak{n}) = p)] \equiv \prod_p [(p = 1) \rightarrow (c_A(\mathfrak{n}) = p)]$$

we see that if $n \geqslant 1$, then $A \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$. For $n = 0$ these equivalences yield $A \in \boldsymbol{P}_1^{(k)} \cdot \boldsymbol{Q}_1^{(k)}$ and hence $A \in \boldsymbol{P}_0^{(k)} \cdot \boldsymbol{Q}_0^{(k)}$ in virtue of 5.51.

It is interesting to observe that if $A \in \boldsymbol{P}_0^{(k)}$ and $f \in \boldsymbol{P}_0^{(k,l)}$, then the set $f(A)$ does not necessarily belong to $\boldsymbol{P}_0^{(l)}$, even if f is one-to-one. We see here another discrepance between our theory and the theory of Borel-sets.

It can be shown, however, that if f is an increasing function, i. e., if $\mathfrak{n} < \overline{\mathfrak{n}} \rightarrow f(\mathfrak{n}) < f(\overline{\mathfrak{n}})$, then $f(A) \in \boldsymbol{P}_0^{(k)}$ ($<$ represents here the lexicographical ordering of k-ads or l-ads of integers) [40]).

5.6. Sets of the class $\boldsymbol{P}_1^{(k)}$ as values of functions $\boldsymbol{P}_0^{(1,k)}$. The theorem 5.51 enables us to give a simple proof of the following theorem which discloses the relationship between the concept of the class $\boldsymbol{P}_1^{(k)}$ and that of recursively enumerable sets:

5.61. *If S fulfills the condition C_0, then the necessary and sufficient condition for a non-empty set A be in $\boldsymbol{P}_1^{(k)}$ is that there is a function $f \in \boldsymbol{P}_0^{(1,k)}$ whose set of values is A.*

[40]) This theorem has been proved by Kleene. See Kleene [9], theorem VII, p. 737, Rosser [14], Corollary I, p. 88, Post [12], p. 291.

Proof[41]). Sufficiency results at once from 5.21. Suppose now that $A \in P_1^{(k)}$ and $\mathfrak{n}_0 \in A$. Let $\varphi(\mathfrak{x}, x)$ be a decidable propositional function with $k+1$ free variables such that

$$\mathfrak{n} \in A \equiv \sum_p \varphi(\mathfrak{n}, p)$$

for any $\mathfrak{n} \in R_k$.

We shall denote by $\mathfrak{s}(n)$ the sequence of k integers $s_1(s_1(n))$, $(s_1(s_1(s_1(n))), \ldots, s_1(s_1 \ldots (s_1(n)) \ldots)$. An easy induction on k shows that for any $\mathfrak{n} \in R_k$ and $p, q \in R_1$ there is an integer h such that $\mathfrak{n} = \mathfrak{s}(h)$, $p = s_1(h)$ and $q = s_2(h)$.

Let l_0 be the Gödel-number of $\varphi(\mathfrak{x}, x)$.

Define now the function $f(n)$ as follows: if $s_1(n)$ is the Gödel-number of a formal proof of $\varphi(\mathfrak{s}(n), s_2(n))$, then $f(n) = \mathfrak{s}(n)$; if not, then $f(n) = \mathfrak{n}_0$.

It is obvious that $f(n) \in A$ for any n. Conversely, if $\mathfrak{n} \in A$, then, for a suitable p, $\vdash \varphi(\mathfrak{n}, p)$. Denoting by q the Gödel-number of a formal proof of $\varphi(\mathfrak{n}, p)$ and by h the integer for which $\mathfrak{s}(h) = \mathfrak{n}$, $s_1(h) = q$, $s_2(h) = p$, we obtain $f(h) = \mathfrak{n}$. Hence A is the set of values of f.

It remains to evaluate the class of f. Remembering the definitions of sets Δ, E and Γ_k given in **3.1** we see that

$$[\mathfrak{m} = f(n)] \equiv \{(\mathfrak{m} = \mathfrak{s}(n)) \cdot [(s_1(n) \in E) \cdot \sum_q (s_1(n), q) \in \Delta \cdot \\ \cdot (l_0, q, \mathfrak{s}(n), s_2(n)) \in \Gamma_{k+1}] + (\mathfrak{m} = \mathfrak{n}_0) \cdot [(s_1(n) \text{ non} \in E) + \\ + \sum_q (s_1(n), q) \in \Delta \cdot (l_0, q, \mathfrak{s}(n), s_2(n)) \text{ non} \in \Gamma_{k+1}]\}.$$

This proves the set $\underset{\mathfrak{m}, n}{E} [\mathfrak{m} = f(n)]$ to be of class $P_1^{(k+1)}$. Remembering further that if $s_1(n) \in E$, then there is exactly one q such that $(s_1(n), q) \in \Delta$, we can rewrite the above equivalence in the following form:

$$[\mathfrak{m} = f(n)] \equiv \\ \equiv ([\mathfrak{m} = \mathfrak{s}(n)] \cdot \{(s_1(n) \in E) \cdot \prod_q [(s_1(n), q) \in \Delta \rightarrow (l_0, q, \mathfrak{s}(n), s_2(n)) \in \Gamma_{k+1}]\} + \\ + (\mathfrak{m} = \mathfrak{n}_0) \cdot \{(s_1(n) \text{ non} \in E) + \prod_q [(s_1(n), q) \in \Delta \rightarrow (l_0, q, \mathfrak{s}(n), s_2(n)) \text{ non} \in \Gamma_{k+1}]\}).$$

The set $\underset{\mathfrak{m}, n}{E} [\mathfrak{m} = f(n)]$ is thus of class $Q_1^{(k+1)}$ and hence by 5.51 it is of class $P_0^{(k+1)}$, q. e. d.

[41]) This proof is essentially due to Kleene [9], theorem III, p. 736.

§ 6. Relations with the theory of general-recursive functions [42]).

6.1. Recursivity conditions. We shall suppose that the system S fulfills the following two conditions:

(R_1) *Primitive recursive subsets of R_k belong to $P_0^{(k)}$;*

(R_2) *If φ is any propositional function with k free variables, then the relation $q B_\varphi \mathfrak{n}$ which holds between q and $\mathfrak{n}$ if and only if q is the Gödel-number of a formal proof of $\varphi(\mathfrak{n})$ is primitive recursive.*

That these both conditions are fulfilled e. g. for the system P has been proved by Gödel [43]).

6.2. Functions of class $P_0^{(k,1)}$ as general recursive functions.

6.21. *If S fulfills the condition R_1, then any general recursive function $f(\mathfrak{n})$ is of class $P_0^{(k,1)}$.*

Proof. If $f(\mathfrak{n})$ is general recursive, then there are: a primitive recursive function $h \in R_1^{R_1}$ and a primitive recursive relation $R(\mathfrak{n}, p)$ such that $\prod_{\mathfrak{n}} \sum_p R(\mathfrak{n}, p)$ and

$$f(\mathfrak{n}) = h(\min_p R(\mathfrak{n}, p)).$$

According to (R_1) the set $A = \underset{(\mathfrak{n}, p)}{E} [R(\mathfrak{n}, p)]$ belongs to $P_0^{(k+1)}$ and the function h to $P_0^{(1,1)}$. The function μ_A is of class $P_0^{(k,1)}$ by 5.42 and hence the compounded function $h(\mu_A(\mathfrak{n}))$ is of class $P_0^{(k,1)}$. This compounded function is equal to $f(\mathfrak{n})$ since $\mu_A(\mathfrak{n}) = \min_p R(\mathfrak{n}, p)$ and hence $f \in P_0^{(k,1)}$.

6.22. *If S fulfills the conditions (R_1) and (R_2) and if $f \in R_k^{R_1}$ is a function for which there is a propositional function $\varphi(\mathfrak{x}, x)$ with $k+1$ free variables such that for any $n \in R_1$ and $\mathfrak{n} \in R_k$*

(21) $$[n = f(\mathfrak{n})] \equiv \vdash \varphi(\mathfrak{n}, n),$$

then $f(\mathfrak{n})$ is a general recursive function and hence $f \in P_0^{(k,1)}$ [44]).

[42]) In this section we suppose the reader to be acquainted with the theory of general-recursive functions. See footnote [2]).

[43]) Gödel [3], p. 186.

[44]) This theorem has been found by Gödel [5], p. 24. See also Rosser [15], final remark, Kleene [9], theorem VIII, p. 738.

Proof. For any $\mathfrak{n}$ there is an integer q such that $s_1(q)B_\varphi(\mathfrak{n},s_2(q))$, hence by (R_2) the function

$$g(\mathfrak{n})=s_2\{\min_q[s_1(q)B_\varphi(\mathfrak{n},s_2(q))]\}$$

is general recursive [45]. Thus it is sufficient to prove that $f(\mathfrak{n})=g(\mathfrak{n})$. To show this put $q_0=\min_q[s_1(q)B_\varphi(\mathfrak{n},s_2(q))]$. Then $s_1(q_0)$ is the Gödel-number of a formal proof of $\varphi(\mathfrak{n},s_2(q_0))$ which implies the existence of at least one formal proof of $\varphi(\mathfrak{n},s_2(q_0))$, i. e. $\vdash\varphi(\mathfrak{n},s_2(q_0))$ and therefore $f(\mathfrak{n})=s_2(q_0)$ by (21). On the other hand $g(\mathfrak{n})=s_2(q_0)$ in virtue of the definition of $g(\mathfrak{n})$ and hence $f(\mathfrak{n})=g(\mathfrak{n})$, q. e. d.

In order to explain the significance of 6.22 it is well to point out that in virtue of this theorem the existence of any propositional function $\varphi(\mathfrak{x},x)$ with the property (21) implies the existence of (possibly another) *decidable* propositional function $\psi(\mathfrak{x},x)$ with the same property. A simple example will elucidate this state of affairs. Let ϑ be any undecidable sentence, $\varphi(x,y)$ and $\psi(x,y)$ the propositional functions

$$(y=2x)+(y=2x+1)\cdot\vartheta \quad \text{and} \quad y=2x.$$

Then $(m=2n)\equiv\vdash\varphi(m,n)\equiv\vdash\psi(m,n)$, $\varphi(x,y)$ is undecidable and $\psi(x,y)$ decidable.

It is remarkable that no theorem analogous to 6.22 holds for sets. We have seen in the proof of 4.21 (formula (14)) that the equivalence

$$n\,\epsilon\,A\equiv\vdash\psi(n)$$

may hold for any n though A does not belong to $P_0^{(1)}$. It is to remark that A must then belong to $P_1^{(1)}$ since

$$n\,\epsilon\,A\equiv\sum_q(qB_\psi n).$$

From 6.21 and 6.22 we obtain the following corollary:

6.23. *If S fulfills the conditions (R_1) and (R_2), then $P_0^{(k,1)}$ is the class of general recursive functions with k arguments.*

6.3. Independence of classes $P_n^{(k)}$ and $Q_n^{(k)}$ from S. Subsets of R_k whose characteristic functions are general recursive may be called general recursive k-adic relations. From 6.23 and 5.53 we obtain therefore:

6.31. *If S fulfills the conditions (R_1) and (R_2), then $P_0^{(k)}$ is the class of general recursive k-adic relations.*

[45]) See e. g. Hilbert-Bernays [8], p. 402.

This theorem is important because it shows that the class $P_0^{(k)}$ though defined in 1.3 with the help of notions dependent from the logical system S taken as basis is in reality independent from S, at least if we limit ourselves to consideration of systems which fulfill the recursivity conditions (R_1) and (R_2). In fact, it is known that the class of general recursive relations can be defined without any reference to formalized logical systems [46]. The independence of $P_0^{(k)}$ from S implies of course the independence of other classes $P_n^{(k)}$ and $Q_n^{(k)}$ from S [47].

We note at last the following corollary from 6.31 and 5.61:

6.32. *If S fulfills the conditions (C_0), (R_1) and (R_2), then $P_1^{(1)}$ is the class of recursively enumerable sets.*

Bibliography.

[1] Rudolf Carnap, *Logische Syntax der Sprache*. Wien (1934).

[2] Alonzo Church, *An unsolvable problem of elementary number theory* American Journal of Mathematics, vol. **58** (1936), pp. 345-363.

[3] Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*. Monatshefte für Mathematik und Physik, vol. **38** (1931), pp. 173-198.

[4] Kurt Gödel, *On undecidable propositions of formal mathematical systems* (mimeographed). The Institute for Advanced Study, Princeton (1934).

[5] Kurt Gödel, *Über die Länge von Beweisen*. Ergebnisse eines mathematischen Kolloquiums, Wien (1936), Heft 7, pp. 23-24.

[6] Kurt Gödel, *Eine Eigenschaft der Realisierungen des Aussagenkalküls*. Ergebnisse eines mathematischen Kolloquiums, Wien (1935), Heft 3, pp. 20-21.

[7] David Hilbert, *Grundlegung der elementaren Zahlentheorie*. Mathematische Annalen, vol. **104** (1931), pp. 485-494.

[8] David Hilbert und Paul Bernays, *Grundlagen der Mathematik*. Bd. 2, Berlin (1939).

[9] S. C. Kleene, *General recursive functions of natural numbers*. Mathematische Annalen, vol. **112** (1936), pp. 727-742.

[10] Kazimierz Kuratowski, *Topologie I*. Warszawa (1933).

[11] Kazimierz Kuratowski et Alfred Tarski, *Les opérations logiques et les ensembles projectifs*. Fundamenta Mathematicae, vol. **17** (1931), pp. 240-248.

[12] Emil L. Post, *Recursively enumerable sets of positive integers and their decision problems*. Bulletin of the American Mathematical Society, vol. **50** (1944), pp. 284-316.

[46]) Hilbert-Bernays [8], pp. 403-416.

[47]) It would be interesting to examine the question of independence of $P_0^{(k)}$ from S under the assumption that S fulfills the condition C_s but not C_{s-1}.

[13] J. B. Rosser, *Gödel's theorems for non constructive logics.* The Journal of Symbolic Logic, vol. **2** (1937), pp. 129-137.

[14] J. B. Rosser, *Extensions of some theorems of Gödel and Church.* Ib., vol. **1** (1936), pp. 87-91.

[15] J. B. Rosser, Review of [5]. Ib., vol. 1 (1936), p. 116.

[16] Alfred Tarski, *Fundamentale Begriffe de Methodologie der deduktiven Wissenschaften I.* Monatshefte für Mathematik und Physik, vol. **37** (1930), pp. 361-404.

[17] Alfred Tarski, *Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und ω-Vollständigkeit.* Ib., vol. **40** (1933), pp. 97-112.

[18] Alfred Tarski, *Der Wahrheitsbegriff in den formalisierten Sprachen.* Studia Philosophica, vol. **1** (1935), pp. 261-405.

[19] Alfred Tarski, *Sur les ensembles définissables de nombres réels.* Fundamenta Mathematicae, vol. **17** (1931), pp. 210-239.

[20] Alfred Tarski, *On undecidable statements in enlarged systems of logic and the concept of truth.* The Journal of Symbolic Logic, vol. **4** (1939), pp. 105-112.

[21] A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem.* Proceedings of the London Mathematical Society (2), vol. **42** (1937), pp. 230-265.

Note. This paper was already under press, when an interesting paper of S. C. Kleene, *Recursive predicates and quantifiers* (Transactions of the American Mathematical Society, vol. **53** (1943), pp. 41-83) became available in Poland.

A considerable part of the theory developed above is to be found in the Kleene's paper. It seems me, however, that some of my results are new (e.g. remarks 4.3) and that my presentation of the theory based on analogies with the theory of projective sets may be of some interest for a mathematical reader.

Professor A. Tarski informed me that he also found already in 1942 results very similar to mines.

Démonstration de l'égalité $2^{\mathfrak{m}}-\mathfrak{m}=2^{\mathfrak{m}}$ pour les nombres cardinaux transfinis.

Par

Wacław Sierpiński (Warszawa).

$\mathfrak{m}$ et $\mathfrak{n}$ étant deux nombres cardinaux, on dit que $\mathfrak{m}-\mathfrak{n}=\mathfrak{p}$ si $\mathfrak{p}$ est le seul nombre cardinal tel que $\mathfrak{m}=\mathfrak{n}+\mathfrak{p}$.

En 1926 M. A. Tarski a énoncé[1]) ce

Théorème: *On peut démontrer sans utiliser l'axiome du choix que, lorsque $\mathfrak{m}$ est un nombre cardinal transfini* (c. à. d. $\geqslant\aleph_0$), *on a*

$$2^{\mathfrak{m}}-\mathfrak{m}=2^{\mathfrak{m}}\,{}^{2}). \tag{1}$$

M. Tarski n'a pas publié la démonstration de ce théorème. Il a seulement indiqué (l. c.) qu'elle s'appuie sur les lemmes 4, 6 et 58 énoncés également sans démonstration (l. c., p. 301 et p. 308).

La démonstration du théorème et des lemmes de M. Tarski m'est inconnue. Dans cette Note je vais démontrer le lemme 5 de M. Tarski et j'en déduirai son théorème (sans utiliser les lemmes 4 et 58).

Lemme 1 (de M. Tarski). *On peut démontrer sans utiliser l'axiome du choix que si A et B sont deux ensembles tels que $A\sim B$, il existe des ensembles C_1, C_2, D_1 et D_2 remplissant les conditions:*

$$A-B=C_1+C_2,\qquad B-A=D_1+D_2,\qquad C_1C_2=0=D_1D_2,$$
$$C_1\sim D_1,\qquad C_2+AB\sim AB\sim D_2+AB.$$

[1]) Comptes rendus des séances de la Soc. des Sciences et des Lettres de Varsovie **19** (1926), Classe III, p. 307, Th. 56. Aussi: Ann. Soc. Polonaise de Math. **5** (1926), p. 101.

[2]) Il résulte du théorème de Zermelo sur le bon ordre qu'on a $\mathfrak{n}-\mathfrak{m}=\mathfrak{n}$ pour tout nombre cardinal transfini $\mathfrak{n}$ et tout nombre cardinal $\mathfrak{m}<\mathfrak{n}$, donc, en particulier, que $2^{\mathfrak{m}}-\mathfrak{m}=2^{\mathfrak{m}}$ pour tout nombre cardinal transfini $\mathfrak{m}$; voir p. e. mes *Leçons sur les nombres transfinis*, Paris 1928, p. 233.