On Infinite Derivates.

By

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In answer to a question proposed by Jarník[1]), Mazurkiewicz[2]) has constructed a function $f(x)$ which is everywhere continuous on the right and whose upper derivate on the right, $D^+f(x)$, is everywhere $+\infty$.

In this note I give a simpler example with the same properties.

Let a number x in $\langle 0,1\rangle$ be expressed in the scale of 3 as

$$x=\frac{a_1}{3}+\frac{a_2}{3^2}+\ldots+\frac{a_m}{3^m}+\ldots$$

where $a_m=0,1$ or 2 $(m=1,2,\ldots)$.

The numbers x can be divided into two classes:

class (i): those for which x has a unique representation;

class (ii): those for which x has double representation, as

$$x=\frac{a_1}{3}+\frac{a_2}{3^2}+\ldots+\frac{a_n}{3^n}$$
$$=\frac{a_1}{3}+\frac{a_2}{3^2}+\ldots+\frac{a_n-1}{3^n}+\frac{2}{3^{n+1}}+\frac{2}{3^{n+2}}+\ldots$$

Let

$$f(x)=\frac{b_1}{2}+\frac{b_2}{2^2}+\ldots+\frac{b_n}{2^n}+\ldots$$

where $b_n=1$ if $a_n=2$, otherwise $b_n=0$ $(n=1,2,\ldots)$ for all points of class (i). For points of class (ii) let $f(x)$ be defined by means of the ending representation of x in the same way as for class (i).

[1]) W. Jarník, Fund. Math. **23** (1934) p. 1.

[2]) S. Mazurkiewicz, ibid., p. 9.

It is easy to see that $f(x)$ is continuous at each point of class (i), while at each point of class (ii) it is continuous on the right but discontinuous on the left.

Let x be any point in $\langle 0,1\rangle$ (expressed as an ending radix fraction if it belongs to class (ii)):

$$x=\frac{a_1}{3}+\frac{a_2}{3^2}+\ldots+\frac{a_n}{3^n}+\ldots$$

Then, for an increasing sequence of values of n $(n=m_1,m_2,\ldots,m_r,\ldots)$, $a_{m_r}=0$ or 1.

Let

$$x'=\frac{a_1}{3}+\ldots+\frac{2}{3^{m_r}}+\frac{a_{m_r+1}}{3^{m_r+1}}+\ldots;$$

then

$$x'-x=\frac{2-a_{m_r}}{3^{m_r}},\quad \text{where } 2-a_{m_r}=1 \text{ or } 2,$$

and

$$f(x')-f(x)=\frac{1}{2^{m_r}}.$$

Hence

$$\lim_{x'\to x}\frac{f(x')-f(x)}{x'-x}=\lim_{m_r\to\infty}\frac{3^{m_r}}{2^{m_r}}=+\infty.$$

Therefore, $D^+f(x)=+\infty$ at every x in $\langle 0,1\rangle$.