Gillespie measure.

By

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1. Remarks. One aim of the theory of measure as introduced by Lebesgue was to furnish a tool for handling questions of length for curves and area for surfaces. While this theory was successful as far as curve length is concerned, the notion of area for surfaces was left in an unsatisfactory state. The fact that some of the properties of lengths for curves have not been successfully extended to areas for surfaces may suggest that our present notion of length for curves is perhaps of too special a nature. In fact even before Lebesgue had given his definition of the measure of a point set on a straight line, Minkowski [9][1]) in 1909 had already considered the question of generalizing curve length by assigning (in the spirit of Peano-Jordan content) a linear measure to point sets lying in the plane.

This notion of assigning a linear measure to point sets not lying on a line has been considered in different ways since Minkowski with varying degrees of success. Among such definitions are those of Young 1905 [13], Janzen 1907 [7], Carathéodory 1914 [3], and Gross 1918 [5]. The Minkowski measure inherits all the faults of Peano-Jordan content. Young himself found inconsistencies for his own measure. Gross measure has the anomaly as shown by Saks [11] of assigning the measure zero to a particular set and a positive (even infinite) measure to the transform of this set by a projective transformation of the plane. Janzen measure is not independent of the coordinate system, e. g. the set constructed by Gross ([5], p. 185) has Janzen measure unity for the axes used in the construction and measure zero when the axes are rotated 45^0.

[1]) Numbers in brackets refer to the bibliography at the end (see p. 154).

To the present Carathéodory measure has received practically no adverse criticism in the literature [8, 10] and seems to have been accepted as an adequate generalization of the notion of length. However, we give incidentally (§ 6) a set for which even this measure is quite inconsistent with our inherent concept of length.

2. Introduction. In this paper we propose still another definition of linear measure for point sets not necessarily lying on a line. For a point set A we shall represent this measure by $G^*(A)$ and call it *Gillespie linear measure* after the late Professor D. C. Gillespie who suggested to us individually definitions similar to the one we have adopted.

Gillespie outer linear measure is defined (§ 3) in such a way that Carathéodory's postulational theory of measure may be used.

It will be obvious that Carathéodory linear measure is never greater than Gillespie linear measure:

$$L^*(A) \leqslant G^*(A)$$

and it is easily shown that $G^*(A) \leqslant \pi L^*(A)$. We prove, however in § 8 that:

$$L^*(A) \leqslant G^*(A) \leqslant \frac{\pi}{2} L^*(A).$$

This relation is the best possible since there are sets for which each equality holds.

In § 6 we establish a relation which seems indispensable to a generalization of euclidean length; namely, if $|P_1|$ and $|P_2|$ are the outer Lebesgue measures of the projections of a plane set A on two perpendicular lines of the plane, then

$$G^*(A) \geqslant \sqrt{|P_1|^2 + |P_2|^2}.$$

The analogous relation for Carathéodory measure is not satisfied. In fact there is a set, see § 5, with $L^*(A) = |P_1| = |P_2| = 1$.

Also (§ 7) if a simple Jordan arc has length in the ordinary sense, then the set of all points of this arc has its Gillespie linear measure equal to the length of the arc.

Gillespie linear measure $G(A)$ is extended (§ 9) to Gillespie area measure $G^{(2)}(B)$ for sets B not necessarily lying in a plane. We prove that if A is a set in the (x,y)-plane and B is the set of all points (x,y,z) with (x,y) a point of A and $0 \leqslant z \leqslant h$, i. e.

$$B = \underset{(x,y,z)}{E}[(x,y)\,\varepsilon\, A,\ 0 \leqslant z \leqslant h],$$

then $G^{(2)}(B)=hG(A)$. This relation between „area“ and „length“ fundamental and simple as it seems, has not been proved [10] foe any of the other measures mentioned above.

3. Definition of $G^*(A)$. In this section A will designatr a plane point set.

If U is a plane convex set with inner points then by $c(U)$ we mean the length of the simple closed curve which is its boundary; if U is a segment then $c(U)$ is twice the distance between its end points.

With ϱ an arbitrary positive number let $U_1, U_2, \dots$ be a sequence of convex sets lying in the plane each with diameter $<\varrho$ whose union contains A, and consider the series of semi-circumferences

$$\sum \frac{c(U_k)}{2}.$$

Designate the greatest lower bound (which may be infinite) of all such numbers by $G_\varrho(A)$. As ϱ decreases, $G_\varrho(A)$ does not decrease. Thus as $\varrho \to 0$ the function $G_\varrho(A)$ approaches a limit (finite or infinite) which is represented by

$$G^*(A)$$

and called the *Gillespie outer linear measure of* A.

If U is a convex set, its closure $\overline{U}$ is also convex and moreover $c(U)=c(\overline{U})$. Thus in the definition of $G^*(A)$ we could have restricted the sets $U_1, U_2, \dots$ to be closed convex sets. Also if only open convex sets $U_1, U_2, \dots$ are used the same number $G^*(A)$ is obtained. For with $\epsilon>0$ arbitrary we may include U_k in an open convex set V_k with $c(U_k)\leqslant c(V_k)\leqslant c(U_k)+\frac{\epsilon}{2^k}$.

One will see that Gillespie outer linear measure possesses the first four properties demanded of an outer measure function in Carathéodory's postulational development of the theory of measure [3, 4]. Thus accepting Carathéodory's general definition of measurability [1]) it follows that the complements of measurable sets are measurable, the intersection of a sequence of measurable sets is measurable, and that open sets are measurable. In particular G_δ's, F_σ's, etc. are measurable. If a set A is measurable, its measure $G(A)$ is defined by the equation $G(A)=G^*(A)$.

[1]) i. e. a set A is *Gillespie linearly measurable* if for every set W with $G^*(W)$ finite the equality $G^*(W)=G^*(AW)+G^*(W-AW)$ holds.

Gillespie linear measure also satisfies Carathéodory's fifth axiom and, in addition, the following modification given by Hahn ([6], p. 444):

For each set A there is a set B which is a G_δ containing A such that $G(B)=G^*(A)$.

For, with $\varrho_1, \varrho_2, \dots$ a decreasing sequence of numbers approaching zero, let $U_{k_1}, U_{k_2}, \dots$ be open convex sets (just shown to exist) each with diameter $<\varrho_k$ whose union contains A and such that $\sum_{n=1}^{\infty}\frac{1}{2}c(U_{k_n})\leqslant G^*(A)+\varrho_k$. The set $B_k=U_{k_1}+U_{k_2}+\dots$ is then open so $B=B_1\cdot B_2\dots$ is a G_δ. Moreover $B\supset A$ so $G(B)\geqslant G^*(A)$. On the other hand $B\subset U_{k_1}+U_{k_2}+\dots$ and the diameter of U_{k_n} is $<\varrho_k$ so $G_{\varrho_k}(B)\leqslant\sum_n \frac{1}{2}c(U_{k_n})\leqslant G^*(A)+\varrho_k$ and thus $G(B)\leqslant G^*(A)$.

The inner measure $G_*(A)$ of a set A is defined as the upper limit of the measures of all measurable subsets of A. From this definition and the above modification of the fifth axiom, Hahn ([6], p. 445) proves a theorem equivalent to the following statement:

If A has $G_*(A)$ finite, then this inner linear measure is the upper limit of the linear measures $G(K)$ of all closed subsets K of A.

Thus since a set with outer measure finite is measurable if and only if its outer and inner measures are the same ([4], p. 263) one will see

Theorem 1. *If $G^*(A)<\infty$, then a necessary and sufficient condition that A be Gillespie linearly measurable is that $\epsilon>0$ imply the existence of a closed subset K of A with $G^*(A-K)<\epsilon$.*

4. Relation of $L^*(A)$ and $G^*(A)$. The definition in § 3 of $G^*(A)$ is patterned closely after Carathéodory's definition [3] of the outer linear measure $L^*(A)$. In fact in Carathéodory's definition we have merely substituted the semi-circumference $\frac{1}{2}c(U_k)$ where he has the diameter $d(U_k)$ of U_k [1]).

Thus, since $d(U_k)\leqslant\frac{1}{2}c(U_k)$ and since $\frac{1}{2}c(U_k)\leqslant\pi d(U_k)$ (because U_k may be included in a circle of radius $d(U_k)$) it follows that

$$L^*(A)\leqslant G^*(A)\leqslant\pi L^*(A). \tag{1}$$

Consequently a set A has $L^*(A)$ and $G^*(A)$ both finite or both infinite and if either is zero the other is also.

[1]) Carathéodery did not actually require the sets $U_1, U_2, \dots$ to be convex, but showed that $L^*(A)$ is not altered if they are so restricted.

Hahn's modification of Carathéodory's fifth axiom is also satisfied by $L^*(A)$ as one may see by follwing arguments similar to those used for $G^*(A)$. Thus the analogue of theorem 1 holds for $L^*(A)$. Consequently

Theorem 2. *If A has $G^*(A)$ (or $L^*(A)$) finite, then a necessary and sufficient condition for A to be Gillespie linearly measurable is that A be Carathéodory linearly measurable.*

For, from (1) a closed subset K of A exists with $G^*(A-K)$ arbitrarily small if and only if $L^*(A-K)$ is arbitrarily small.

5. A particular set. It might appear, from the similarity of the results in § 4 for Gillespie and Carathéodory linear measures, that the difference between these two measures is so slight as to be of no significance. However, in this section we construct a set A whose projection on the x-axis is the closed segment $\underset{(x,y)}{E}[0\leqslant x\leqslant 1, y=0]$ and whose projection on the y-axis is $\underset{(x,y)}{E}[x=0, 0\leqslant y\leqslant 1]$, but nevertheless (instead of having $L^*(A)$ at least $\sqrt{2}$) has $L^*(A)=1$. In the next section we show that for $G^*(A)$ this irregularity cannot occur.

Toward constructing this set A we first define an operation of order n on a circle C [1]) of radius r and with respect to a fundamental coordinate system.

Circumscribe a square about C with sides parallel to the axes. Draw a chord of C from the upper to the right hand point of tangency and another chord from the left hand to the lower point of tangency. Divide the square into $(2n)^2$ equal squares with sides parallel to the axes. Of the smaller squares thus formed consider *only* those which have a diagonal lying along one of the chords just drawn. From these squares select those which are subsets of C; those which are not entirely subsets of C redivide into n^2 equal squares. Of these still smaller squares which lie along either of the above chords, select those which are subsets of C and those which are not entirely subsets of C redivide into n^2 equal squares etc. The selected squares are then infinite in number, no two overlap, each is a subset of C, and their union contains all except the end points of the two chords of C indicated above. In each of these squares inscribe a circle. The infinite set of circles thus obtained is to be considered the result of the operation of order n on C.

[1]) i. e. all points whose distance from a fixed point is $\leqslant r$.

Notice in particular that for any n:

(1) The sum of the diameters of the circles obtained is $2r$.

(2) The union of the circles obtained may be included in two rectangles each of length $r\sqrt{2}$ and width r/n.

Start with a circle of diameter unity tangent to both exes in the first quadrant and perform the operation of order 3. Call the union of the circles thus obtained Γ_3. Then the closure $\overline{\Gamma}_3$ is Γ_3 together with the four points $(0,\frac{1}{2})$, $(\frac{1}{2},0)$, $(1,\frac{1}{2})$ and $(\frac{1}{2},1)$. On each circle of Γ_3 perform the operation of order 4 and call the union of all circles thus obtained Γ_4. Then $\overline{\Gamma}_4$ contains only a countable set of points more than Γ_4. In general, on each circle of Γ_{n-1} we perform the operation of order n and obtain a set $\Gamma_n\subset\Gamma_{n-1}$, and note $\overline{\Gamma}_n-\Gamma_n$ is countable.

The set A we shall consider is the intersection

$$A=\overline{\Gamma}_3\overline{\Gamma}_4\ldots\overline{\Gamma}_n\ldots=\lim_{n\to\infty}\overline{\Gamma}_n.$$

Thus A is a closed (and even perfect) set and is thus measurable.

For any non-negative constant $a\leqslant 1$, the line $x=a$ (and also the line $y=a$) intersects $\overline{\Gamma}_n$ in a non-empty closed set so this line contains a point of A. Thus the projection of A on either axis is the unit interval $[0,1]$.

Since the diameter of any set U is greater than or equal to the diameter of the projection of U on a line one will see that $L(A)\geqslant 1$. On the other hand $\overline{\Gamma}_n$ is clearly covered by a countable number of convex sets, each with diameter $<1/n$, such that the sum of the diameters is unity, so $L(A)\leqslant 1$. Hence $L(A)=1$.

For later use in connection with the Gillespie linear measure of this particular set A we make a further observation. From property (2) above, the subset of A in any one circle of Γ_n may be included in two rectangles the sum of whose semi-circumferences is $(\sqrt{2}+1/n)$ multiplied by the diameter (which is $\leqslant 1/2n$) of that circle. Thus since $A-A\Gamma_n$ is countable we may include all of A in a countable number of convex sets each with diameter less than $1/n$, such that the sum of the semi-circumferences is not greater than $(\sqrt{2}+1/n)$ multiplied by the sum (which is unity) of the diameters of the circles of Γ_n. Thus

$$G(A)\leqslant\sqrt{2}.$$

6. Projection properties. Let P be a closed polygon with $c(P)=s_1+\ldots+s_n$ where s_k is the distance between consecutive vertices of P, and let a_k be the distance between the projections of these two vertices on the x-axis and b_k on the y-axis. Let $|P^x|$ be the distance between the end points of the projection of P on the x-axis and $|P^y|$ for the y-axis. One will see that

$$c(P)=\sum_{k=1}^{n}\sqrt{a_k^2+b_k^2}\geqslant\sqrt{(\sum a_k)^2+(\sum b_k)^2}\geqslant\sqrt{(2|P^x|)^2+(2|P^y|)^2}.$$

Thus if U is any convex set

(1) $$c(U)\geqslant 2\sqrt{|U^x|^2+|U^y|^2}.$$

We now prove the following theorem, the analogue of which is seen, by the example of § 5, not to hold for Carathéodory measure.

Theorem 3. *If A is any plane set, $m^*(A^x)$ and $m^*(A^y)$ the outer Lebesgue measures of the projections of A on the x- and y-axes respectively, then*

$$G^*(A)\geqslant\sqrt{[m^*(A^x)]^2+[m^*(A^y)]^2}\ ^{1)}.$$

For let $U_1, U_2, \ldots$ be a covering of A by convex sets such that $G^*(A)+\epsilon>\sum\frac{c(U_k)}{2}$. From (1) this sum is $\geqslant\sum\sqrt{|U_k^x|^2+|U_k^y|^2}$ which from Minkowski's inequality is $\geqslant\sqrt{(\sum|U_k^x|)^2+(\sum|U_k^y|)^2}$. But $\sum U_k^x$ is a covering of A^x so $\sum|U_k^x|\geqslant m^*(A^x)$ and we see the desired result.

The set A constructed in § 5 was seen to have

$$A^x=\underset{(x,y)}{E}[0\leqslant x\leqslant 1, y=0] \quad\text{and}\quad A^y=\underset{(x,y)}{E}[x=0, 0\leqslant y\leqslant 1]$$

and thus from theorem 3 to have $G(A)\geqslant\sqrt{2}$. But at the end of § 5 we saw that $G(A)\leqslant\sqrt{2}$ so for this, particular set we have the exact relations

$$m(A^x)=m(A^y)=L(A)=1,\quad G(A)=\sqrt{2}.$$

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1) For Janzen measure $J^*(A)$ one will sometimes see the relation

$$J^*(A)\geqslant\sqrt{[m^*(A^x)]^2+[m^*(A^y)]^2}$$

e. g. ([12], p. 5). However, Janzen measure is defined with respect to, and may not be independent of, the coordinate system, so in this relation it is mandatory that the x and y refer to the coordinate system with respect to which $J^*(A)$ is defined. On the other hand $G^*(A)$ is independent of the coordinate system.

7. Length of a curve. Let γ be a curve without double points, i. e. a unique image of the closed interval $0\leqslant t\leqslant 1$ by continuous functions φ and ψ such that $\varphi(t)=\varphi(t')$ and $\psi(t)=\psi(t')$ implies $t=t'$. We shall let (γ) represent the set of points on the curve γ. The set (γ) is then closed and thus Gillespie linearly measurable.

In this section we shall prove the

Theorem 4. *The Gillespie linear measure $G(\gamma)$ of the set (γ) is the upper limit λ of numbers of the form*

(1) $$\sum_{i=1}^{n}\sqrt{[\varphi(t_i)-\varphi(t_{i-1})]^2+[\psi(t_i)-\psi(t_{i-1})]^2}$$

where $0=t_0<t_1<\ldots<t_n=1$, i. e. $G(\gamma)$ is the length of the curve γ.

If $0=t_0<\ldots<t_n=1$ and γ_i is the part of γ corresponding to the open interval (t_{i-1},t_i), then by theorem 3

$$G^*(\gamma_i)\geqslant\sqrt{[\varphi(t_{i-1})-\varphi(t_i)]^2+[\psi(t_{i-1})-\psi(t_i)]^2}$$

for $i=1,2,\ldots,n$; hence

$$G^*(\gamma)\geqslant\sum_{i=1}^{n}G^*(\gamma_i)\geqslant\sum_{i=1}^{n}\sqrt{[\varphi(t_{i-1})-\varphi(t_i)]^2+[\psi(t_{i-1})-\psi(t_i)]^2}$$

and

(2) $$G^*(\gamma)\geqslant\lambda.$$

We now prove the reverse inequality.

Since λ is the upper limit of numbers of the form (1) we have, for $0\leqslant t_0\leqslant 1$,

$$\sqrt{[\varphi(0)-\varphi(t_0)]^2+[\psi(0)-\psi(t_0)]^2}+\sqrt{[\varphi(t_0)-\varphi(1)]^2+[\psi(t_0)-\psi(1)]^2}\leqslant\lambda,$$

i. e. the set (γ) is included in an ellipse with one axis of length γ and (using d for the distance between the end points of γ) the other axis of length $\sqrt{\lambda^2-d^2}$. We may thus include the set (γ) in a rectangle with dimensions λ and $\sqrt{\lambda^2-d^2}$.

Let $\varrho>0$ be an arbitrary number. Choose $0=t_0<t_1<\ldots<t_n=1$ such that if d_i is the distance between the points $(\varphi(t_{i-1}),\psi(t_{i-1}))$ and $(\varphi(t_i),\psi(t_i))$ and λ is the length of the arc γ_i of γ joining these two points, then

$$\lambda_i<\frac{\varrho}{2}\quad\text{and}\quad\left[\lambda^2-\left(\sum_{i=1}^{n}d_i\right)^2\right]^{1/2}<\varrho.$$

Now enclose (γ_i) in a rectangle U_i of dimensions λ_i and $(\lambda_i^2-d_i^2)^{1/2}$ as we have just shown is possible. The rectangle U_i has diagonal $<\varrho$ and is a convex set, so we have

$$G_\varrho(\gamma)\leqslant\sum_{i=1}^{n}\frac{c(U_i)}{2}=\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}(\lambda_i^2-d_i^2)^{1/2}\leqslant{}^{1)}\ \lambda+\Big[\lambda^2-\Big(\sum_{i=1}^{n}d_i\Big)^2\Big]^{1/2}\leqslant\lambda+\varrho.$$

Since ϱ is arbitrary we thus have

$$G(\gamma)\leqslant\lambda$$

the inequality reverse from (2).

8. $G^*(A)\leqslant\frac{\pi}{2}L^*(A)$**.** From the fact that a point set of diameter d may be included in a circle of diameter $2d$, i. e. semi-circumference πd, we have already seen that $G^*(A)\leqslant\pi L^*(A)$. A theorem by Young ([1], p. 463) states that a set of diameter d may be included in a circle of diameter $\frac{2d}{\sqrt{3}}$ and thus the sharper inequality $G^*(A)\leqslant\frac{\pi}{\sqrt{3}}L^*(A)$ may be obtained. In this section we prove still more; we prove

Theorem 5. If A is an arbitrary plane point set, then

$$G^*(A)\leqslant\frac{\pi}{2}L^*(A).$$

With $\varrho>0$, let $U_1,U_2,\dots$ be a covering of A by convex sets such that the diameter $d(U_i)<\varrho$ and $\sum d(U_i)<L^*(A)+\varrho$. Then $G_\varrho(A)\leqslant\sum\frac{c(U_i)}{2}$. But, (see [2], p. 65) a plane convex set U of diameter d has $c(U)\leqslant\pi d$. Thus $G_\varrho(A)<\frac{\pi}{2}[L^*(A)+\varrho]$ and since ϱ is arbitrary we have the desired result.

Furthermore, this result is the best possible for there are sets for which the equality holds. A set constructed for a different purpose by Besicovitch ([1], p. 431) is one such set, but the proof will not be given.

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[1]) This inequality follows since $0\leqslant a_i\leqslant b_i$ implies

$$\sum(b_i^2-a_i^2)^{1/2}\leqslant[(\sum b_i)^2-(\sum a_i)^2]^{1/2}$$

and because $\sum_{i=1}^{n}\lambda_i\leqslant\lambda$.

9. Gillespie area measure. Let A be a point set in 3-dimensional space.

The surface area $s(U)$ of a convex set U in 3-dimensional space is defined as $\inf H$ where H is the set of real numbers determined by: $p\in H$ if there exist an open convex set P and closed triangles $\Delta_1,\Delta_2,\dots,\Delta_n$ for which

$$U\subset P,\qquad\text{boundary of } P\subset\sum_{j=1}^{n}\Delta_j,\qquad p=\sum_{j=1}^{n}\ (\text{area of } \Delta_j).$$

The definition of Gillespie outer area measure $G^{*2}(A)$ is obtained from the definition of Gillespie outer linear measure (§ 3) by replacing the words "the plane" by "3-dimensional space" and $c(U_k)$ by $s(U_k)$.

Furthermore the statements made in § 3 about Gillespie linear measure all have analogues for Gillespie area measure.

10. Relation between Gillespie linear and area measures. In the Lebesgue theory of integration if M is a bounded Lebesgue measurable set on the x-axis and f is a non-negative bounded Lebesgue measurable function on M, then the plane set $N_f=\underset{(x,y)}{E}[x\in M, 0\leqslant y\leqslant f(x)]$ is Lebesgue plane measurable and $m^{(2)}(N_f)=\int_M f(x)\,dx$. This relation is equivalent to the fact that if f is a constant k then the set N_f is Lebesgue plane measurable with

$$m^{(2)}(N_f)=km(M). \tag{1}$$

To each of the linear measures mentioned in the introduction corresponds an analogues area measure for sets not necessarily lying in the plane. It has not been shown that any of these measures preserve, as does Lebesgue, the euclidean relation that area is the product of length by length.

In this section we show, however, that if A is a set in the (x,y)-plane and B is the cylindrical set

$$B=\underset{(x,y,z)}{E}[(x,y)\,\varepsilon\, A, 0\leqslant z\leqslant h]$$

then the Gillespie area of B is the Gillespie length of A multiplied by h.

Throughout this section A and B will be used to designate the sets given here.

We first prove, using $|U|$ for the Lebesgue 2-dimensional measure of the plane convex set U,

Lemma 1. *If $G^*(A)$ is finite, $\varrho_1>\varrho_2>\ldots,\varrho_n\to 0$, and $U_{n1}, U_{n2},\ldots$ a sequence of coverings of A by plane convex sets with $d(U_{nk})<\varrho_n$ such that $\lim_{n\to\infty}\sum_k \frac{1}{2}c(U_{nk})=G^*(A)$, then $\lim_{n\to\infty}\sum_k |U_{nk}|=0$.*

For U_{nk} may be included in a circle of radius $d(U)_{nk}$ and thus $\sum_k |U_{nk}|\leqslant\sum_k \pi d(U_{nk})^2<\pi\varrho_n\sum d(U_{nk})\leqslant\pi\varrho_n\sum_k\frac{1}{2}c(U_{nk})$ which approaches zero with $1/n$.

We now obtain an inequality for outer measures.

Theorem 6. $G^{*(2)}(B)\leqslant hG^*(A)$.

Given $\varrho>0$, take N an integer so large that $h/N<\varrho/\sqrt{2}$. Then from the lemma we may cover A by convex sets $U_1, U_2,\ldots$ with $d(U_k)<\varrho/(2\sqrt{2})$, $\sum\frac{1}{2}c(U_k)<G^*(A)+\varrho/h$ and $\sum|U_k|<\varrho/N$. Now let

$$V_{kn}=\underset{(x,y,z)}{E}[(x,y)\,\varepsilon\, U_k, nh/N\leqslant z\leqslant(n+1)h/N].$$

Then $d(V_{kn})<\varrho$ and $\sum_k\sum_{n=0}^N V_{kn}\supset B$. Thus

$$G_\varrho^{(2)}(B)\leqslant\sum_k\sum_{n=0}^N\frac{1}{2}s(V_{kn})\leqslant\sum\frac{2N|U_k|}{2}+N\cdot\frac{h}{N}\cdot\frac{1}{2}c(U_k)\leqslant$$
$$\leqslant N(\varrho/N)+h[G^*(A)+\varrho/h]=hG^*(A)+2\varrho.$$

Therefore since ϱ is arbitrary the theorem holds.

In particular

Corollary. *If A is Gillespie linearly measurable with $G(A)<\infty$, then B is Gillespie area measurable and $G^{(2)}(B)\leqslant hG(A)$.*

For, from theorem 1, $\epsilon>0$ implies a closed subset K of A exists with $G(A-K)<\epsilon$. Thus the set

$$H=\underset{(x,y,z)}{E}[(x,y)\,\varepsilon K, 0\leqslant z\leqslant h]$$

is a closed subset of B such that, from (2), $G^{*(2)}(B-H)\leqslant h\epsilon$. Thus B is Gillespie area measurable since, as one will see, the area analogue of theorem 1 holds.

If A is a set in 3-dimensional space we use the notation

$$A^z=\underset{(x,y)}{E}[(x,y,z)\,\varepsilon\, A].$$

Lemma 2. *If Δ is a closed triangle in 3-dimensional space, then*

$$\int_{-\infty}^{\infty}G(\Delta^z)\,dz\leqslant\text{area of }\Delta.$$

Lemma 3. *If W is a bounded convex set in 3-dimensional space, then*

$$\int_{-\infty}^{\infty}c(W^z)\,dz\leqslant s(W).$$

Let $\epsilon>0$ be arbitrary, and let P be an open convex set with boundary P^* such that

(i) P is contained in W.

(ii) There exists closed triangles $\Delta_1,\Delta_2,\ldots,\Delta_n$ for which

$$P^*\subset\sum_{j=1}^n\Delta_j,\qquad\sum_{j=1}^n(\text{area of }\Delta_j)\leqslant s(W)+\epsilon.$$

Now W^z and P^z are convex, $W^z\subset P^z$ and the boundary of P^z is contained in $\sum_{j=1}^n\Delta_j^z$ so we have[1])

$$c(W^z)\leqslant c(P^z)\leqslant\sum_{j=1}^n G(\Delta_j^z).$$

Hence, using the previous lemma, we obtain

$$\int_{-\infty}^{\infty}c(W^z)\,dz\leqslant\sum_{z=1}^n\int_{-\infty}^{\infty}G(\Delta_j^z)\,dz\leqslant\sum_{j=1}^n(\text{area of }\Delta_j)\leqslant s(W)+\epsilon.$$

The proof is complete.

Theorem 7. $G_*^{(2)}(B)\geqslant hG_*(A)$.

First let K be a closed subset of A with $G(K)<\infty$. Then the set $H=\underset{(x,y,z)}{E}[(x,y)\,\varepsilon\, K, 0\leqslant z\leqslant h]$ is a closet subset of B with, from theorem 6, $G^{(2)}(H)<\infty$.

From the definition of Gillespie outer linear measure, corresponding to an $\epsilon>0$ there exists a $\varrho>0$ such that $G(K)-\epsilon<\sum\frac{1}{2}c(U_k)$ if $U_1, U_2,\ldots$ is a covering of K by plane convex sets with $d(U_k)<\varrho$. We then choose a covering $W_1,W_2,\ldots,W_n$ of H by three dimensional open convex sets with $d(W_k)<\varrho$ such that

[1]) [2], p. 47.

$$G^{(2)}(H)+\epsilon>\sum_{k=1}^{n}\frac{1}{2}s(W_k);$$

the numbers of the covering reduced to being finite in number since H is closed. Then W_k^z is convex with diameter $<\varrho$ and, for $0\leqslant z\leqslant h$, the union $\sum_{k=1}^{n}W_k^z$ contains the plane set H^z which is the set K. Thus

$$\sum_{k=1}^{n}\frac{1}{2}c(W_k^z)>G(K)-\epsilon, \qquad 0\leqslant z\leqslant h.$$

But, from lemma 3,

$$s(W_k)\geqslant\int_{-\infty}^{\infty}c(W_k^z)\,dz\geqslant\int_{0}^{\infty}c(W_k^z)\,dz.$$

Also since we have a finite number of terms

$$\sum_{k=1}^{n}\int_{0}^{h}\frac{1}{2}c(W_k^z)\,dz=\int_{0}^{h}\Big[\sum_{k=1}^{n}\frac{1}{2}c(W_k^z)\Big]dz.$$

Thus $G^{(2)}(H)+\epsilon>\int_{0}^{h}[G(K)-\epsilon]\,dz$, and since ϵ is arbitrary

(2) $$G^{(2)}(H)\geqslant hG(K).$$

Now since $B\supset H$ we have $G_*^{(2)}(B)\geqslant G^{(2)}(H)$. If $G^*(A)$ is finite, then $G(K)$ may be taken arbitrarily close to it so in this case the desired result $G_*^{(2)}(B)\geqslant hG_*(A)$ holds. On the other hand if $G_*(A)$ is infinite then $G(K)$ and consequently $G^{(2)}(H)$ may be taken arbitrarily large so the desired relation holds.

From the corollary to theorem 6 and theorem 7 we have

Corollary 1. *If A is Gillespie linearly measurable with $G(A)$ finite, then B is Gillespie areally measurable and $G^{(2)}(B)=hG(A)$.*

We may now prove

Theorem 8. *If A is an arbitrary plane set, then*

$$G_*^{(2)}(B)=hG_*(A) \quad \textit{and} \quad G^{*(2)}(B)=hG^*(A).$$

For the first relation, we have already seen that if $G_*(A)$ is infinite, then $G_*^{(2)}(B)$ is also infinite. With $G_*(A)<\infty$, let $\underset{\sim}{H}$ be a bounded closed subset of B with $G_*^{(2)}(B)-\epsilon<G^{(2)}(\underset{\sim}{H})$, such a set

existing from the definition of inner measure and the analogue for area measure of theorem 2. Being bounded and closed, the set $K=\underset{(x,y)}{E}[(x,y,z)\,\varepsilon\,\underset{\sim}{H}]$ is also closed. Thus the set

$$H=\underset{(x,y,z)}{E}[(x,y)\,\varepsilon\,K,\ 0\leqslant z\leqslant h]$$

contains $\underset{\sim}{H}$ so $G(\underset{\sim}{H})\leqslant G^{(2)}(H)$ and since H is measurable (closed) $G^{(2)}(H)=hG(K)$ from corollary 1. But $\underset{\sim}{H}\subset B$ so $K\subset A$ and $G(K)\leqslant G_*(A)$ But ϵ is arbitrary so

$$G_*^{(2)}(B)\leqslant hG_*(A)$$

which with theorem 7 gives the desired equality for inner measure.

For outer measures, if $G^*(A)<\infty$ let $\tilde{A}$ be a measurable set containing A with $G(\tilde{A})<\infty$. Then (see [4] p. 262, theorem 4)

$$G(\tilde{A})=G^*(A)+G_*(\tilde{A}-A).$$

Then multiplying by h and using the corollary 1 and the first equality of the corollary under proof we have

$$G^{(2)}(\tilde{B})=hG^*(A)+G_*^{(2)}(\tilde{B}-B)$$

where $\tilde{B}=\underset{(x,y,z)}{E}[(x,y)\,\epsilon\,\tilde{A},0\leqslant z\leqslant h]$. But $G^{(2)}(\tilde{B})<\infty$ and

$$G^{(2)}(\tilde{B})=G^{*(2)}(B)+G_*^{(2)}(\tilde{B}-B)$$

so we have the desired equality

$$G^{*(2)}(B)=hG^*(A)$$

when $G^*(A)<\infty$. If $G^*(A)$ is infinite, $G^{*(2)}(B)$ is also infinite so we have the desired result.

In 3-dimensional space a set which is areally measurable need not project orthogonally on a plane into a linearly measurable set. However, with A and B the related sets of this section, the converse of corollary 1 holds, i. e.

Corollary 3. *If B is a Gillespie areally measurable set with $G^{(2)}(B)$ finite, then A is Gillespie linearly measurable with*

$$G^{(2)}(B)=hG(A).$$

For $G^{*(2)}(B)=G_*^{(2)}(B)$ so from theorem 8, $G^*(A)=G_*(A)$ and moreover the desired equality holds.

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On Characteristic Functions of Families of Sets.

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In an interesting paper[1]) Szpilrajn has employed the characteristic function to develop a certain method of dealing with the algebraic structure of sequences of sets; and has established with the aid of this method a variety of specific theorems and equivalences in the domain of set-theoretical topology. He attributes to Kuratowski the first use of the characteristic function of a sequence of sets.

In the present note, I shall trace certain connections between the content of Szpilrajn's paper and the general theory of abstract Boolean algebras which I have developed in two memoirs published elsewhere[2]). In doing so, I deem my chief purpose to be that of reconciling two independent points of view which prove, upon examination, to present a considerable analogy so far as the theory of the algebraic structure of sequences of sets is concerned.

As I shall point out below, an obvious but theoretically desirable generalization of Szpilrajn's work leads to the introduction of the characteristic function of an arbitrary transfinite sequence, or well-ordered family, of sets. It seems to me of more importance, perhaps, to observe that the rôle of order, which is essential to the definition of the characteristic function, appears to be artificial

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[1]) E. Szpilrajn, *The characteristic function of a sequence of sets and some of its applications*, Fundam. Math. **31** (1938), p. 207–233; see also Fundam. Math. **26** (1935), p. 302.

[2]) M. H. Stone, *The Theory of Representations for Boolean Algebras*, Trans. Amer. Math. Soc. **40** (1936), pp. 37–111 (cited here by the letter R); and *Applications of the Theory of Boolean Rings to General Topology*, ibidem **41** (1937), pp. 375–481 (cited here by the letter A).