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Institute for Advanced Study
Princeton, N. J.

On Characteristic Functions of Families of Sets.

By

M. H. Stone (Cambridge, Mass., U. S. A.).

In an interesting paper [1]) Szpilrajn has employed the characteristic function to develop a certain method of dealing with the algebraic structure of sequences of sets; and has established with the aid of this method a variety of specific theorems and equivalences in the domain of set-theoretical topology. He attributes to Kuratowski the first use of the characteristic function of a sequence of sets.

In the present note, I shall trace certain connections between the content of Szpilrajn's paper and the general theory of abstract Boolean algebras which I have developed in two memoirs published elsewhere [2]). In doing so, I deem my chief purpose to be that of reconciling two independent points of view which prove, upon examination, to present a considerable analogy so far as the theory of the algebraic structure of sequences of sets is concerned.

As I shall point out below, an obvious but theoretically desirable generalization of Szpilrajn's work leads to the introduction of the characteristic function of an arbitrary transfinite sequence, or well-ordered family, of sets. It seems to me of more importance, perhaps, to observe that the rôle of order, which is essential to the definition of the characteristic function, appears to be artificial

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[1]) E. Szpilrajn, *The characteristic function of a sequence of sets and some of its applications*, Fundam. Math. **31** (1938), p. 207–233; see also Fundam. Math. **26** (1935), p. 302.

[2]) M. H. Stone, *The Theory of Representations for Boolean Algebras*, Trans. Amer. Math. Soc. **40** (1936), pp. 37–111 (cited here by the letter R); and *Applications of the Theory of Boolean Rings to General Topology*, ibidem **41** (1937), pp. 375–481 (cited here by the letter A).

so far as the majority of applications is concerned. In principle, therefore, one is tempted to seek an order-free theory of the algebraic relations envisaged. I shall show here that such a theory is already in existence and that, through the adjunction of elementary considerations of order, it leads back to the theory of the characteristic function due to Szpilrajn.

§ 1. The Space $\mathfrak{B}_\mathfrak{c}$. If $\mathfrak{c}$ is any cardinal number, we shall denote by $\mathfrak{B}_\mathfrak{c}$ the cartesian product of $\mathfrak{c}$ two-point Hausdorff spaces. It is a totally-disconnected bicompact Hausdorff space; in the particular case where $\mathfrak{c}=\aleph_0$ it is homeomorphic with the Cantor discontinuum. In the sequel we shall suppose that $\mathfrak{c}$ is an infinite cardinal.

A more detailed description of $\mathfrak{B}_\mathfrak{c}$, adapted to later discussions, may be given in the following terms. Starting from the class A of ordinals α preceding some fixed ordinal β, subject to the restriction that A shall have cardinal number $\mathfrak{c}$, we take as the points of $\mathfrak{B}_\mathfrak{c}$ the real functions $\mathfrak{s}$ defined over A and assuming only the values $0,1$. We then take as a basis (of open sets) for $\mathfrak{B}_\mathfrak{c}$ the family composed of all sets $\mathfrak{E}_\alpha$ and $\mathfrak{E}'_\alpha$, $\alpha<\beta$, where $\mathfrak{E}_\alpha$ consists of all $\mathfrak{s}$ with $\mathfrak{s}(\alpha)=1$ and $\mathfrak{E}'_\alpha$ is the complement of $\mathfrak{E}_\alpha$ in $\mathfrak{B}_\mathfrak{c}$ (i.e. $\mathfrak{E}'_\alpha$ consists of all $\mathfrak{s}$ with $\mathfrak{s}(\alpha)=0$). The algebraic and topological properties of the space $\mathfrak{B}_\mathfrak{c}$ so obtained are fully discussed in R Chapter I, especially Theorems 9-13; they include those noted above. For our later convenience we have replaced the sets $\mathfrak{U}_\alpha$, $\mathfrak{U}'_\alpha$ of R by $\mathfrak{E}'_\alpha$, $\mathfrak{E}_\alpha$ respectively in the present description.

The sets $\mathfrak{E}_\alpha$ bear a special relationship to the topology of $\mathfrak{B}_\mathfrak{c}$ which proves to be of importance in the sequel. This relationship is most concisely expressed in terms of the Boolean algebra $A_\mathfrak{c}$ generated from the sets $\mathfrak{E}_\alpha$ by the formation of all possible complements, finite unions, and finite intersections. According to R Definitions 1, 2 and Theorems 1, 9 the sets belonging to $A_\mathfrak{c}$ are characterized as the open-and-closed subsets of $\mathfrak{B}_\mathfrak{c}$. Moreover, R Theorem 12 discloses that the algebra $A_\mathfrak{c}$ is a free Boolean algebra generated by the $\mathfrak{c}$ elements $\mathfrak{E}_\alpha$ and is thus completely characterized in algebraic terms. The proof of the theorem cited consists partly in establishing the independence (to use Szpilrajn's terminology) of the sequence $\{\mathfrak{U}_\alpha\}=\{\mathfrak{E}'_\alpha\}$, which is obviously equivalent to that of $\{\mathfrak{E}_\alpha\}$. The constituents of $\{\mathfrak{E}_\alpha\}$ are thus non-void closed subsets of $\mathfrak{B}_\mathfrak{c}$. By virtue of the bicompactness of $\mathfrak{B}_\mathfrak{c}$, all the atoms of $\{\mathfrak{E}_\alpha\}$ are therefore non-void; and, since the sets $\mathfrak{E}_\alpha$, $\mathfrak{E}'_\alpha$, $\alpha<\beta$, constitute an open basis for $\mathfrak{B}_\mathfrak{c}$, no atom can contain more than one point. The sequence $\{\mathfrak{E}_\alpha\}$ is therefore completely independent. As a matter of fact it is quite easy to see, directly from the definition of the sets $\mathfrak{E}_\alpha$ and without recourse to the topology of $\mathfrak{B}_\mathfrak{c}$, not only that the constituents of $\{\mathfrak{E}_\alpha\}$ are non-void, the sequence consequently being independent, but also that the atoms of $\{\mathfrak{E}_\alpha\}$ are precisely the one-point subsets of $\mathfrak{B}_\mathfrak{c}$, the sequence thus being completely independent. And the bicompactness of $\mathfrak{B}_\mathfrak{c}$ can then be deduced essentially from this direct observation in the manner indicated in the proof of R Theorem 9.

§ 2. Definition of the Characteristic Function. Let $e=\{E_\alpha\}$, where α ranges over the class A of § 1, be a transfinite sequence of subsets of a fixed non-void set X. The characteristic function c_e of this sequence is defined as that single-valued function from X into $\mathfrak{B}_\mathfrak{c}$ determined by assigning to each x in X the image $\mathfrak{s}=c_e(x)$ where $\mathfrak{s}(\alpha)$ is 0 or 1 according as $x\,\epsilon\,E'_\alpha$ or $x\,\epsilon\,E_\alpha$, $\alpha<\beta$. It is evident that an arbitrary single-valued function c from X into $\mathfrak{B}_\mathfrak{c}$ is the characteristic function of a unique sequence $e=\{E_\alpha\}$ determined by taking E_α as the set of all x such that $\mathfrak{s}=c(x)$ satisfies the condition $\mathfrak{s}(\alpha)=1$. In short, the characteristic function c_e is determined by the relations

$$c_e(E_\alpha)=\mathfrak{E}_\alpha c_e(X),\qquad E_\alpha=c_e^{-1}(\mathfrak{E}_\alpha),\qquad X=c_e^{-1}(\mathfrak{B}_\mathfrak{c}).$$

These relations evidently serve to establish the two following properties, which summarize the information essential to the majority of applications of the characteristic function:

(1) the correspondence $\mathfrak{E}_\alpha c_e(X)\leftrightarrow E_\alpha$ induces an isomorphism between the Boolean algebra $A_\mathfrak{c}(X)$ generated in $c_e(X)$ by the sets $\mathfrak{E}_\alpha c_e(X)$ and the Boolean algebra E_0 generated in X by the sets E_α;

(2) the sets in $A_\mathfrak{c}(X)$ are both open and closed in the relative topology of $c_e(X)$ considered as a subspace of $\mathfrak{B}_\mathfrak{c}$.

From A Theorem 56 we know that the correspondence $\mathfrak{E}_\alpha\to\mathfrak{E}_\alpha c_e(X)$ induces a homomorphism $A_\mathfrak{c}\to A_\mathfrak{c}(X)$. Hence the correspondence $\mathfrak{E}_\alpha\leftrightarrow E_\alpha$ induces a homomorphism $A_\mathfrak{c}\to E_0$.

It is convenient to determine when two elements x_1 and x_2 in X satisfy the condition $c_e(x_1)=c_e(x_2)$. Without difficulty we find that the following statements are equivalent: $c_e(x_1)=c_e(x_2)$; x_1 and x_2 are not separated by $\{E_\alpha\}$; x_1 and x_2 are not separated by E_0; x_1 and x_2 belong to the same atom of $\{E_\alpha\}$. Consequently we may construct c_e in two steps, first identifying points of X belonging to the same atom of $\{E_\alpha\}$ and then forming the characteristic function of the reduced sequence $\{E^*_\alpha\}$ in the resulting set X^*. It is evident that the first step here is precisely that of reducing the algebra $\boldsymbol{E}_0$ in accordance with A Theorem 54.

§ 3. Alternative Definition of the Characteristic Function. We shall now rephrase the definition of the characteristic function in algebraic terms.

We begin by observing that the correspondence $\mathfrak{E}_\alpha \leftrightarrow E_\alpha$, $\alpha<\beta$, induces a homomorphism $A_\mathfrak{c} \to \boldsymbol{E}_0$. Since $A_\mathfrak{c}$ is a free Boolean algebra each of its elements is expressible as a finite union of terms of the general form

$$\mathfrak{E}_{\alpha_1} \cdots \mathfrak{E}_{\alpha_n} \mathfrak{E}'_{\alpha_{n+1}} \cdots \mathfrak{E}'_{\alpha_{n+p}}, \qquad n \geqslant 0,\ p \geqslant 0,\ n+p \geqslant 1;$$

and, as is shown in the proof of R Theorem 12, two such expressions can be equal *only* in consequence of the fundamental Boolean identities. On replacing each $\mathfrak{E}_\alpha$ in every such expression by its correspondent E_α, we obtain a correspondence from $A_\mathfrak{c}$ to $\boldsymbol{E}_0$ which carries equal elements into equal elements, complements into complements, unions into unions, and intersections into intersections, by virtue of the fundamental Boolean identities. We thus have a homomorphism $A_\mathfrak{c} \to \boldsymbol{E}_0$, which evidently becomes an isomorphism if and only if $\{E_\alpha\}$ is independent.

With each point x in X we now associate by virtue of A Theorem 34, the prime ideal $\mathfrak{p}(x)$ in $\boldsymbol{E}_0$ which consists of those sets in $\boldsymbol{E}_0$ *not* containing x. From $\mathfrak{p}(x)$ we pass by A Theorem 48 to the prime ideal $\mathfrak{p}_\mathfrak{c}(x)$ of all those sets in $A_\mathfrak{c}$ which are carried by the homomorphism $A_\mathfrak{c} \to \boldsymbol{E}_0$ into sets in $\mathfrak{p}(x)$. And from $\mathfrak{p}_\mathfrak{c}(x)$ we pass to a uniquely determined point $\mathfrak{s}_x$ in $\mathfrak{B}_\mathfrak{c}$ with the help of R Theorem 9.

The characteristic function c_e of $\{E_\alpha\}$ is defined by the equation $c_e(x) = \mathfrak{s}_x$ for all x in X.

It is easy to show this definition equivalent to that of § 2. The proof of R Theorem 9 discloses that $\mathfrak{s}_x(\alpha)$ is 0 or 1 according as $\mathfrak{U}'_\alpha = \mathfrak{E}_\alpha \,\epsilon\, \mathfrak{p}_\mathfrak{c}(x)$ or $\mathfrak{U}_\alpha = \mathfrak{E}'_\alpha \,\epsilon\, \mathfrak{p}_\mathfrak{c}(x)$. Clearly the relations $\mathfrak{E}_\alpha \,\epsilon\, \mathfrak{p}_\mathfrak{c}(x)$, $\mathfrak{E}'_\alpha \,\epsilon\, \mathfrak{p}_\mathfrak{c}(x)$ are equivalent respectively to the relations $E_\alpha \,\epsilon\, \mathfrak{p}(x)$, $E'_\alpha \,\epsilon\, \mathfrak{p}(x)$; and hence are equivalent respectively to the relations $x \,\epsilon\, E'_\alpha$, $x \,\epsilon\, E_\alpha$. Thus we see that $c_e(x) = \mathfrak{s}$ is determined by putting $\mathfrak{s}(\alpha)$ equal to 0 or to 1 according as $x \,\epsilon\, E'_\alpha$ or $x \,\epsilon\, E_\alpha$.

§ 4. Unordered Families. The construction of the characteristic function c_e described in the preceding paragraph reveals that the ordering of $\{E_\alpha\}$ has significance only insofar as it determines the homomorphism $A_\mathfrak{c} \to \boldsymbol{E}_0$ through the correspondence $\mathfrak{E}_\alpha \leftrightarrow E_\alpha$. If we replace this homomorphism by any other, the construction can still be carried through and still provides a mapping of X in $\mathfrak{B}_\mathfrak{c}$ with the essential properties (1) and (2) of § 2. By considering the reduced algebra determined by $\boldsymbol{E}_0$, as suggested at the close of § 2, and applying A Theorem 69 we obtain an immediate verification of property (1); and property (2) then follows from the results summarized in § 1.

The most direct treatment of unordered families on the basis of A and R is, however, the following. Let $\{E\}$ be a family of subsets, distinct or not, of a fixed non-void set X; and let $\boldsymbol{E}_0$ be the Boolean algebra generated in X by the given sets E. By A Theorem 67 and R Theorem 1 there is associated with $\boldsymbol{E}_0$ a totally-disconnected bicompact Hausdorff space $\mathfrak{B}(\boldsymbol{E}_0)$. If the algebra $\boldsymbol{E}_0$ be reduced in accordance with A Theorem 54, the resulting isomorphic algebra of sets is, by virtue of A Theorem 69, equivalent to a certain algebra of subsets of a fixed set $\mathfrak{X} \subset \mathfrak{B}(\boldsymbol{E}_0)$; and, in view of information, summarized in A Theorem 69 and R Theorem 1, the latter algebra consists precisely of the sets $\mathfrak{G}\mathfrak{X}$ where $\mathfrak{G}$ is both open and closed in $\mathfrak{B}(\boldsymbol{E}_0)$, the set $\mathfrak{X}$ itself being everywhere dense in $\mathfrak{B}(\boldsymbol{E}_0)$. Thus we obtain a single-valued function c from X into $\mathfrak{X}$ in $\mathfrak{B}(\boldsymbol{E}_0)$; and we see that c carries $\boldsymbol{E}_0$ isomorphically into the algebra of sets $\mathfrak{G}\mathfrak{X}$, the inverse c^{-1} serving to invert the isomorphism determined by c. If we now appeal to R Theorem 10, we can imbed $\mathfrak{B}(\boldsymbol{E}^0)$ topologically, as a closed set, in the space $\mathfrak{B}_\mathfrak{c}$. The function c therefore maps X into $\mathfrak{B}_\mathfrak{c}$, with $c(X) = \mathfrak{X}$ as before; and the algebra into which it carries $\boldsymbol{E}_0$ can now be characterized as consisting of all sets $\mathfrak{G}_\mathfrak{c}\mathfrak{X}$ where $\mathfrak{G}_\mathfrak{c}$ is both open and closed in $\mathfrak{B}_\mathfrak{c}$.

§ 5. An Application. To illustrate the applicability of the results of § 4 we shall consider a generalization of a theorem of Kuratowski proved by Szpilrajn with the aid of the characteristic function. Let X be a non-void topological space, more specifically, a T_0-space (in the terminology of Alexandroff and Hopf) of infinite character $\mathfrak{c}$. Let $\{E\}$ be an arbitrary (open) basis for X. We may in particular choose $\{E\}$ so that its cardinal number is $\mathfrak{c}$. Since any two distincts points in X are separated by the basis $\{E\}$, the Boolean algebra E_0 is a reduced algebra of sets. Thus the map c of X onto $\mathfrak{X}$ is biunivocal. Now, if G is any non-void open subset of X, it is the union of certain sets E; and $c(G)$ is the union of the corresponding sets $c(E)$. Since $c(E)$ is open in the relative topology of $\mathfrak{X}$, considered as a subspace of $\mathfrak{B}_{\mathfrak{c}}$, we conclude that $c(G)$ is also open in this topology. We thus obtain the following result:

Theorem. *If X is any T_0-space of infinite character $\mathfrak{c}$, it is a biunivocal continuous image (by the map c^{-1}) of an appropriate subspace $\mathfrak{X}$ of a totally-disconnected bicompact Hausdorff space of character $\mathfrak{c}$ (the space $\mathfrak{B}_{\mathfrak{c}}$).*

Of course, the map c of X into $\mathfrak{X}$ is continuous only if X is homeomorphic with $\mathfrak{X}$. The conditions under which a given space X is homeomorphic with a subspace of a totally-disconnected bicompact Hausdorff space (Boolean space) are discussed in R Theorem 55. Even though c fail to be continuous, its topological character must still be comparatively simple. In fact we can show that, if $\mathfrak{G}$ is any (relatively) open subset of $\mathfrak{X}$, then $c^{-1}(\mathfrak{G})$ is the union of at most $\mathfrak{c}$ sets of the form GF where G is open and F closed in X. We include here the cases $G=X$, $F=X$. From § 4 we know that the sets $c(H)$, $H \epsilon E_0$, constitute a basis for $\mathfrak{X}$. Hence $\mathfrak{G}$ is the union of (at most $\mathfrak{c}$) sets $c(H)$; and $c^{-1}(\mathfrak{G})$ is the union of the corresponding sets $H=c^{-1}c(H)$. Now each set H, being expressible in terms of the basis $\{E\}$ as a finite union of sets of the form

$$E_1 \dots E_n E'_{n+1} \dots E'_{n+p}, \qquad n \geqslant 0,\ p \geqslant 0,\ n+p \geqslant 1,$$

is a finite union of sets of the form GF described above. Thus $c^{-1}(\mathfrak{G})$ has the property asserted. In case X is a regular space we can sharpen the preceding result, asserting now that $c^{-1}(\mathfrak{G})$ is the union of at most $\mathfrak{c}$ closed sets. To establish this proposition we associate with each point x of a non-void open set $G \subset X$ an open set $G(x)$ such that $x \epsilon G(x) \subset G^{-}(x) \subset G$, this being possible by the regularity of X. Since $\{E\}$ is a basis for X, it must contain a set $E(x)$ such that $x \epsilon E(x) \subset G(x)$. The relations $E^{-}(x) \subset G^{-}(x) \subset G$ now show that G is the union of the closed sets $E^{-}(x)$, of which at most $\mathfrak{c}$ are distinct. It follows immediately that $c^{-1}(\mathfrak{G})$, being the union of at most $\mathfrak{c}$ sets GF, can be expressed as the union of at most $\mathfrak{c}$ closed sets. In particular, if X is regular and separable — in other words, is metric and separable —, the set $c^{-1}(\mathfrak{G})$ is an F_σ-set.

In conclusion we point out that the representation provided by the Theorem proved above is entirely distinct from the representations provided in the theory of "Boolean maps" given in R. Indeed the present representation of X in $\mathfrak{B}_{\mathfrak{c}}$ defines a Boolean map $m(\mathfrak{B}_{\mathfrak{c}}, \mathfrak{X}^*, X)$, where $\mathfrak{X}^*$ is the family of one-point subsets of $\mathfrak{X}=c(X)$, only in the extremely special case where X and $\mathfrak{X}$ are homeomorphic by the map c.