Le problème s'impose si l'on peut tirer du théorème de Hausdorff une décomposition de la droite en $\aleph_1$ ensembles G_δ non vides et disjoints.

Vu que tout ensemble $G_{\delta\sigma}$ linéaire est somme de $\aleph_0$ ensembles G_δ disjoints[7]), ce serait évidemment le cas si tous les ensembles (2a) seraient des $G_{\delta\sigma}$. Comme d'après (2a) et (1) $\sum_{\xi<\alpha}\Gamma_\xi=\Gamma_\alpha-\Xi_\alpha$ pour $\alpha<\Omega$, les ensembles $\sum_{\xi<\alpha}\Gamma_\xi$ devraient être (pour $\alpha<\Omega$) des ensembles $F_{\delta\sigma}$. Or, d'après un théorème que j'ai démontré en généralisant un théorème de M. Lusin[8]), la suite transfinie $\{\Gamma_\alpha\}_{\alpha<\Omega}$ serait alors stationnaire, ce qui est impossible, les ensembles $\Gamma_\alpha(\alpha<\Omega)$ étant tous distincts.

Le problème de démontrer sans faire appel à l'hypothèse du continu que la droite est somme de $\aleph_1$ ensembles G_δ non vides et disjoints reste donc ouvert.

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[7]) W. Sierpiński, Fund. Math. **10** (1927), p. 324 (Lemme 4).

[8]) W. Sierpiński, Fund. Math. **24**, (1935), p. 309 (Théorème II, dans lequel il faut passer aux complémentaires).

On the decomposition of manifolds into products of curves and surfaces.

By

Karol Borsuk (Warszawa).

1. The sets $E_1, E_2, \dots, E_n$ constitute a product-decomposition of a space E, if the product $E_1\times E_2\times\dots E_n$ [1]) is a homeomorph of E. We then call the sets E_i *topological divisors of* E. A topological divisor of E containing more than one point and not homeomorphic to E is called a *real topological divisor of* E. A space which has no real topological divisor is called *topologically first*.

Two decompositions $E_1, E_2, \dots, E_n$ and $E'_1, E'_2, \dots, E'_{n'}$ of E will be considered as *identical*, if after cancelling their one-puncting terms, they may differ only by their order[2]). It is easy to notice[3]) that every compact space of finite dimension and finite number of components is decomposable into a product of topological first sets. The problem whether this decomposition is possible in one manner only is, in the general case, unsolved. Except some trivial cases[4]), so far as I know, only one partial result concerning this problem has been obtained. We mean the theorem that no polyhedron

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[1]) That is the space whose elements are all ordered n-tuples $(x_1, x_2, \dots, x_n)$ with $x_i \in E_i$ for $i=1,2,\dots,n$, and whose metric is given by the formula:

$$\varrho[(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)] = \sqrt{\sum_{i=1}^{n}[\varrho(x_i, y_i)]^2}.$$

[2]) Compare my paper *Sur la décomposition des polyèdres en produits cartésiens*, Fund. Math. **31** (1938), p. 138.

[3]) l. c., pp. 138 and 139.

[4]) In many simple cases the fact that E is topologically first is an immediate consequence of its simplest topological properties. So it is, for example, for a continuum containing points in which it is 1-dimensional, or for E consisting of two non homeomorphic components.

(of arbitrary dimension) can have more than one decomposition into a product of sets of dimension $\leqslant 1$ [5]).

It is the purpose of this paper to give a further contribution to this question. Especially, to show that a decomposition of any n-dimensional manifold [6]) into a product of sets of dimension $\leqslant 2$ is possible in one manner at most.

2. We begin by some elementar lemmas concerning open subsets of R_n [7]) and their topological divisors.

Lemma 1. *Let Γ be an open subset of R_n and $p \in \Gamma$. If f is a continuous transformation of Γ into a subset of R_n and ε a positive number such that for every point $x \in R_n$ with $\varrho(x,p) \leqslant \varepsilon$ it is $x \in \Gamma$ and $\varrho(x,f(x)) < \varepsilon$, then $p \in f(\Gamma)$.*

Proof. Let R denote the open n-dimensional sphere in R_n of center p and radius ε. We denote by S the boundary of R. Then $S+R \subset \Gamma$. Let us put for every $x \in S$ and $0 \leqslant t \leqslant 1$

$$f_t(x) = x + 2t \cdot [f(x) - x], \qquad \text{if} \quad 0 \leqslant t \leqslant \tfrac{1}{2},$$
$$f_t(x) = f[x + (2t-1)\cdot(p-x)] \quad \text{if} \quad \tfrac{1}{2} \leqslant t \leqslant 1.$$

It is easy to observe that $f_t(x)$ constitutes a homotopic deformation of S over R_n into the point $f(p)$. It follows [8]) that there exists $x_0 \in S$ and $0 \leqslant t_0 \leqslant 1$ such that $f_{t_0}(x_0) = p$. But, for $0 \leqslant t \leqslant \frac{1}{2}$ and $x \in S$, we have

$$\varrho[f_t(t),p] \geqslant \varrho(x,p) - \varrho[f(x),x] > \varrho(x,p) - \varepsilon = 0.$$

Hence $t_0 > \frac{1}{2}$ and consequently

$$p = f_{t_0}(x_0) = f[x_0 + (2t_0 - 1)\cdot(p - x_0)] \in f(\Gamma).$$

Thus the lemma is proved.

[5]) l. c., p. 139.

[6]) An *n-dimensional manifold* is such a continuum M, that for every $p \in M$ there exists a neighbourhood V_p of p in M homeomorphic to the Euclidean n-dimensional space R_n.

[7]) We denote by R_n the n-dimensional Euclidean number-space. The point x of R_n with the Cartesian coordinates $t_1, t_2, \ldots, t_n$ will be denoted by $(t_1, t_2, \ldots, t_n)$. If a is a number, $a \cdot x$ denotes the point $(a \cdot t_1, a \cdot t_2, \ldots, a \cdot t_n)$. If $x' = (t_1', t_2', \ldots, t_n')$ is also a point of R_n, we denote by $x + x'$ the point $(t_1 + t_1', t_2 + t_2', \ldots, t_n + t_n')$.

[8]) K. Borsuk, *Sur un espace des transformations continues et ses applications topologiques*, Monatsh. für Math. und Phys. **38** (1931), p. 383.

Lemma 2. *Let Γ be a homeomorph of an open subset of R_n and $\{f_n\}$ a sequence of continuous transformations of Γ into its subsets, convergent uniformely to the identity-transformation. Then for every point $p \in \Gamma$ there exists a such number N, that $p \in f_n(\Gamma)$ for every $n > N$.*

Proof. Let h denote a homeomorphic transformation of Γ into an open subset Γ' of R_n and h^{-1} the transformation inverse to h. Then the transformations hf_nh^{-1} transform Γ' into its subsets and in every compact subset of Γ' converge uniformely to the identity-transformation. Thus, if we put $p' = h(p)$ and establish ε sufficiently small, then there exists such a number N, that for every $n > N$ the transformation hf_nh^{-1} satisfies the hypothesis of the preceding lemma. From this we conclude that $h(p) = p' \in hf_nh^{-1}(\Gamma') = hf_n(\Gamma)$. Hence $p \in f_n(\Gamma)$ and the lemma is proved.

3. ***Theorem.*** *Every 1-dimensional topological divisor of an open subset of R_n is locally homeomorphic to R_1* [9]).

Proof. Let $A \times B$ be a homeomorph of an open subset Γ of R_n and let $\dim A = 1$. Since $A \times B$ is locally connected and locally compact, we conclude easily that both sets A and B are also locally connected and locally compact. Since $\dim(A \times B) = n$ and $\dim A = 1$, we infer [10]) that $\dim B = n-1$. We see at once that

(1) *A is homogenously 1-dimensional and B is homogenously $(n-1)$-dimensional.*

For, if for any point $(x,y) \in A \times B$ the dimension of A in x would be less than 1, or the dimension of B in y less than $n-1$, then the dimension of $A \times B$ in (x,y) would be less than n, which is not true.

In order to show that a locally connected and locally compact set A is locally homeomorphic to R_1, we have to prove [11]) that a has the Menger-Urysohn order equal to 2 in each of its points. We decompose this proof into two parts:

[9]) That is, each point of it has a neighbourhood homeomorphic to R_1.

[10]) W. Hurewicz, *Sur la dimension des produits cartésiens*, Annals of Mathematics **36** (1935), p. 194. In his paper it is supposed, that A is compact, but the hypothesis of local compactness of A is evidently also sufficient.

[11]) K. Menger, *Kurventheorie*, Leipzig-Berlin 1932, p. 267.

1. *The Menger-Urysohn order of A in $a \in A$ is* >1.

Since A is homogenously 1-dimensional, the order of A in a is $\geqslant 1$. Let us suppose that it is equal to 1. Since A is locally connected, there will exist, for every $n=1,2,\ldots$, an open and connected neighbourhood U_n of a in A of the diameter $<1/n$ bounded in A by one point a_n. Let us put

$$\varphi_n(x)=x \quad \text{for} \quad x \in A-U_n,$$
$$\varphi_n(x)=a_n \quad \text{for} \quad x \in U_n.$$

Thus we obtain a continuous transformation φ_n of A into $A-U_n \subset A-(a)$. The sequence $\{\varphi_n\}$ converges uniformely to the identity-transformation of A. Hence, putting

$$f_n(x,y)=(\varphi_n(x),y) \quad \text{for} \quad (x,y) \in A \times B,$$

we obtain a sequence $\{f_n\}$ of continuous transformations of $A \times B$ into subsets of $[A-(a)] \times B$ uniformely convergent to the identity-transformation. But this is impossible in view of the lemma 2 of Nr. 2. Thus the supposition that the Menger-Urysohn order of A in a is $\leqslant 1$ leads to a contradiction. Thus 1 is proved.

2. *The Menger-Urysohn order of A in $a \in A$ is* $\leqslant 2$.

For otherwise A would contain 3 simple arcs L_1, L_2, L_3 having, except their endpoint a, no other points in common[12]). Let $A_1, A_{1,2}$ and A_3 denote respectively the interiors of the arcs L_1, L_1+L_2 and L_3. Since $\dim(A_1 \times B)=n$ and every n-dimensional subset of R_n contains inner points[13]), there exist such points $c \in A_1$ and $b \in B$, that the point $q=(c,b)$ is an inner point of $A_1 \times B$. Let φ denote a homeomorphic transformation of A_1 into $A_{1,2}$, such that $\varphi(c)=a$. If we put $\psi(x,y)=(\varphi(x),y)$ for each $(x,y) \in A_1 \times B$, we obtain a homeomorphic transformation of $A_1 \times B$ into $A_{1,2} \times B$ mapping $q=(c,b)$ into $p=(a,b)$. Applying Brouwer's theorem concerning the invariance of region in R_n[13]) we conclude that p is an inner point of $A_{1,2} \times B$. But this is false, because $p=(a,b)$ lies on the boundary of the set $A_3 \times B$, having no common points with $A_{1,2} \times B$. Thus the supposition that 2 is not true leads to a contradiction.

Hence the theorem is proved.

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[12]) K. Menger, l. c., p. 214.

[13]) see, for example, K. Menger, *Dimensionstheorie*, Leipzig-Berlin 1928, p. 244.

4. Corollary 1. *Every* 1-*dimensional topological divisor of any Euclidean region*[14]) *is either a simple closed curve or a homeomorph of* R_1.

This results from theorem 3 and from the theorem that a separable connected space, having the Menger-Urysohn order equal to 2 in all its points, is either a simple closed curve or a homeomorph of R_1[15]). This last possibility falls off if we restraint ourselves to the continua. Consequently, we derive also from the theorem 3 the following

Corollary 2. *Every* 1-*dimensional topological divisor of a n-dimensional manifold*[6]) *is a simple closed curve.*

5. Theorem. *If A is a topological divisor of dimension* $\geqslant 2$ *of a n-dimensional Euclidean region, then no simple arc cuts A.*

Proof. Since A is locally compact and $\dim A \geqslant 2$, there exists a *curve*[16]) C nowhere dense in A. Therefore $C \times B$ is not dense in $A \times B$, so that $\dim(C \times B) \leqslant n-1$[13]). Since further $\dim(C \times B) > \dim B$[10]), we conclude that

$$\dim B \leqslant n-2. \tag{2}$$

It is easily seen that if a simple arc L cuts A between two points a and a', then $L \times B$ cuts $A \times B$ between the points (a,β) and (a',β'), where β and β' are arbitrary points of B. Thus it remains to show that $L \times B$ does not cut $A \times B$ between two points a,b given arbitrarily in $(A-L) \times B$.

We can obviously assume that $A \times B$ is a subset of the boundary S_n of a $(n+1)$-dimensional Euclidean sphere and that the simple arc L is identical with the interval $0 \leqslant t \leqslant 1$.

Now we suppose, contrary to our theorem, that $L \times B$ cuts the Euclidean region $A \times B$ between certain two points a and b. Since $\dim B \leqslant n-2$, $A \times B$ is not dissected by $(0) \times B$ between a and b. Therefore there exists a simple arc K joining a and b in $(A-(0)) \times B$. Let us put

$$P = \underset{y \in B}{E}[(x,y) \in K \text{ for some } x \in A]. \tag{3}$$

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[14]) We understand by *Euclidean region* a homeomorph of an open and connected subset of R_n.

[15]) F. Frankl, *Über die zusammenhängenden Mengen von höchstens zweiter Ordnung*, Fund. Math. **11** (1928), p. 97.

[16]) That is a continuum of dimension 1.

Then P is a compact subset of B (the „projection" of the arc K into B). Since B is locally compact, we infer that there exists such a positive number ε, that the set

$$U = \underset{y \in B}{E}[\varrho(y,P) < \varepsilon]$$

has a compact closure $\overline{U}$.

Now let T denote a boundary (in S_n) of $A \times B$. We put $M = (L \times B) + T$. Since $L \times B$ is closed in $A \times B$, the set M is closed in S_n. Then M is compact and it cuts S_n between a and b. Let us put, for $0 \leqslant t \leqslant 1$ and $p \in M$,

(4) $\quad f_t(p) = p \quad$ if $\quad p \in T \quad$ or $\quad p = (x,y) \quad$ with $\quad y \in B - U$,

(5) $\quad f_t(p) = [(1 - t + \frac{t}{\varepsilon} \cdot \varrho(y,P)) \cdot x,\ y] \quad$ if $\quad p = (x,y) \quad$ with $\quad y \in U$.

It is readily seen that $f_0(p) = p$ for each $p \in M$, and that f_t is a continuous deformation of M over itself into the set $f_1(M)$. But this last set does not cut S_n between a and b. In order to prove it, let us show that the simple arc K joining a and b has no common points with it. For otherwise there would exist two points $p = (x,y) \in K$, and $p' = (x',y') \in M$, so that $p = f_1(p')$. By (4), (5) and (3) it is $y = y' \in P$. In virtue of (5) we conclude that $p = f_1(x',y') = (0,y')$, contradicting the hypothesis that $p \in K \subset [A - (0)] \times B$.

On the other hand, it is known [17]) that there exists such a continuous transformation φ of $S_n - (a) - (b)$ into the boundary S_{n-1} of the n-dimensional Euclidean sphere, that φ is essential on every compact subset of S_n, which cuts S_n between a and b, and which is not essential on every compact subset of S_n, which does not cut S_n between a and b. In particular, there exists such a continuous family $\{\varphi_t\}$ of continuous transformations of $f_1(M)$ into S_{n-1}, that $\varphi_0 = \varphi$ and $\varphi_1 = \text{const}$. Let us put, for every $p \in M$

$$\psi_t(p) = \varphi f_{2t}(p) \qquad \text{if} \quad 0 \leqslant t \leqslant \tfrac{1}{2},$$
$$\psi_t(p) = \varphi_{2t-1} f_1(p) \qquad \text{if} \quad \tfrac{1}{2} \leqslant t \leqslant 1.$$

The family $\{\psi_t\}$ is continuous and it joins the transformation $\psi_0 = \varphi$ with the transformation $\psi_1 = \varphi_1 f_1 = \text{const}$. Hence φ is not essential on M, and consequently M does not cut S_n between a and b, contradicting our hypothesis. Thus the theorem is established.

[17]) K. Borsuk, *Sur les coupures locales des variétés*, Fund. Math. **32** (1939), p. 292.

6. *Corollary.* *Let A be a topological divisor of dimension $\geqslant 2$ of an Euclidean region. If U is an open connected subset of A, and E a compact subset of a simple arc $L \subset U$, then $U - E$ is connected.*

For, let B denote such a point-set that $A \times B$ is an Euclidean n-dimensional region. Since $U \times B$ is connected and open in $A \times B$, it is also an Euclidean n-dimensional region. Hence, by theorem **5**, $U - L$ is connected. Let us suppose now that $U - E$ is not connected, and let G and G' denote two different components of it. Because $\dim B \leqslant n-2$, the set U is of dimension $\geqslant 2$ in each of its points. Then L is not dense in U. Consequently there exist two points, $p \in G - L$, and $p' \in G' - L$, between which E, and much more so L, cuts U, which is impossible.

7. Let M be a metric space. Every set composed of $n+1$ points (vertices) of M will be called an *n-dimensional simplex of M*. Each of its subsets will be called a *face* of it. If its diameter is $< \varepsilon$, it is called a *ε-simplex* of M. By a *oriented simplex* we understand a simplex for which an order of its vertices is chosen, while two orders are considered as identical if they differ by an even permutation. The oriented simplex of vertices $a_0 a_1 \ldots a_n$ (in the given order) will be designated by $\Delta(a_0 a_1 \ldots a_n)$. If we reverse its orientation, Δ will be replaced by $-\Delta$.

A finite aggregate of simplexes of M, containig also all faces of its simplexes, will be called an *absolute complex of M*. Every absolute complex is *geometrically realisable*, i. e. we can assume that all vertices of its simplexes are points of an Euclidean space (of sufficiently high dimension), and that the corresponding geometrical simplexes do not cross one another, i. e. the common part of two such simplexes is identical with the geometrical simplex determined by their common vertices.

A linear combination of oriented n-dimensional ε-simplexes of M with rational coefficients will be called a *n-dimensional algebraic ε-complex of M*. In the well known manner [18]), we attach to every n-dimensional ε-complex $\varkappa$ of M a $(n-1)$-dimensional algebraic ε-complex $\dot{\varkappa}$ of M, called the *boundary of $\varkappa$*. When the boundary $\dot{\varkappa}$ identically vanishes, $\varkappa$ is a *n-dimensional ε-cycle of M*. A cycle γ of M which is the boundary of an algebraic η-complex of M is called *η-homologous on M to zero*.

[18]) See, for example, P. Alexandroff and H. Hopf, *Topologie*, Berlin 1935, p. 167.

8. Let now M and M' be compact spaces, $\varkappa=\sum_{i=1}^{l}a_i\cdot\Delta_i$ be an algebraic k-dimensional ε-complex of M, and $\varkappa'=\sum_{i'=1}^{l'}a'_{i'}\cdot\Delta'_{i'}$ be an algebraic k'-dimensional ε-complex of M. We denote by $|\Delta_i|$, respectively by $|\Delta'_{i'}|$, the simplexes corresponding to oriented simplexes Δ_i and $\Delta'_{i'}$. These simplexes, together with their faces, constitute an absolute complex $|\varkappa|$ of M, respectively an absolute complex $|\varkappa'|$ of M'. By virtue of the above remark on the geometrical realisation of the complexes, we may assume that $|\varkappa|$ and $|\varkappa'|$ are the triangulations of certain polyhedrons. Let P denote the polyhedron being the product of these two polyhedrons. To simplicial decompositions $|\varkappa|$ and $|\varkappa'|$ of factors, corresponds a decomposition of P into convex cells being products of simplexes $|\Delta_i|$ and $|\Delta'_{i'}|$. Let us orient the cell $|\Delta_i|\times|\Delta'_{i'}|$ in the following manner: if $\Delta=(a_0a_1\dots a_k)$ and $\Delta'=(a'_0a'_1\dots a'_{k'})$, we attribute to the absolute cell $|\Delta|\times|\Delta'|$ the same orientation as the orientation of the $(k+k')$-dimensional simplex $[(a_0,a'_0)(a_1,a'_0)\dots(a_k,a'_0)(a_k,a'_1)\dots(a_k,a'_{k'})]$. Let us denote the cell $|\Delta|\times|\Delta'|$ oriented in this manner by (Δ,Δ'). Now we define the *product of algebraic complexes $\varkappa$ and $\varkappa'$* as the $(k+k')$-dimensional cellular complex $(\varkappa,\varkappa')$ given by the formula

$$(\varkappa,\varkappa')=\sum_{i=1}^{l}\sum_{i'=1}^{l'}a_i\cdot a'_{i'}\cdot(\Delta_i,\Delta'_{i'}).$$

But this cellular complex may be replaced by a simplicial algebraic complex of $M\times M'$. Indeed, by a result due to H. Freudenthal[19]), every absolute cell $|(\Delta_i,\Delta'_{i'})|$ is simplicially decomposable into simplexes of the form

$$(6)\qquad |[(a_{i_0},a'_{i'_0})(a_{i_1},a'_{i'_1})\dots(a_{i_{k+k'}},a'_{i'_{k+k'}})]|,$$

where both sequences of indices $i_0,i_1,\dots,i_{k+k'}$, and $i'_0,i'_1,\dots,i'_{k+k'}$, are not decreasing, and $i_\nu\leqslant k$, $i'_\nu\leqslant k'$ for $\nu=1,2,\dots,k+k'$. By replacing in $(\varkappa,\varkappa')$ every cell $(\Delta_i,\Delta'_{i'})$ by the correspondent sum of these simplexes suitably oriented, we arrive to a simplicial algebraic complex of M, which we will denote by $\varkappa\times\varkappa'$. Thus we have associated with each pair $\varkappa,\varkappa'$, in which $\varkappa$ is a k-dimensional algebraic complex of M, and $\varkappa'$ a k'-dimensional algebraic complex of M', a $(k+k')$-dimensional algebraic complex $\varkappa\times\varkappa'$ of the product-space $M\times M'$. If we recall the metric defined in the product $M\times M'$, we infer from (6), that

(7) *If $\varkappa$ and $\varkappa'$ are ε-complexes, then $\varkappa\times\varkappa'$ is a $\varepsilon\cdot\sqrt{2}$-complex.*

Let K be such an absolute complex of M, that among its simplexes appears every simplex of the given algebraic complex $\varkappa$, and let Q denote the geometrical realisation of K. Similarly, let K' be an absolute complex of M' containing all simplexes of $\varkappa'$, and let Q' denote its geometrical realisation. Then the simplexes of $\varkappa\times\varkappa'$ belong to a simplicial decomposition of the polyhedron $Q\times Q'$, and the following conditions are accomplished:

1. *If $\varkappa$ and $\varkappa'$ are cycles, $\varkappa\times\varkappa'$ is also a cycle*[20]).
2. *If $\varkappa$ is a cycle homologous to zero on K, and $\varkappa'$ an arbitrary cycle of K', then $\varkappa\times\varkappa'$ is a cycle homologous to zero on $Q\times Q'$*[21]).
3. *If $\varkappa$ is a cycle not homologous to zero on K, and $\varkappa'$ a cycle not homologous to zero on K', then $\varkappa\times\varkappa'$ is a cycle not homologous to zero on $Q\times Q'$*[22]).

9. A sequence $\{\varkappa_i\}$ of cycles is called a *k-dimensional true cycle of M*, if there exists such a sequence of positive numbers $\{\varepsilon_i\}$ convergent to zero, that $\varkappa_i$ is a k-dimensional ε_i-complex of M. If $\underline{\underline{\varkappa}}^1=\{\varkappa_i^1\}$, $\underline{\underline{\varkappa}}^2=\{\varkappa_i^2\}$ are two true cycles of M, and a^1,a^2 two rational numbers, we denote by $a^1\underline{\underline{\varkappa}}^1+a^2\underline{\underline{\varkappa}}^2$ the true cycle $\{a^1\varkappa_i^1+a^2\varkappa_i^2\}$ on M.

A true cycle $\{\varkappa_i\}$ is homologous to zero on M, whenever there exists a sequence $\{\eta_i\}$ of positive numbers, convergent to zero, and such that $\varkappa_i$ is η_i-homologous to zero on M. If there exists such a positive number ε, that no cycle $\varkappa_i$ is ε-homologous to zero on M, then the true cycle $\{\varkappa_i\}$ is called *totally unhomologous to zero on M*. In view of (7) and the condition 1 of Nr. 8, we infer that if $\underline{\underline{\varkappa}}=\{\varkappa_i\}$ is a k-dimensional true cycle of M, and $\underline{\underline{\varkappa}}'=\{\varkappa'_i\}$ is a k'-dimensional true cycle of M', the sequence $\underline{\underline{\varkappa}}\times\underline{\underline{\varkappa}}'=\{\varkappa_i\times\varkappa'_i\}$ is a $(k+k')$-dimensional true cycle of $M\times M'$.

A true cycle $\underline{\underline{\gamma}}=\{\gamma_i\}$ is called a *Vietoris cycle on M*, if the cycles $\varkappa_i=\gamma_{i+1}-\gamma_i$ constitute a true cycle homologous to zero on M. Since $(\gamma_{i+1}\times\gamma'_{i+1})-(\gamma_i\times\gamma'_i)=\gamma_{i+1}\times(\gamma'_{i+1}-\gamma'_i)+(\gamma_{i+1}-\gamma_i)\times\gamma'_i$,

[19]) H. Freudenthal, *Eine Simplizialzerlegung des Cartesischen Produktes zweier Simplexe*, Fund. Math. **29** (1937), p. 139.

[20]) See, for example, P. Alexandroff and H. Hopf, l. c., p. 304.

[21]) l. c., p. 307.

[22]) l. c., p. 306 and 307.

we infer from 2 of Nr. 8, that the product $\underline{\gamma}\times\underline{\gamma}'$ of two Vietoris cycles $\underline{\gamma}$ and $\underline{\gamma}'$ is also a Vietoris cycle.

The Vietoris cycles $\underline{\gamma}_1,\underline{\gamma}_2,\dots,\underline{\gamma}_n$ are *homologically dependent on* M if there exist rational numbers $a_1,a_2,\dots,a_n$ not all equal to zero, and such that the cycle $a_1\cdot\underline{\gamma}_1+a_2\cdot\underline{\gamma}_2+\dots+a_n\cdot\underline{\gamma}_n$ is homologous to zero on M. The maximal number (finite or not) of the k-dimensional Vietoris cycles homologically independent on M is called the *k-dimensional Betti number of* M. We will denote it by $p_k(M)$.

10. Lemma. *Let $\underline{\varkappa}=\{\varkappa_i\}$ be a k-dimensional true cycle of M, and $\underline{\varkappa}'=\{\varkappa_i'\}$ a k'-dimensional true cycle of M'. If $\underline{\varkappa}$ is totally unhomologous to zero on M, and $\underline{\varkappa}'$ total unhomologous to zero on M', then the $(k+k')$-dimensional true cycle $\underline{\varkappa}\times\underline{\varkappa}'$ is unhomologous to zero on $M\times M'$. On the other hand, if either $\underline{\varkappa}$ is homologous to zero on M, or $\underline{\varkappa}'$ homologous to zero on M', then $\underline{\varkappa}\times\underline{\varkappa}'$ is homologous to zero on $M\times M'$.*

Proof. At first, let us suppose that both cycles $\underline{\varkappa}$ and $\underline{\varkappa}'$ are totally unhomologous to zero. Hence there exists a sequence $\{\varepsilon_i\}$ of positive numbers convergent to zero, and an $\varepsilon>0$ such that $\varkappa_i$ and $\varkappa_i'$ are ε_i-complexes, and that none of them is ε-homologous to zero on M, respectively on M'. We may admit that $\varepsilon_i<\varepsilon$ for $i=1,2,\dots$ Let us denote by M_i (respectively by M_i') a finite subset of M (respectively of M') containing all vertices of $\varkappa_i$ (respectively of $\varkappa_i'$), and such that

(8) *each point $p\in M$ (respectively $p'\in M'$) is distant from M_i (respectively from M_i') by less than ε_i.*

We may consider M_i (respectively M_i') as an absolute complex of M (respectively of M') in which all subsets of diameter $<\varepsilon$ are simplexes. Now, we see at once that the cycle $\varkappa_i$ is unhomologous to zero on M_i, and the cycle $\varkappa_i'$ unhomologous to zero on M_i'.

If the $(k+k')$-dimensional true cycle $\underline{\varkappa}\times\underline{\varkappa}'$ were homologous to zero on $M\times M'$, there would exist such a sequence $\{\eta_i\}$ of positive numbers, that for every i there would exist in $M\times M'$ an η_i-complex λ_i having $\varkappa_i\times\varkappa_i'$ as his boundary. In virtue of (8), we can further assume (replacing, if it is necessary, λ_i by a complex obtained from it by a sufficiently small dislocation of vertices not belonging to $M_i\times M_i'$), that all vertices of λ_i lie on $M_i\times M_i'$. For every i sufficiently great each simplex of λ_i is of diameter $<\varepsilon$. We conclude, from the definition of M_i and M_i', that if $(x_{i0},x_{i0}'),(x_{i1},x_{i1}'),\dots,(x_{im},x_{im}')$ are vertices of a simplex of λ_i, then the points $x_{i0},x_{i1},\dots,x_{im}$ constitute a simplex of the absolute complex M_i, and the points $x_{i0}',x_{i1}',\dots,x_{im}'$ — a simplex of M_i'. Then, denoting by Q_i and Q_i' the geometrical realisations of M_i and M_i', we conclude that the geometrical realisation of λ_i is on the polyhedron $Q_i\times Q_i'$. Thus the cycle $\varkappa_i\times\varkappa_i'=\dot\lambda_i$ would be homologous to zero on $Q_i\times Q_i'$, which is impossible, in view of the property 3 of Nr. 8. Thus, the first part of the lemma is proved.

Suppose now that one of the true cycles $\underline{\varkappa}$ and $\underline{\varkappa}'$ is homologous to zero, for example $\underline{\varkappa}$ is homologous to zero on M. Let ε be a positive number given arbitrarily. Thus for every i sufficiently great the cycle $\varkappa_i$ is homologous to zero on Q_i, hence, in view of property 2 of Nr. 8, $\varkappa_i\times\varkappa_i'$ is homologous to zero on $Q_i\times Q_i'$. Let us notice that $\varkappa_i\times\varkappa_i'$ is a cycle of any simplicial subdivision Σ_i of cellular decomposition of the polyhedron $Q_i\times Q_i'$ into cells being products of simplexes of the absolute complexes M_i and M_i'. We conclude that in the subdivision Σ_i there exists an algebraic complex λ_i having $\varkappa_i\times\varkappa_i'$ as its boundary. Since the diameters of the simplexes of Σ_i are $<\sqrt{2}\cdot\varepsilon_i$, we infer that the true cycle $\{\varkappa_i\times\varkappa_i'\}$ is homologous to zero on $M\times M'$, and the proof of the lemma is finished.

11. Theorem. *If $A\times B$ is in the point (a,b) locally homeomorphic to R_n and $\dim A=m$, then there exists an open and connected neighbourhood U of a in A so that each compact subset E of U, containing a true cycle $\underline{\varkappa}$ of dimension $m-1$ not homologous to zero on E, cuts U.*

Proof. Let U' be an open and connected neighbourhood of a in A, and V an open and connected neighbourhood of b in B. If U' and V are sufficiently small, the product $U'\times V$ is an Euclidean n-dimensional region G. There exists further an open and connected neighbourhood $U\subset U'$ of a in A, so that every compact subset of U is homotopically deformable to a on U'. We conclude that if E is a compact subset of U containing a $(m-1)$-dimensional true cycle $\underline{\varkappa}$ not homologous to zero on E, there exists a compact subset X of U', so that $E\subset X$ and $\underline{\varkappa}$ is homologous to zero on X.

Let $\Delta^0(C)$ denote, for every locally compact space C, the Alexandroff-Pontrjagin dimension modulo 0 of C. We have

$\Delta^0(G)=\dim G=n;\quad \Delta^0(U')\leqslant\dim A=m;\quad \Delta^0(U')+\Delta^0(V)=\Delta^0(G)$ [23].

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[23]) See L. Pontrjagin, C. R. Paris **190** (1930), p. 1105-7.

Consequently we have $\Delta^0(V)=n-\Delta^0(U')\geqslant n-m$. Thus there exist in V such compact subsets Y and $F\subset Y$, that F contains a true cycle $\underline{\varkappa}'$ of dimension $n-m-1$ totally unhomologous to zero on F, and homologous to zero on Y. By the lemma **10**, $\underline{\varkappa}\times\underline{\varkappa}'$ is a true cycle of $E\times F$ of dimension $m-1+(n-m-1)=n-2$ which is unhomologous to zero on $E\times F$, and homologous to zero on every one of the sets $X\times F$ and $E\times Y$. By the theorem of Phragmen-Brouwer, there exists in the set

$$C=(X\times F)+(E\times Y)\subset G$$

a true cycle of dimension $n-2+1=n-1$ unhomologous to zero on C [24]). It follows [25]) that C cuts R_n and consequently it cuts also the region G. Hence there exist in $G=U'\times V$ two such points (α,β) and (α',β') that C cuts G between them. We can suppose at once that $\beta'\epsilon V-F$. Since if $\beta'\epsilon F$ then $\alpha'\epsilon U'-X$ and we can carry β' beyond P without cutting C by (α',β') during this dislocation.

Now let us suppose that E does not cut U' and prove on this assumption, that the points (α,β) and (α',β') may be chosen in such a manner that

(9) $$\alpha,\alpha'\epsilon U'-X.$$

Indeed, if $\beta\epsilon V-Y$, then we can carry α beyond X without cutting C by (α,β) during this dislocation of α. If, however, $\beta\epsilon Y$ and $\alpha\epsilon X$, then $\alpha\epsilon U'-E$ and $\beta\epsilon Y-F$. Now, we choose a point $\overline{\alpha}\epsilon U'-X$ in an arbitrary manner and we carry α to $\overline{\alpha}$ in $U'-E$. During this dislocation of α the point (α,β) will be beyond C, since α remains in $U'-E$ and $\beta\epsilon Y-F$. Thus we can replace (α,β) by $(\overline{\alpha},\beta)$ with $\overline{\alpha}\epsilon U'-X$, without change of (α',β'). In the same manner (α',β') may be replaced by $(\overline{\alpha}',\beta')$ with $\overline{\alpha}'\epsilon U'-X$, without changing $(\overline{\alpha},\beta)$. Consequently, we may admit that (9) is accomplished.

Now let us denote by I a continuum joining α with α' in $U'-E$ and by J a continuum joining β with $\beta'\epsilon V-F$ in V. The set

$$D=\big((\alpha)\times J\big)+\big(I\times(\beta')\big)+\big((\alpha')\times J\big)$$

is evidently a subcontinuum of $(U'\times V)-C$, joining (α,β) with (α',β'), which is impossible, since C cuts $U'\times V$ between these points.

[24]) See K. Borsuk, *Über sphäroidale und H-sphäroidale Räume*, Recueil Mathém. Moscou **1** (**43**), 1936, p. 646.

[25]) See P. Alexandroff, *Dimensionstheorie*, Math. Ann. **106** (1932), p. 185.

Therefore the assumption that E does not cut U' leads to a contradiction. Hence E cuts U'. There exist in $U'-E$ two different components W_1 and W_2. Since U' is connected and locally connected, each of them contains on its boundary the points of the compact set $E\subset U$ [26]), consequently there exist two points $p_1\epsilon W_1\cdot U-E$ and $p_2\epsilon W_2\cdot U-E$, between which E cuts U.

Thus our theorem is proved.

12. It is easy to see, that in the case when $A\times B$ is homeomorphic to R_n, the latter proof holds if we put $U=A$. Consequently we can formulate the following

Theorem. *If $A\times B$ is homeomorphic to R_n and $\dim A=m$, then each compact subset E of A containing a true cycle $\underline{\varkappa}$ of dimension $m-1$ not homologous to zero on E cuts A.*

13. ***Theorem.*** *Every 2-dimensional topological divisor of an open subset of R_n is locally homeomorphic to R_2.*

Proof. Let A be a 2-dimensional topological divisor of an open subset of R_n, and a some point of A. By the theorem **11**, there exists an open and connected neighbourhood U of a in A so that each compact subset E of U, containing a true 1-dimensional cycle $\{\varkappa_i\}$ not homologous to zero on E, cuts U. In particular:

(10) *Every simple closed curve Ω lying in U cuts U.*

On the other hand, it results from the corollary **6** that no closed subset E of Ω, different from Ω, cuts U. Then Ω is an irreducible cut of U. We conclude [27]):

(11) *Every simple closed curve Ω lying in U is common boundary in U for every component of $U-\Omega$.*

Now, to prove our theorem, we apply a theorem of L. Zippin [28]), by which, in order that a metric locally connected and locally compact space U be a 2-dimensional Euclidean region, it is necessary

[26]) C. Kuratowski, *Une définition topologique de la ligne de Jordan*, Fund. Math. **1** (1920), p. 43.

[27]) C. Kuratowski, *Sur les coupures irréductibles du plan*, Fund. Math. **6** (1924), p. 133.

[28]) L. Zippin, *On Continuous Curves and the Jordan Curve*, Amer. Journ. of Math. **52** (1930), p. 340.

and sufficient, that U should satisfy besides both conditions (10) and (11) the following condition:

(12) *$U-\Omega$ contains exactly two components.*

In order to prove this last condition, we will show that (12) is a consequence of (10) and (11). More precisely, we will show:

(13) *Let E be a connected, locally connected and complete space so that every simple closed curve $\Omega\subset E$ is an irreducible cut of E. Then $E-\Omega$ contains exactly two components.*

Let us suppose that (13) is false. There exist in $E-\Omega$ besides two different components G_1, G_2 other components; their sum be designated by G_3. Since the boundary of every component of $E-\Omega$ is identical with Ω, there exist two simple arcs L_1 and L_2 having all their endpoints on Ω and their innerpoints respectively on G_1 and G_2. There exists in $L_1+\Omega$ a simple arc $L_1'\subset L_1$ so that $\Omega'=L_1'+L_2$ is a simple closed curve. Every component of G_1-L_1 has its boundary contained in $\Omega+L_1=\Omega+L_1'$, but not contained in L_1', because no subset of L_1' cuts E. Consequently, every point of G_1-L_1 can bec onnected in G_1-L_1 with Ω outside of L_1'. Similarly, every component of G_2-L_2 has its boundary contained in $\Omega+L_2$, but not contained in the set $L_1'\cdot\Omega+L_2=(L_1'-G_1)+L_2$, being a true subset of Ω'. Consequently every point of G_2-L_2 can be connected in G_2-L_2 with Ω outside of L_1'. All points of Ω being on the boundary of every component of G_3, we conclude that the set

$$G_3+(G_1-L_1')+(G_2-L_2)+(\Omega-L_1')=G_3+[(G_1+G_2+\Omega)-\Omega']=E-\Omega'$$

is connected, which is impossible, since Ω' cuts E. Thus (13), and consequently also our theorem, are proved.

14. In the case when $A\times B$ is homeomorphic to R_n, we may apply the theorem **12** instead of the theorem **11**, and replace in the proof of Nr. **13** the neighbourhood U by A. In this manner we conclude that A is a 2-dimensional Euclidean region. On the other hand, the set A, as a topological divisor of R_n, is *unicoherent*[29]). Consequently, A is homeomorphic to R_2. Thus we arrive to the following

Theorem. *The only one 2-dimensional topological divisor of an Euclidean space is the Euclidean plane.*

[29]) That is, by all decomposition of A into two connected closed subsets A_1 and A_2, the set $A_1\cdot A_2$ is connected. It is evident that each topological divisor of any unicoherent space is also unicoherent.

15. From the theorem 13 there follows:

Corollary 1. *Every 2-dimensional topological divisor of a n-dimensional manifold is a surface*[30]).

Corollary 2. *If $E=A\times B$ is in the point $p=(a,b)$ locally homeomorphic to the Euclidean n-dimensional space R_n, and if a has in A arbitrarily small neighbourhoods of dimension 2, then b has in B arbitrarily small neighbourhoods of dimension $n-2$.*

It follows evidently from our hypothesis, that there exist arbitrarily small open and connected neighbourhoods of a in A and of b in B, and that every sufficiently small neighbourhood of a in A is of dimension 2. On the other hand, for every sufficiently small open and connected neighbourhood U of a in A and V of b in B, the product $U\times V$ is an Euclidean n-dimensional region. Since $\dim U\leqslant 2$, we have

(14) $$\dim V\geqslant n-2.$$

By the theorem 13, U is a 2-dimensional Euclidean region. Hence there exists a neighbourhood $U_0\subset U$ of a in A being a product of two simple open arcs L_1 and L_2. There $L_1\times L_2\times V$ is a n-dimensional Euclidean region. It follows[10]) that

(15) $$\dim V\leqslant n-2.$$

Both relations (14) and (15) give our corollary.

16. We shall now investigate the algebraic side of the problem of the decomposition into products, by establishing a simple relation between the Betti numbers of space and the Betti numbers of its topological divisors. We begin by some elementar lemmas.

Let P and $P'\subset P$ be two polyhedrons. A cycle α (with rational coefficients) of P will be called *P'-cycle on P*, if there exists in P' a cycle α' homologous to α on P.

Cycles $\alpha_1,\alpha_2,\dots,\alpha_k$ of P will be called *homologically independent on P*, whenever a homology of the form $a_1\alpha_1+a_2\alpha_2+\dots+a_k\alpha_k\sim 0$ on P, with rational coefficients a_i, implies that all a_i vanish.

A set $\alpha_1,\alpha_2,\dots,\alpha_k$ of P'-cycles homologically independent on P will be called a *P'-system on P*. This P'-system is *complete in the dimension r*, if all cycles α_i are r-dimensional and every r-dimensional P'-cycle on P is homologous on P to some linear combination (with

[30]) That is a 2-dimensional manifold.

rational coefficients) of $a_1, a_2, \dots, a_k$. By omission in this definition of the hypothesis concerning the dimension, we obtain the notion of the *complete P'-system on P*.

It is easy to see that the number k of the cycles constituting a complete r-dimensional P'-system on P is equal to the rank of the factor-group $\mathfrak{L}'_r/\mathfrak{H}'_r$, where $\mathfrak{L}'_r$ denotes the group of r-dimensional P'-cycles on P, and $\mathfrak{H}'_r$ its subgroup constitued by the cycles homologous to zero on P.

Hence all complete r-dimensional P'-systems on P are equally numerous. Similarly all complete P'-systems on P are equally numerous.

17. Let $P, Q, P' \subset P$ and $Q' \subset Q$ be polyhedrons. We will now prove the two following lemmas:

Lemma 1. *If $a_1, a_2, \dots, a_k$ is a P'-system on P and $\beta_1, \beta_2, \dots, \beta_l$ a Q'-system on Q, then the cycles $a_i \times \beta_j$ $(i=1,2,\dots,k;\ j=1,2,\dots,l)$ constitute a $(P' \times Q')$-system on $P \times Q$.*

Proof. Since two homologies $a \sim a'$ on P and $\beta \sim \beta'$ on Q imply the homology $a \times \beta \sim a' \times \beta'$ on $P \times Q$ [31]), we see that the cycles $a_i \times \beta_j$ are $(P' \times Q')$-cycles on $P \times Q$. On the other hand, the homological independence of the cycles a_i on P and the cycles β_j on Q imply [32]) that the cycles $a_i \times \beta_j$ are homologically independent on $P \times Q$.

Lemma 2. *Let $a_1^r, a_2^r, \dots, a_{k}^r$ denote (for $r=0,1,\dots$) a complete r-dimensional P'-system on P, and $\beta_1^s, \beta_2^s, \dots, \beta_{l_s}^s$ denote (for $s=0,1,\dots$) a complete s-dimensional Q'-system on Q. Then the cycles $a_i^r \times \beta_j^s$, with $r+s=t$; $i=1,2,\dots,k_r$ and $j=1,2,\dots,l_s$ constitute a complete t-dimensional $(P' \times Q')$-system on $P \times Q$, and the cycles $a_i^r \times \beta_j^s$, with $i=1,2,\dots,k_r$; $j=1,2,\dots,l_s$ and r and s arbitrary, constitute a complete $(P' \times Q')$-system on $P \times Q$.*

Proof. By hypothesis, the cycles $a_1^0, a_2^0, \dots, a_{k_0}^0, a_1^1, a_2^1, \dots, a_{k_1}^1, a_1^2, \dots$ constitute a P'-system on P, and the cycles $\beta_1^0, \beta_2^0, \dots, \beta_{l_0}^0, \beta_1^1, \beta_2^1, \dots, \beta_{l_1}^1, \beta_1^2, \dots$ a Q'-system on Q. It results, in view of the preceding lemma, that the cycles $a_i^r \times \beta_j^s$, with $0 \leqslant i \leqslant k_r$; $0 \leqslant j \leqslant l_s$; $r, s = 0,1,\dots$ constitute a $(P' \times Q')$-system on $P \times Q$. In particular, the cycles $a_i^r \times \beta_j^s$ with $r+s=t$ constitute a t-dimensional $(P' \times Q')$-system on $P \times Q$. Then it remains to prove, that every t-dimensional cycle γ lying in $P' \times Q'$ is homologous on P to some linear combination of the last cycles.

Obviously we can suppose that all P'-cycles a_i^r lie in P' and all Q'-cycles β_j^s lie in Q'. The cycles $a_1^r, a_2^r, \dots, a_{k_r}^r$ being homologically independent on P and constituting a complete P'-system, we see at once that this system can be completed by some r-dimensional cycles homologous to zero on P to some system $a_1^r, a_2^r, \dots, a_{m_r}^r$, $(m_r \geqslant k_r)$ homologically independent on P' and such that every r-dimensional cycle lying in P' is homologous on P' to some linear combination of $a_1^r, a_2^r, \dots, a_{m_r}^r$. Similarly, the system $\beta_1^s, \beta_2^s, \dots, \beta_{l_s}^s$ can be completed by some s-dimensional cycles homologous to zero on Q to some system $\beta_1^s, \beta_2^s, \dots, \beta_{n_s}^s$, $(n_s \geqslant l_s)$ of s-dimensional cycles in Q', homologically independent on Q' and such that every s-dimensional cycle lying in Q' is homologous on Q' to some linear combination of $\beta_1^s, \beta_2^s, \dots, \beta_{n_s}^s$. We conclude [32]) that every t-dimensional cycle γ lying in $P' \times Q'$ is homologous on $P' \times Q'$ to some linear combination Γ of the cycles $a_i^r \times \beta_j^s$, where $i \leqslant m_r$; $j \leqslant n_s$; $t=r+s$. Since $a_i^r \sim 0$ on P for $i > k_r$, and $\beta_j^s \sim 0$ on Q for $j > l_s$, we infer [32]) that all cycles $a_i^r \times \beta_j^s$ with $i > k_r$ or $j < l_s$ are homologous to zero on $P \times Q$. Consequently, we can cancel in Γ these cycles and we arrive in this manner to the linear combination Γ' of the cycles $a_i^r \times \beta_j^s$ with $r \leqslant k_r$, $s \leqslant l_s$, which is homologous to γ on $P \times Q$.

Since each cycle is a sum of cycles of homogenous dimension, we infer at once that the cycles $a_i^r \times \beta_j^s$ with $i \leqslant k_r$; $j \leqslant l_s$ and r, s arbitrary, constitute a complete $(P' \times Q')$-system in $P \times Q$. Then the lemma 2 is proved.

18. A continuous transformation φ of a space E is called *retraction*, if $\varphi(E) \subset E$ and $\varphi(p)=p$ for every $p \in \varphi(E)$. Then we say that $\varphi(E)$ is a *retract of E*. It is known that many topological properties of E pass from E on its retracts. So it is for example with separability, compactness, local compactness, connectedness, local connectedness and so on. If $E = A \times B$ and $a \in A$, we obtain a retraction of E if we put $\varphi(x,y)=(a,y)$ for every $(x,y) \in A \times B$. Since the set $\varphi(E)=(a,B)$ is a homeomorph of B, we infer that *every topological divisor of E is homeomorphic to some retract of E*.

[31]) See P. Alexandroff and H. Hopf, l. c., p. 304, formula (4).

[32]) l. c., p. 306 and 307, formula (10) and (11). In that book are investigated merely the cycles with integral coefficients. But the cycles with rational coefficients, which appear to us, differ from the latter merely by a numerous coefficient, which has no influence on its independence.

A compact set A is called an *absolute neighbourhood-retract*, if for every space $E \supset A$ it is a retract of some neighbourhood U of A in E. In particular, all polyhedrons are absolute neighbourhood-retracts[33]). It is known[34]) that the property „being an absolute neighbourhood-retract" is a topological invariant, and that every retract of an absolute neighbourhood-retract is also an absolute neighbourhood-retract. In view of the preceding we infer that all topological divisors of any absolute neighbourhood-retract are also absolute neighbourhood-retracts. On the other hand, the product of two absolute neighbourhood-retracts is also an absolute neighbourhood-retract[35]).

19. Let us denote by Q_ω the Hilbert parallelotype[36]) and by φ_n the following transformation of Q_ω:

(16) $$\varphi_n(x_1, x_2, \dots, x_n, x_{n+1}, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots).$$

Then φ_n is a retraction of Q_ω into the n-dimensional parallelotype $Q_n = \varphi_n(Q_\omega)$.

Lemma. *Let P and $P' \subset P$ be two polyhedrons lying in Q_ω, and r a retraction of $P \times Q_\omega$ to a subset C of $P' \times Q_\omega$, so that for every $p \in P' \times Q_\omega$ the segment $\overline{pr(p)}$ lies in $P \times Q_\omega$. Then the m-dimensional Betti number of C is equal to the number k of the cycles constituting a complete m-dimensional P'-system on P.*

Proof. We can identify every point y of P with the point $(y; 0, 0, \dots) \in P \times Q_\omega$. Let us put, at once, for every

$$p = (y; x_1, x_2, \dots, x_n, \dots) \in P \times Q_\omega$$

and $0 \leqslant t \leqslant 1$

$\varrho_t(p) = (y; (1-t)\cdot x_1, (1-t)\cdot x_2, \dots, (1-t)\cdot x_n, \dots)$,

$r_t(p) =$ point dividing the segment $\overline{pr(p)}$ in the relation $t/(1-t)$.

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[33]) K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), p. 227, **10**.

[34]) l. c., p. 223, 2.

[35]) l. c., p. 226, 9.

[36]) The *Hilbert parallelotype* is the compact subset of the Hilbert space composed by all sequences $\{x_i\}$ with $0 \leqslant x_i \leqslant 1/i$, $i = 1, 2, \dots$

It is evident that ϱ_t is a continuous deformation of the product $P \times Q_\omega$ on itself into the polyhedron P, during which the product $P' \times Q_\omega$ is deformed on itself into P'. Similarly we see that r_t is a continuous deformation of the product $P' \times Q_\omega$ on $P \times Q_\omega$ into the set C.

Let γ be a cycle, its simplexes belonging to some simplicial decomposition of P. By barycentrical subdivision of its simplexes we obtain the cycle $\gamma^{(1)}$ homologous to γ on P. The iteration of this process leads to a sequence $\gamma^{(1)}, \gamma^{(2)}, \dots$ of the cycles, constituting a Vietoris cycle on P, of the same dimension as γ. We denote its Vietoris cycle by V^γ. It is evident that

If the cycle γ is on P', the Vietoris cycle V^γ is also on P'.

(17) *If γ, γ' are two cycles of P and a, a' two rational numbers, then* $V^{a\gamma + a'\gamma'} = a \cdot V^\gamma + a' \cdot V^{\gamma'}$.

(18) *If $\gamma \sim \gamma'$ on P, then $V^\gamma \sim V^{\gamma'}$ on P and vice versa.*

We see at once that

(19) *For every n-dimensional Vietoris cycle Γ of $P' \times Q_\omega$, there exists in P' a n-dimensional cycle γ so, that $\Gamma \sim V^\gamma$ on $P' \times Q_\omega$.*

For, by the deformation ϱ_t the Vietoris cycle Γ is carried to some Vietoris cycle on P' homologous to Γ on $P' \times Q_\omega$. Then there exist n-dimensional cycles γ'_i of P' so that $\Gamma' = \{\gamma'_i\} \sim \Gamma$ on $P' \times Q_\omega$. We can assume at once (replacing, if it is necessary, γ'_i by a cycle obtained from it by a sufficiently small dislocation of its vertices), that the simplexes of γ'_i belong to some simplicial decomposition of P. But for an index i sufficiently great, the cycles γ'_i are mutually homologous in P'. Hence it suffices to put $\gamma = \gamma'_i$, with i sufficiently great, in order to obtain $\Gamma \sim \Gamma' \sim V^\gamma$ on $P' \times Q_\omega$.

(20) *If the cycles $\gamma_1, \gamma_2, \dots, \gamma_k$ are homologically independent on P, then the Vietoris cycles $V^{\gamma_1}, V^{\gamma_2}, \dots, V^{\gamma_k}$ are also homologically independent on $P \times Q_\omega$.*

For the relation $a_1 \cdot V^{\gamma_1} + a_2 \cdot V^{\gamma_2} + \dots + a_k V^{\gamma_k} \sim 0$ on $P \times Q_\omega$ leads by the deformation ϱ_t to the relation $a_1 \cdot V^{\gamma_1} + a_2 \cdot V^{\gamma_2} + \dots + a_k \cdot V^{\gamma_k} \sim 0$ on P and, by (17), to the relation

$$V^{a_1\gamma_1 + a_2\gamma_2 + \dots + a_k\gamma_k} \sim 0 \text{ on } P.$$

In view of (18), we conclude that $a_1 \cdot \gamma_1 + a_2 \cdot \gamma_2 + \dots a_k \cdot \gamma_k \sim 0$ on P and, since $\gamma_1, \gamma_2, \dots, \gamma_k$ are homologically independent, we infer that all coefficients a_i vanish.

(21) *If every n-dimensional P'-cycle of P is homologous on P to some linear combination of the cycles $\gamma_1, \gamma_2, \dots, \gamma_k$, then every n-dimensional Vietoris cycle Γ of $P' \times Q_\omega$ is homologous on $P \times Q_\omega$ to some linear combination of $V^{\gamma_1}, V^{\gamma_2}, \dots, V^{\gamma_k}$.*

For by (19) there exists in P' an n-dimensional cycle γ so that $\Gamma \sim V^{\gamma}$ on $P' \times Q_\omega$. By hypothesis there exist rational coefficients $a_1, a_2, \dots, a_k$ so that $\gamma \sim a_1 \cdot \gamma_1 + a_2 \cdot \gamma_2 + \dots a_k \cdot \gamma_k$ on P. In view of (18) and (17) we conclude that

$$\Gamma \sim V^{\gamma} \sim V^{a_1 \cdot \gamma_1 + a_2 \cdot \gamma_2 + \dots + a_k \cdot \gamma_k} = a_1 \cdot V^{\gamma_1} + a_2 \cdot V^{\gamma_2} + \dots + a_k \cdot V^{\gamma_k}$$

on $P \times Q_\omega$.

Let now $\gamma_1, \gamma_2, \dots, \gamma_k$ be a complete m-dimensional P'-system on P. Then there exist m-dimensional cycles $\gamma'_1, \gamma'_2, \dots, \gamma'_k$ in P' so that $\gamma_i \sim \gamma'_i$ on P for $i = 1, 2, \dots, k$. Then $\gamma'_1, \gamma'_2, \dots, \gamma'_k$ is also a complete m-dimensional P'-system on P. The corresponding m-dimensional Vietoris cycles $V^{\gamma'_1}, V^{\gamma'_2}, \dots, V^{\gamma'_k}$ lie on P' and constitute, by (20) and (21), a homologically independent system, so that every m-dimensional Vietoris cycle Γ of $P' \times Q_\omega$ is homologous on $P \times Q_\omega$ to some linear combination of the form $a_1 \cdot V^{\gamma'_1} + a_2 \cdot V^{\gamma'_2} + \dots + a_k \cdot V^{\gamma'_k}$. Let us denote by $V_r^{\gamma'_i}$ the Vietoris cycle in which the retraction r maps $V^{\gamma'_i}$. If we suppose that the Vietoris cycle Γ is on C, the retraction r leaves it invaried, and the homology

$$\Gamma \sim a_1 \cdot V^{\gamma'_1} + a_2 \cdot V^{\gamma'_2} + \dots + a_k \cdot V^{\gamma'_k}$$

leads to the homology

(22) $$\Gamma \sim a_1 \cdot V_r^{\gamma'_1} + a_2 \cdot V_r^{\gamma'_2} + \dots + a_k \cdot V_r^{\gamma'_k} \quad \text{on} \quad C.$$

Thus we have proved that every m-dimensional Vietoris cycle on C is homologous on C to some linear combination of the m-dimensional Vietoris cycles $V_r^{\gamma'_1}, V_r^{\gamma'_2}, \dots, V_r^{\gamma'_k}$. Hence the m-dimensional Betti number of C is not greater than k.

On the other hand, the Vietoris cycles $V_r^{\gamma'_i}$ can be obtained from $V^{\gamma'_i}$ by the continuous deformation r_t. Consequently $V_r^{\gamma'_i} \sim V^{\gamma'_i}$ on $P \times Q_\omega$. Since $V^{\gamma'_1}, V^{\gamma'_2}, \dots, V^{\gamma'_k}$ are homologically independent on $P \times Q_\omega$, the Vietoris cycles $V_r^{\gamma'_i}$ are homologically independent on $P \times Q_\omega$ and consequently also on C. Hence the m-dimensional Betti number of C is not smaller than k. This completes the proof of the lemma.

20. *Theorem.* *If A and B are two absolute neighbourhood retracts, then $p_m(A \times B) = \sum_{k=0}^{m} p_k(A) \cdot p_{m-k}(B)$ for every $m = 0, 1, \dots$*

Proof. We can assume that A and B are subsets of the Hilbert parallelotype Q_ω [37]. The transformation φ_n, defined by the formula (16), maps Q_ω into an n-dimensional parallelotype $Q_n = \varphi_n(Q_\omega)$, and A into some closed subset $\varphi_n(A)$ of Q_n. Let A' denote a subpolyhedron of Q_n, constituing a neighbourhood of $\varphi_n(A)$ in Q_n so that every point of A'_n is distant from $\varphi_n(A)$ less than $1/n$. It is evident that the sets

$$A''_n = \underset{p \in Q_\omega}{E} [\varphi_n(p) \in A'_n]$$

are neighbourhoods of A in Q_ω, and that

(23) $$\lim_{n=\infty} A''_n = A.$$

Similarly, there exists for every $n = 1, 2, \dots$ a polyhedron $B'_n \subset Q_n$ so that the sets

$$B''_n = \underset{p \in Q_\omega}{E} [\varphi_n(p) \in B'_n]$$

are neighbourhoods of B in Q_n, and that

(24) $$\lim_{n=\infty} B''_n = B.$$

By hypothesis, A and B are absolute neighbourhood retracts. Then we infer from (23) and (24) that for a sufficiently great $n = n_0$ there exists a retraction φ of A''_{n_0} into A and a retraction ψ of B''_{n_0} into B. But A''_{n_0} is a neighbourhood of A and B''_{n_0} of B. It follows, in view of (23) and (24) that, for a sufficiently great natural k_0, we have

(25) $$\overline{p\varphi(p)} \subset A''_{n_0} \quad \text{for every} \quad p \in A''_{n_0+k_0},$$
$$\overline{q\psi(q)} \subset B''_{n_0} \quad \text{for every} \quad q \in B''_{n_0+k_0}.$$

[37]) By the known „Einbettungssatz" of P. Urysohn. See, for example, P. Alexandroff and H. Hopf, l. c., p. 81.

Now let us notice that the set $A''_{n_0+k_0}$ can be considered as the product of the polyhedron $P'=A'_{n_0+k_0}$ and of some subset Q^*_ω of the Hilbert space, homeomorphic to Q_ω itself. On the other hand, the set A''_{n_0} can be considered as the product of the polyhedron $P\supset P'$, being a product of A'_{n_0} and k_0 segments, and the set Q^*_ω itself. Similarly the set B''_{n_0} is a product of some polyhedron Q by Q^*_ω, and the set $B''_{n_0+k_0}$ is a product of $Q'=B'_{n_0+k_0}\subset Q$ by Q^*_ω. Thus, we infer from (25) and the lemma **19** that the Betti numbers of the sets $A=\varphi(A''_{n_0})$ and $B=\psi(B''_{n_0})$ are given by the relations:

(26) $p_m(A)=$ *the number of the cycles constituting a complete m-dimensional P'-system on P.*

(27) $p_m(B)=$ *the number of the cycles constituting a complete m-dimensional Q'-system on Q.*

Now, let us put

$$\Theta(p)=[\varphi(x),\psi(y)] \quad \text{for} \quad p=(x,y)\in A''_{n_0}\times B''_{n_0}.$$

Evidently Θ is a retraction of $A''_{n_0}\times B''_{n_0}$ into $A\times B$ and, in view of (25), we have

$$\overline{p\Theta(p)}\subset A''_{n_0}\times B''_{n_0} \quad \text{for every} \quad p\in A''_{n_0+k_0}\times B''_{n_0+k_0}.$$

We see at once that there exists a homeomorphism transforming $A''_{n_0}\times B''_{n_0}$ into the product of the polyhedron $P\times Q$ and of the set $Q^*_\omega\times Q^*_\omega$ in such a manner, that the subset $A''_{n_0+k_0}\times B''_{n_0+k_0}$ of $A''_{n_0}\times B''_{n_0}$ gets into the product of the polyhedron $P'\times Q'\subset P\times Q$ and the set $Q^*_\omega\times Q^*_\omega$ itself. Thus, the lemma **19** leads us to the following conclusion:

(28) $p_m(A\times B)=$ *the number of the cycles constituting a complete m-dimensional $(P'\times Q')$-system on $P\times Q$.*

By confrontation of the relations (26), (27) and (28) with the lemma 2 of Nr. **17**, we obtain our theorem.

21. Let us put, for every compact space A and complex number x,

$$\mathfrak{P}_A(x)=\sum_{n=0}^{\infty}p_n(A)\cdot x^n;$$

$\mathfrak{P}_A(x)$ will be called the *characteristic series of A*. If all $p_n(A)$ are finite and almost all vanish, $\mathfrak{P}_A(x)$ is a polynomial — the *characteristic polynomial of A*. In particular, we have this last eventuality in the case when A is an absolute neighbourhood retract [38]). Evidently, all coefficients of each characteristic polynomial are non negative integral numbers.

Using the notion of the characteristic polynomial, we can give to the theorem **20** the following very simple form:

Theorem. *The characteristic polynomial of the product of two absolute neighbourhood retracts is identical with the product of the characteristic polynomials of factors.*

22. Two compact spaces with the same characteristic series will be called *homologically affined*. In particular, compact spaces homologically affined with the point coincide with those which have the characteristic polynomial identical with 1.

Since, as we have already noticed, all topological divisors of any absolute neighbourhood retract are also absolute neighbourhood retracts, we obtain, from the theorem **21**, the following

Corollary 1. *Topological divisors of any absolute neighbourhood retract A are absolute neighbourhood retracts, their characteristic polynomials are divisors of $\mathfrak{P}_A(x)$.*

Corollary 2. *If an absolute neighbourhood retract has real topological divisors not affined with a point, then its characteristic polynomial is decomposable into the product of two polynomials $\neq 1$ having natural coefficients* [39]).

23. Among the absolute neighbourhood retracts are contained all polyhedrons and also all manifolds [40]). Then the just obtained results concern in particular the decomposition of these important sets into products. In particular, we may apply these results to investigate the decomposition of the n-dimensional manifolds into products. We will prove the following

[38]) S. Lefschetz, *On locally connected and related sets*, Annals of Math. **35** (1934), p. 128. See also K. Borsuk, *Zur kombinatorischen Eigenschaften der Retrakte*, Fund. Math. **21** (1933) p. 98.

[39]) It is to be noticed, that there exist polynomials decomposable in more than one manner into product of polynomials, which are not decomposable in this manner. For example, we have $(1+x)\cdot(1+x+4x^3)\equiv(1+2x)(1+x^2+2x^3)$, while none of the factors on both sides of this identity is decomposable into product of polynomials with natural coefficients.

[40]) See S. Lefschetz, l. c., p. 121, th. II and K. Borsuk, l. c., p. 227, **10** and p. 240, **32**.

Theorem. No n-dimensional manifold M can be decomposed in two different manners[41]) *into a product of topologically first sets of dimension* $\leqslant 2$.

Proof. In view of the corollary 2 of Nr. **4** and the corollary 1 of Nr. **15**, it remains to prove that a decomposition of M into product of simple closed curves and topologically first surfaces is possible in no more than one manner[41]). Let α denote the number of closed curves (that is of terms homeomorphic to the circumference S) appearing in the given product-decomposition of M, β — the number of terms homeomorphic to the *projective plane* P[42]), and γ — the number of terms homeomorphic to the so called *tube of Klein* K[43]). The other terms are either orientable surfaces $M_1, M_2, \ldots, M_\delta$ different from the torus-surface (which is decomposable into a product of two simple closed curves) or the non-orientable surfaces $N_1, N_2, \ldots, N_\eta$, topologically different from P and Q. Thus M is homeomorphic to the product

$$S^\alpha \times P^\beta \times K^\gamma \times \prod_{i=1}^{\delta} M_i \times \prod_{j=1}^{\eta} N_j. \tag{29}$$

Now, let us notice that the Betti numbers of the orientable surface with the Euler-Poincaré characteristic χ are $1, 2-\chi, 1, 0, \ldots$ and those of the non-orientable surface, with the same Euler-Poincaré characteristic χ, are $1, 1-\chi, 0, 0, \ldots$[44]). Consequently the one- and two-dimensional topological divisors of M are topologically determined by their Betti numbers, except in the two following cases: 1° the homeomorphs of projective plane P, being homologically affined with the point, 2° the homeomorphs of the tube of Klein K and of the circumference S, being homologically affined with one another. In other words, we have $\mathfrak{P}_P(x) = 1$ and

[41]) That is, the not one-puncting terms in two such decompositions of M can differ only by their order.

[42]) *Projective plane* is the non-orientable surface with the Euler-Poincaré characteristic $\chi = 1$.

[43]) *Tube of Klein* is the non-orientable surface with the Euler-Poincaré characteristic $\chi = 0$.

[44]) See, for example, P. Alexandroff and H. Hopf, l. c., p. 269.

$\mathfrak{P}_K(x) = \mathfrak{P}_S(x) = 1 + x$. The characteristic polynomials of the remaining topological divisors of M are the following:

$$\mathfrak{P}_{M_i}(x) = 1 + p_1(M_i)\cdot x + x^2, \quad \text{with} \quad p_1(M_i) \neq 2,$$
$$\mathfrak{P}_{N_j}(x) = 1 + p_1(N_j)\cdot x, \quad \text{with} \quad 0 \neq p_1(N_j) \neq 1.$$

M being homeomorphic to the product (29), we conclude from this and from the theorem **21** that

$$\mathfrak{P}_M(x) = (1+x)^{\alpha+\gamma} \cdot \prod_{i=1}^{\delta}[1 + p_1(M_i)\cdot x + x^2] \cdot \prod_{j=1}^{\eta}[1 + p_1(N_j)\cdot x]. \tag{30}$$

Now it is easy to notice that every factor in this product is indecomposable in the body of all rational numbers. Then the decomposition of the form (30) is determined in one manner only. Consequently, the terms $M_1, M_2, \ldots, M_\delta$ and $N_1, N_2, \ldots, N_\eta$ are topologically determined by M. It remains to show that the numbers α, β and γ are also determined by M. For this purpose let us observe that the degree m of the polynomial $\mathfrak{P}_M(x)$ satisfies the relation

$$m = (\alpha + \gamma) + 2\delta + \eta.$$

But the numbers m, δ and η are determined by the polynomial $\mathfrak{P}_M(x)$. Thus the number $c_1 = m - 2\delta - \eta$ is determined by M, and we have

$$\alpha + \gamma = c_1. \tag{31}$$

Another relation between the numbers α, β and γ follows from the relation between the dimension n of M and the dimension of topological divisors of M. We have namely $n = \alpha + 2\beta + 2\gamma + 2\delta + 2\eta$. Then, denoting by c_2 the number $n - 2(\delta + \eta)$ determined by M, we obtain

$$\alpha + 2\beta + 2\gamma = c_2. \tag{32}$$

In order to obtain a third relation between α, β and γ, let us observe that the Künneth formula[45]) determining the Betti groups of the product of two polyhedrons implies that the 1-dimensional Betti group of product of connected polyhedrons is isomorphic to the direct product of the 1-dimensional Betti groups of factors. Consequently, the 1-dimensional group of torsion $T_1(M)$ of M is

[45]) See, for example, P. Alexandroff and H. Hopf, l. c., p. 308.

isomorphic to the direct product of the 1-dimensional groups of torsion of the divisors of M. But, as we have already shown, the product $M_0=\prod_{i=1}^{\delta} M_i \times \prod_{j=1}^{\eta} N_j$ is determined by M, hence his 1-dimensional group of torsion $T_1(M_0)$ is also determined. We have at once[46]: $T_1(S)=0$, $T_1(P)=T_1(K)=G_2$, where G_2 denotes the group of the rest modulo 2, containing two elements 0, and 1. Therefore

$$T_1(M)=[G_2]^{\beta+\gamma}\times T_1(M_0). \tag{33}$$

Let us denote by μ the number of elements in the (finite) group $T_1(M)$ and by μ_0 the number of elements in $T_1(M_0)$. By (33), we have $\mu=2^{\beta+\gamma}\cdot\mu_0$, and consequently

$$\beta+\gamma=c_3, \tag{34}$$

where the number $c_3=\dfrac{\lg\mu-\lg\mu_0}{\lg 2}$ is determined by the manifold M. Thus we have established three linear equations, (31), (32) and (34), involving the numbers α, β, γ. Since the determinant of these equations is equal to 1, the numbers α, β, γ are determined. Thus the theorem is completely proved.

We have determined all topological divisors of M using only homological properties of M: the Betti numbers $p_k(M)$ and the 1-dimensional group of torsion $T_1(M)$. Consequently, we may state the following

Corollary. *Every n-dimensional manifold M decomposable into product of sets of dimension $\leqslant 2$ is topologically determined by his Betti numbers $p_k(M)$ and his 1-dimensional group of torsion $T_1(M)$.*

[46]) l. c., p. 208, (2′) and p. 265, (4 b).

Sur un problème de la théorie générale des ensembles.

Par

Wacław Sierpiński (Warszawa).

1. A et B étant deux ensembles, désignons par $A\times B$ l'ensemble de tous les couples ordonnés (a,b) où $a\in A$ et $b\in B$.

$\mathcal{A}$ et $\mathcal{B}$ étant deux familles d'ensembles, désignons par $\mathcal{A}\dot{\times}\mathcal{B}$ la famille de tous les ensembles $A\times B$ où $A\in\mathcal{A}$ et $B\in\mathcal{B}$.

Nous dirons qu'une famille d'ensembles *jouit de la propriété de Souslin* [1]) si elle ne contient aucune sous-famille indénombrable d'ensembles disjoints (non-vides). M. E. Szpilrajn a posé récemment le problème suivant:

$\mathcal{A}$ et $\mathcal{B}$ étant deux familles d'ensembles jouissant chacune de la propriété de Souslin, la famille $\mathcal{A}\dot{\times}\mathcal{B}$ jouit-elle toujours de ladite propriété?

En utilisant l'axiome du choix, je vais démontrer que *la réponse à ce problème est négative.*

En résolvant un problème de M. B. Knaster, j'ai démontré à l'aide de l'axiome du choix [2]) qu'il existe une relation symétrique R dont le champ E est indénombrable et telle que tout sous-ensemble indénombrable de E admet deux éléments différents a et b pour lesquels on a aRb, et deux éléments différents a_1 et b_1 pour lesquels on a a_1 non Rb_1 [3]).

[1]) D'après la dénomination de M. E. Szpilrajn; cf. le problème de M. Souslin, Fund. Math. **1**, (1920), p. 223 (problème 3).

[2]) W. Sierpiński, Ann. Ec. Norm. Sup. Pisa **2** (1933), p. 285.

[3]) Il résulte de l'axiome du choix l'existence d'une suite transfinie $E=\{x_\xi\}_{\xi<\Omega}$ de type Ω formée de nombres réels différents. Soit R la relation définie dans E comme il suit: $x_\alpha R x_\beta$ signifie que $\alpha<\beta$ et $x_\alpha<x_\beta$ ou bien que $\alpha>\beta$ et $x_\alpha>x_\beta$. On démontre que la relation R satisfait aux conditions requises.