M. Mostowski a remarqué que notre théorème résulte sans peine du théorème suivant que MM. A. Lindenbaum et A. Tarski ont publié sans démonstration en 1926 [4]):

Si A, B, C, A_1 et C_1 sont des ensembles tels que $A \supset B \supset C$, $A_1 \supset C_1$, $A \sim A_1$ et $C \sim C_1$, il existe un ensemble B_1 tel que $A_1 \supset B_1 \supset C_1$ et $B \sim B_1$.

En effet, soit $M \supset P$, $\overline{\overline{M}} = \mathfrak{m}$, $\overline{\overline{P}} = \mathfrak{p}$ et $\mathfrak{m} \geqslant \mathfrak{n} \geqslant \mathfrak{p}$. D'après $\mathfrak{m} \geqslant \mathfrak{n}$ il existe un sous-ensemble Q de M de puissance $\mathfrak{n}$ et, d'après $\mathfrak{n} \geqslant \mathfrak{p}$, il existe un sous-ensemble R de Q de puissance $\mathfrak{p}$. On a donc $M \supset Q \supset R$ et $R \sim P$ et, comme $M \supset P$, il existe, d'après le théorème de MM. Lindenbaum et Tarski, un ensemble N tel que $M \supset N \supset P$ et $N \sim Q$, donc $\overline{\overline{N}} = \mathfrak{n}$, c. q. f. d.

Il est à remarquer que notre théorème cesse d'être vrai lorsqu'on y remplace les nombres cardinaux par les nombres ordinaux. En effet, soit M un ensemble ordonné du type $\omega + 1$ et soit P son sous-ensemble formé du dernier élément de M. On a donc $\overline{M} = \omega + 1$, $\overline{P} = 1$ et $\omega + 1 > \omega > 1$; or, il n'existe évidemment aucun ensemble N tel que $M \supset N \supset P$ et $\overline{N} = \omega$.

Notre théorème est également en défaut lorsqu'on y remplace les nombres cardinaux par les nombres de dimension de M. Fréchet. En effet, soient A, B, C et D quatre segments (fermés) de droites dans le plan ayant une extrémité commune, p, et soit $M = A + B + C + D$, $P = M - \{p\}$, $Q = A + B + C$. dX désignant le nombre de dimension de l'ensemble X, on a, comme on voit sans peine, $dM > dQ > dP$ (puisque $dP = 1$, 1 désignant le nombre de dimension de la droite), mais il n'existe aucun ensemble N tel que $M \supset N \supset P$ et $dN = dQ$.

Voici encore un autre exemple de ce genre formé d'ensembles linéaires dénombrables. Soit P l'ensemble formé des nombres $1 - \frac{1}{n}$ et $2 - \frac{1}{n}$, où $n = 2, 3, \ldots$, et soit M l'ensemble qu'on obtient en adjoignant à l'ensemble P le nombre 1. Soit Q l'ensemble formé des nombres $1 - \frac{1}{n}$, où $n = 2, 3, \ldots$ et du nombre 1. On voit sans peine que $dM > dQ > dP$ (puisque P est homéomorphe à $Q - \{1\}$) et qu'il n'existe aucun ensemble N tel que $M \supset N \supset P$ et $dN = dQ$.

[4]) C. R. Soc. Sciences et Lettres Varsovie Cl. III, XIX (1926), p. 303, th. 15. Quant à l'idée de la démonstration, voir mon livre *Zarys teorii mnogości*, t. I, 3-me éd. Warszawa 1928, p. 90, renvoi [2]).

On choices from finite sets.

By

Wanda Szmielew (Łódź).

This paper is closely connected with a paper published by Mostowski in the previous volume of this journal [1]). We shall call a class $\boldsymbol{S}$ of sets a n-class, if every set of this class has exactly n elements. A function $f(X)$ defined for $X \in \boldsymbol{S}$ and such that $f(X) \in X$ for $X \in \boldsymbol{S}$ will be called a choice-function for $\boldsymbol{S}$. Any set A such that $A \cdot X$ has exactly one element for every $X \in \boldsymbol{S}$ will be called a choice-set for $\boldsymbol{S}$.

We consider the following particular cases of the multiplicative axiom:

For every n-class of mutually disjoint sets there is a choice-set.

This proposition will be abbreviated as $[n]$ and it will be supposed that n is a natural number (i. e. a finite cardinal number different from 0). M being a finite non-empty set of natural numbers

$$M = (m_1, \ldots, m_r),$$

we shall abbreviate the conjuction $[m_1] \& [m_2] \& \ldots \& [m_r]$ as $[M]$.

Mostowski established in 1939 a necessary condition for the derivability of the implication

$$[M] \rightarrow [n] \qquad (1)$$

on the base of the Zermelo's axioms of set-theory and he asked whether this condition is at the same time sufficient. This problem is not solved as yet. The purpose of this paper is to establish another condition (S) which is sufficient for the derivability of the foresaid

[1]) A. Mostowski, Fund. Math. **33** (1945), p. 137-168. Quoted below as M.

implication but which is stronger than the Mostowski's necessary condition [2]).

Theorem 1. *Let M be a finite set of natural numbers and n a natural number and let us suppose that M and n satisfy the following condition*

(S) *In every decomposition of n in a sum of primes*

$$n=p_1+\dots+p_s$$

there is at least one p_i which belongs to M.

Then the implication (1) *is derivable.*

Proof[3]). First of all, let us remember, that the axiom of choice is equivalent to an apparently much stronger proposition namely to the so called principle of choice[4]). This equivalence holds also between the corresponding particular cases of the axiom and principle of choice[5]). The proposition $[n]$ is thus for any n equivalent to the following particular case of the principle of choice:

For any n-class $\boldsymbol{S}$ there is a function of choice.

In what follows we shall use simply the same symbol $[n]$ as an abbreviation of the just formulated proposition and we shall interpret the symbol $[M]$ in a corresponding way.

Let now M be a finite non-empty set of natural numbers and let n be a natural number such that M and n together satisfy the supposition of the theorem 1. It is to show that the implication (1) is derivable. To do it we proceed by induction on n.

For $n=1$ the theorem is evidently true, because not only the implication (1) but already the proposition $[n]$ is in this case derivable.

[2]) This was written in 1939. During the war Mostowski found another condition (D) which is weaker than (S) but is still sufficient for the derivability of (1). See M, theorems I and II. As these investigations of Mostowski are quite complicated and use concepts of group-theory, I think it not unnecessary to give a direct proof of sufficiency of my condition (S).

[3]) The first result of this kind, namely the proof of the implication $[2]\to[4]$ was obtained by Mostowski. Tarski found later a simple and elementary proof for this implication. The idea of Tarski's proof is used in the proof of theorem 1.

[4]) See e. g. A. Schoenflies, *Entwickelung der Mengenlehre und ihrer Anwendungen*. Leipzig and Berlin 1913, p. 174.

[5]) Proof thereof is quite the same as the proof of equivalence of the axiom of choice and principle of choice in general case. See preceding footnote.

We suppose now that $n>1$ and that the theorem 1 holds for every natural number $k<n$. Two cases are to be distinguished according as n is prime or not.

In the first case we immediately infer from the assumption of the theorem that $n\in M$. The derivability of (1) is therefore again evident.

We pass now to the second case: n is not prime. We assume that the proposition $[M]$ is true:

(2) *if $m\in M$, then for every m-class there is a choice-function.*

It will be convenient to introduce a set N defined as follows. A natural number r belongs to N if in every decomposition of r into a sum of primes $r=p_1+\dots+p_s$ there is at least one p_i which belongs to M.

From the inductive assumption and (2) we infer now that in order to prove the theorem 1 it is sufficient to show that

(3) *if $n\in N$, then* $[n]$.

It is further easy to see that the inductive assumption implies that

(4) *if $\boldsymbol{S}$ is a class such that the cardinal number of any $X\in\boldsymbol{S}$ is $<n$ and belongs to N, then there is a choice-function for $\boldsymbol{S}$.*

Indeed let $k_1,\dots,k_l$ be all natural numbers which are $<n$ and belong to N. Let $\boldsymbol{S}_i$ for $1\leqslant i\leqslant l$ be the class of sets $X\in\boldsymbol{S}$ whose cardinal number is $\overline{\overline{X}}=k_i$. The class $\boldsymbol{S}$ is thus decomposed in mutually disjoint subclasses $\boldsymbol{S}_1,\dots,\boldsymbol{S}_l$. According to the inductive assumption there is a choice function g_i for any of these subclasses and we get a choice function for the whole class $\boldsymbol{S}$ putting

$$g(X)=g_i(X)\quad\text{for}\quad X\in\boldsymbol{S}_i,\quad i=1,2,\dots,l.$$

In order to prove (3) let us suppose that

$$n\in N. \tag{5}$$

We have to show that $[n]$ is true, i. e. that there is a choice function for every n-class $\boldsymbol{T}$.

As $n \neq 1$ there is a prime p which is a divisor of n. Let $\boldsymbol{U}$ be a class of all sets X of the power $\overline{\overline{X}}=p$ such that every of them is contained in a set $Y \epsilon \boldsymbol{T}$. From (5) we immediately see that $p \epsilon M$ because n may be decomposed into a sum of $\frac{n}{p}$ primes $=p$. Hence $[p]$ occurs in the conjunction $[M]$. The truth of $[M]$ being postulated at the beginning, we infer that $[p]$ is true too. Now $\boldsymbol{U}$ is a p-class and consequently there is a choice function, say g_0, for $\boldsymbol{U}$.

Consider now any set $Y \epsilon \boldsymbol{T}$. As $\overline{\overline{Y}}=n$ there are of course $\binom{n}{p}$ subsets $X \subset Y$ which have the cardinal number p and belong therefore to $\boldsymbol{U}$. For every such X the value $g_0(X)$ is defined as an element of Y.

If y is any element of Y we denote by $q(y)$ the number of such $X \subset Y$ that $\overline{\overline{X}}=p$ and $g_0(X)=y$. We decompose now the set Y into a sum of two disjoint sets Y' and Y'' as follows: an element $y \epsilon Y$ will be assigned to Y' or Y'' according as $q(y) \geqslant q(x)$ for every $x \epsilon Y$ or not. It is plain that Y' is non-empty. Y'' would be empty only then, if $q(y)$ were constant; this would imply that $\binom{n}{p}$ is divisible by n. It is an easy matter to show that this is impossible because p is a prime which divides n [6]) thence Y'' is non-empty.

Put $\overline{\overline{Y'}}=n'$ and $\overline{\overline{Y''}}=n''$. We have $n=n'+n''$ and $n' \neq 0$ and $n'' \neq 0$. It follows that one at least of the numbers n' and n'' belongs to N. Otherwise there would exist by (2) and (5) decompositions

$$n'=p'_1+\ldots+p'_{s'} \quad \text{and} \quad n''=p''_1+\ldots+p''_{s''}$$

[6]) Indeed, we have the following probably known lemma belonging to the theory of numbers: *If n is a natural number, and p a prime $\leqslant n$, then n divides $\binom{n}{p}$ if and only if p does not divide n.* To prove the necessity, remark that $\binom{n}{p}:n=[(n-1)\ldots(n-p+1)]:p!$. If $\binom{n}{p}$ is divisible by n, i. e. if $\binom{n}{p}:n$ is a natural number, then p must be a divisor of the product $(n-1)\ldots(n-p+1)$, i. e., of one of its factors; evidently n cannot be then divisible by p. On the other side we have $p\binom{n}{p}=n\binom{n-1}{p-1}$. Hence, if n is not divisible by p, so must be $\binom{n-1}{p-1}$. Putting $\binom{n-1}{p-1}=lp$ we obtain $\binom{n}{p}=ln$, i. e., $\binom{n}{p}$ is divisible by n, q. e. d. In the proof of theorem 1 we use this lemma in one direction only.

of n' and n'' into sums of primes which do not belong to M and we would obtain a decomposition of n

$$n=p'_1+\ldots+p'_{s'}+p''_1+\ldots+p''_{s''}$$

with the same property which contradicts (5).

We put now $h(Y)=Y'$ if $n' \epsilon N$ and $h(Y)=Y''$ if $n'' \epsilon N$. To every $Y \epsilon \boldsymbol{T}$ corresponds thus a non-empty subset $h(Y) \subset Y$ and the cardinal number of this subset is $<n$ and belongs to N. Hence denoting by $\boldsymbol{S}$ the class of all these subsets $h(Y)$ where $Y \epsilon \boldsymbol{T}$ we infer from (4) that there is a choice function g for the class $\boldsymbol{S}$. Putting $f(Y)=g(h(Y))$ for every set $Y \epsilon \boldsymbol{T}$ one obtains finally a choice function for the class $\boldsymbol{T}$. The proposition $[n]$ is consequently true and the number n satisfies the condition (3).

The inductive proof of theorem 1 is thus finished.

The theorem 1 can be strengthened a little:

Theorem 2. *Let M be a finite set of natural numbers and n a natural number and let us suppose that M and n satisfy the following condition*

(S') For every decomposition of n into a sum of primes

$$n=p_1+p_2+\ldots+p_s$$

there is a number i, $1 \leqslant i \leqslant s$ and a natural number k such that $k \cdot p_i$ belongs to M.

The implication $[M] \rightarrow [n]$ is then derivable.

Proof. In order to derive this theorem from theorem 1 it is of course sufficient to prove the following lemma:

The implication $[k \cdot n] \rightarrow [n]$ holds for every natural k and n.

Indeed, suppose that $[k \cdot n]$ is true and consider any n-class $\boldsymbol{S}$. For every $X \epsilon \boldsymbol{S}$ denote by X^* the set of ordered pairs $\langle x,i \rangle$, where $x \epsilon X$ and $1 \leqslant i \leqslant k$. Let $\boldsymbol{S}^*$ be the class of all these sets X^*. As this is evidently an $k \cdot n$-class, there is by $[k \cdot n]$ a choice function f^* for $\boldsymbol{S}^*$. If now X is any set from $\boldsymbol{S}$ and if $f^*(X^*)=\langle x,i \rangle$, we put $f(X)=x$ and obtain so a choice function for the class $\boldsymbol{S}$. The proposition $[n]$ is therefore true.

From theorems 1 and 2 we may infer for instance that implications

$$[2] \rightarrow [4], \quad [(2,3)] \rightarrow [6], \quad [(2,3)] \rightarrow [8],$$
$$[(2,5)] \rightarrow [8], \quad [6] \rightarrow [8], \quad [10] \rightarrow [8]$$

are all true.

For any natural number m we use according to Mostowski the symbol $[m]!$ as an abbreviation of the conjuction $[1]\&[2]\&\ldots\&[m]$.

If n is a natural number >1, we denote by $\mu(n)$ the number p determined by two following conditions:

(i) there is a decomposition of n into a sum of primes in which p is the smallest term;

(ii) there is no decomposition of n into a sum of primes greater than p.

From theorem 1 we obtain now immediately the following

Corollary 3. *If m and n are natural numbers such that $m\geqslant\mu(n)$, then the implication $[m]!\to[n]$ is true.*

Mostowski has shown that also the converse theorem is true [7]). The inequality $m\geqslant\mu(n)$ is not only sufficient but also necessary for the derivability of the implication $[m]!\to[n]$. Mostowski derived further from the corollary 3 a sufficient condition for the truth of the implication $[m]!\to[n]!$ and proved that this condition is at the same time necessary [8]).

We conclude with the remark that conditions given in theorem 1 and 2 are by no means necessary for the derivability of the implication $[M]\to[n]$. For instance the implication $[(3,7)]\to[9]$ is true [9]) but neither the assumptions of theorem 1 nor that of theorem 2 are satisfied.

[7]) See M, p. 162.

[8]) M, theorem VIII.

[9]) M, theorem IX.

On definable sets of positive integers *).

By

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The celebrated paper of K. Gödel on undecidable statements [1]) had (among others) the effect that several writers began to analyze the notion of functions of natural argument taking on integer values as well as related with them sets of positive integers. The chief purpose of these endeavours was to formulate an exact definition of what may be called „calculable function", i. e. such function $f(n)$ that there exists a method permitting to calculate the value of $f(n)$ for any given n in a finite number of steps. For sets we have the corresponding notion of „calculable sets" for which there is a finite method permitting to decide whether any given integer is in set or not. The solution of this problem given by Herbrand, Gödel, Church, Kleene and Turing [2]) suggested still other types of sets and of functions. So e. g. Rosser and Kleene found an interesting class of sets which they called „recursively enumerable" [3]).

The aim of this paper is to show that the two above mentioned classes of sets (and of functions) form the beginning of an infinite sequence of classes whose properties closely resemble those of projective sets [4]). For convenience of readers not acquainted with papers referred to in footnotes [2]) and [3]) I shall develop the theory without using the notion of general recursivity (the final section **6** is the only exception).

*) See note on the page 112.

[1]) Gödel [3]. Numbers in brackets refer to bibliography given at the end of this paper.

[2]) Gödel [4], [5], Church [2], Kleene [9], Turing [21]. It is now customary to call calculable functions and sets „general recursive". An excellent exposition of the theory of these functions is to be found in Hilbert-Bernays [8], Supplement II, 392-421.

[3]) Kleene [9], Rosser [14]. Further development will be found in Post [12].

[4]) I shall refer to the exposition of the theory of these sets given by Kuratowski [10].