On transformations having periodic properties

By

W. L. Ayres (Ann Arbor, Mich., U. S. A.)

1. The purpose of this paper is to study the relations between properties of four types of periodic and semi-periodic transformations. It is shown that these types are identical only for quite special spaces such as linear graphs and dendrites, but that they give always identical properties relative to the cyclic elements of a Peano space. These properties are studied in detail. The principal results are that the set of cyclic elements which are invariant in the large form a non-vacuous Peano space, and that the components of the space minus this invariant set are permuted among themselves in a definite manner.

2. We consider a metric space $X$ and a single-valued transformation (function) $f$ of $X$ into itself, i.e., for each $x \in X$, $f(x) \in X$ and there is an $x_0 \in X$ such that $f(x_0) = x$. We denote by $f^x$ the point $f(f(x))$, and by $f^k$ the result of repeating $f$ $k$ times.

**Property $P_r$** $f$ is said to be **periodic** if there is a positive integer $n$ such that $f^n = I$, where $I$ is the identity transformation.

**Property $P_{p}$** $f$ is said to be **point-wise periodic** if for each $x \in X$ there is a positive integer $n = n(x)$ such that $f^n(x) = x$.

**Property $P_{a}$** $f$ is said to be **almost periodic** if for each $\varepsilon > 0$ there exists a positive integer $n = n(\varepsilon)$ such that $d(f^n(x), x) < \varepsilon$ for every $x \in X$.

**Property $P_{c}$** $f$ is said to be **point-wise almost periodic** if for each $x \in X$ and any $\varepsilon > 0$ there is a positive integer $n = n(x, \varepsilon)$ such that $d(f^n(x), x) < \varepsilon$.

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in the first and alternate places still blank in the \(x_i\) sequence. We define \(f\) as follows:

\[
\begin{align*}
    f(x_i) &= x_{i+1}, \\
    f(y_i) &= y_{i+1}, \\
    f(y_{-i}) &= y_0.
\end{align*}
\]

Then \(f\) is point-wise almost periodic, does not have a unique inverse since \(f(x_{-i}) = f(y_{-i}) = x_0\), and is neither periodic, point-wise periodic nor almost periodic.

3.2. Due to this example we define

Property \(P_3\), \(f\) is said to be a point-wise almost periodic correspondence if \(f\) has Property \(P_2\) and a single-valued inverse.

3.3. Theorem. If \(X\) is compact and \(f\) is continuous and has any one of the Properties \(P_3\), \(P_4\), \(P_5\), \(P_6\), then \(f\) is a homeomorphism.

4. In this section we wish to consider the role of the continuous transformations \(f\) with one of the periodic properties in the Whyburn cyclic element theory \(^1\). Each of the four properties \(P_1\), \(P_2\), \(P_5\), \(P_6\) give precisely the same results here so the proofs will be carried out with Property \(P_1\) and the other results follow from 3.1. Hereafter in section 4 we assume \(X\) is a compact Peano space and \(f\) is continuous and has Property \(P_1\).

Directly from 3.3 we have

4.1. Theorem. If \(C\) is a cyclic element of \(X\), then \(f(C)\) is a cyclic element of \(X\) and there is a cyclic element \(C'\) such that \(f(C') = C\).

4.2. Theorem. If \(C_i\) is a cyclic element of the cyclic chain \(\langle C_1, C_2, \ldots \rangle\) between the cyclic elements \(C_1\) and \(C_2\), then \(f(C_i)\) is a cyclic element of \(\langle f(C_1), f(C_2) \rangle\).

4.3. Theorem. If \(C_1\) and \(C_2\) are invariant cyclic elements, under \(f\), then the Chain \(\langle C_1, C_2 \rangle\) is invariant under \(f\).

By invariant cyclic elements we mean \(f(C) = C\) only and individual points may vary. Similarly by invariant chain \(A\) we mean \(f(A) = A\) only and individual cyclic elements may vary. However we shall show later (4.5) that if the end-elements of a chain are invariant, then every cyclic element of the chain is invariant.

\(^1\) We presuppose a knowledge of the cyclic element theory, See C. Kuratowski and T. Whyburn, Sur les éléments cycliques et leurs applications, Fund. Math. 16 (1930), pp. 305-31 and bibliography of earlier papers found therein.
4.4. Theorem. There exists a cyclic element \( C \) which is invariant under \( f \).

Since \( f \) is a homeomorphism by 3.3, \( X \) contains a cyclic element \( C \) such that \( f(C) = C \). Then \( f(C) = C \) by 4.1.

4.4.1. Corollary. If \( X \) is a dendrite, then \( f \) leaves at least one point invariant.

4.5. Theorem. If the cyclic elements \( C_1 \) and \( C_2 \) of \( X \) are invariant under \( f \), then every cyclic element of the Chain\((C_1, C_2)\) is invariant under \( f \).

Suppose \( f(C_1) = C_0 + C_3 \). Then \( C_1 + C_2 = C_0 \). By 4.2, \( C_2 \) belongs to Chain\((C_1, C_2)\). Since \( f \) is a homeomorphism, we have the order \( C_1, C_2, C_3 \) for all \( C \) in Chain\((C_1, C_2)\) and we assume the former 4. Since order is preserved under homeomorphisms, we have the order \( C_1, C_2, C_3 \) for all \( C \) in Chain\((C_1, C_2)\). There is a point \( y \) such that \( X - y = X_1 + X_2 \), where \( X_1 \) and \( X_2 \) are mutually separated and \( X_1 \cap C_1 = X_2 \cap C_0 \). Let \( x \in C_1 \) and let \( \varepsilon \) be any positive number less than \( \varepsilon(x, X_2) \). Then \( f(x) \in X_2 \) for all \( x \) and thus \( \varepsilon(x, f(x)) < \varepsilon \) for all \( x \).

4.5.1. Corollary. If \( X \) is a dendrite and \( x \) and \( y \) are points of \( X \) which are invariant under \( f \), then every point of the arc \( xy \) of \( X \) is invariant under \( f \).

4.6. Theorem. The set of invariant cyclic elements under \( f \) forms a Peano space \( I_f \).

By 4.5, \( I_f \) is a connected collection of cyclic elements of \( X \) and it remains to prove that \( I_f \) is closed. Let \( x \in I_f - I_f \). From the cyclic element theory, \( x \) is a cut point or end point of \( X \), and, in either case, is a limit point of cut points which are cyclic elements of \( I_f \). Since these cyclic elements are points and invariant, we have \( x \) invarian under the continuity of \( f \).

4.6.1. Corollary. If \( X \) is a dendrite, the set of points invariant under \( f \) forms a non-arc sub-dendrite.

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2) If \( C_1 \) and \( C_2 \) are distinct cyclic elements of Chain\((C_1, C_2)\), then either Chain\((C_1, C_2) \supset Chain(C_1, C_2)\) or Chain\((C_1, C_2) = Chain(C_0, C_2)\). The former defines order \( C_1, C_2, C_3 \) and the latter order \( C_1, C_2, C_3 \).
3) If \( C_1 \) and \( C_2 \) are distinct cyclic elements of Chain\((C_1, C_2)\), then either Chain\((C_1, C_2) \supset Chain(C_1, C_2)\) or Chain\((C_1, C_2) = Chain(C_0, C_2)\).
Let $X$ be a circle and let $f$ be a rotation of $X$ through an angle of $\pi/2\pi$ radians, where $i$ is an irrational number between 0 and 1. The transformation $f^*$ is then a rotation of $mi/2\pi$ radians and $f$ is neither periodic nor point-wise periodic. But $f$ is almost periodic since for any $\varepsilon > 0$ there exist integers $m$ and $n$ so that $|mi-n|<\varepsilon$.

6.2. There exist point-wise periodic transformations (and thus point-wise almost periodic) which are neither periodic nor almost periodic.

For each integer $n>1$ let $X_n$ consist of $n$ triangular 2-cells $T_m$, each having the interval from $(1,0,0)$ to $(-1,0,0)$ as one side and the third vertex on the ellipse $x^2+y^2=1$, $z=1/n$. Let these vertices be arranged on the ellipse so as to divide the ellipse into arcs of equal length, so that the vertices are in the order $T_s, T_{s+1}, T_{s+2}, \ldots, T_{s+n}$ around the ellipse, and so that no triangle of $X_n$ intersects any triangle of $X_n$ $(m<n)$ except in the base-line $y=z=0$. Let $X=\sum_{n=1}^\infty X_n$.

Let $f$ be defined on each $T_n$ as the collineation carrying $T_m$ into $T_{m+1}$ (mod $n$) and keeping the $z$-coordinate of each point and the points of the base-line invariant. Then $f$ is point-wise periodic on $X$, periodic of period $n$ on $X_n$; and thus clearly not periodic on $X$.

Now let $m$ be any positive integer and $0<\varepsilon<1$. Then if $s$ is the ellipse-vertex of that triangle $T_{m+j}$ $(0<i<2\pi)$ which is nearest the point $(-1,0,0)$, $f^*(s)$ will be the ellipse-vertex of that triangle $T_{m+j}$ which is nearest $(1,0,0)$. Hence $f^*(s), f^*(n) > \varepsilon$ and $f$ is not almost periodic.

6.3. Even in case $X$ is a dendrite, the transformations may not be equivalent if there are an infinite number of non-invariant end points. For consider the following example: For each integer $n>1$ let $X_n$ consist of $n$ intervals of length $1/n$ radiating out from the point $(0,0)$, dividing the plane into equal angles, and so chosen that $X_1$ and $X_2$ have only the point $(0,0)$ in common for $i=j$. Let $X=\sum X_n$, and let $f$ be a rotation of angle $2\pi/n$ on each $X_n$. Then $f$ is point-wise periodic but not periodic.

1) This is a simplification of an example due to my student, Mr. Ralph Phillips. My earlier example was for a non-Peanoian space $X$ and raised the question of the existence of such an example in Peano space, which problem Mr. Phillips solved.
In the above example $X$ has a point of increasing order and $X - I$ has infinitely many components. However the example may be modified so that $X$ has points of finite order only and $X - I$ has just two components.

6.4. Theorem. If $X$ is a dendrite, $f$ is a continuous point-wise almost periodic correspondence, and only a finite number of end points of $X$ are non-invariant under $f$, then $f$ is periodic.

Let $e_{1}, e_{2}, \ldots, e_{n}$ denote the non-invariant end points. For each $i$ let $e_{i}$ denote the smallest of the numbers $e(e_{i}, e_{j})$, $i \neq j$. By hypothesis there exists an integer $n_{i}$ such that $e(e_{i}, e_{j}) < e_{i}$. Now $I_{i} \subseteq X_{i}$ and $f^{n}(e_{i})$ is an end point of $X$ by 3.3. Hence $f^{n}(e_{i}) = e_{i}$. Then if $m = \sum n_{i}$, the transformation $f^{m}$ leaves all end points of $X$ invariant and thus is the identity transformation by 4.5.1.

6.4.1. Corollary. If $f$ is a continuous point-wise almost periodic correspondence on a set $X$ and $xy$ is an arc of $X$ such that $f(\text{arc } xy) = \text{arc } xy$, $f(x) = x$, and $f(y) = y$, then $f(z) = z$ for every point $z$ of the arc $xy$.

6.5.1. Lemma. If $f$ is a continuous point-wise almost periodic correspondence on a simple closed curve $X$, then if $f$ leaves one point of $X$ invariant it leaves every point of $X$ invariant.

This follows from the fact that order is preserved on an arc under a homeomorphism.

6.5.2. Theorem. If $X$ is a finite linear graph containing at least one branch point $^{1}$ and $f$ is a continuous point-wise almost periodic correspondence on $X$, then $f$ is periodic on $X$.

Let $\{A_{i}\}$ be the maximal free arcs of $X$ (we allow the possibility of the two end points of $A_{i}$ being identical making $A_{i}$ a simple closed curve which is a node or end element of $X$) and $\{n_{i}\}$ be the set of points which are end points of the arcs of $\{A_{i}\}$. As the arcs $A_{i}$ are maximal free arcs, the set $\Sigma_{n_{i}}$ is the set of end points and branch points of $X$ and $f(\Sigma_{n_{i}}) = \Sigma_{n_{i}}$, since $f$ is a homeomorphism.

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1) This condition is necessary to prevent $X$ being a simple closed curve. See example 6.1.

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Let $A_{i}$ be any one of the free arcs which is a simple closed curve and let $\nu_{i}$ be its single end point. Since $X - A_{i}$, $\nu_{i}$ is a branch point of $X$. Let $p_{i}$ be an interior point of $A_{i}$ and $\nu = f(p_{i} - A_{i} + \nu_{i})$. Then for the integer $n_{i}$ such that $e(p_{i} - A_{i} + \nu_{i}) < e_{i}$

(a) $f^{n_{i}}(A_{i}) = A_{i}$

(b) $f^{n_{i}}(\nu_{i}) = \nu_{i}$

Let $n_{i}$ denote the smallest integer for which $f^{n_{i}}(A_{i}) = A_{i}$. Now for each interior point $x$ of $A_{i}$ and each $n$ for which $f^{n}(x) \in A_{i}$ we have $f^{n}(A_{i}) = A_{i}$ and it follows that $n_{i}$ is a multiple of $n_{i}$ so that $f^{n_{i}}$ is a point-wise almost periodic correspondence on $A_{i}$. As $\nu_{i}$ is invariant under $f^{n_{i}}$, every point of $A_{i}$ is invariant under $f^{n_{i}}$ by 6.5.1.

Now consider any $A_{i}$ which is a true arc and let $\nu_{i}$ and $\nu_{i}$ denote its end points. As above there is an integer $n_{i}$ such that $f^{n_{i}}(A_{i}) = A_{i}$, $f^{n_{i}}(\nu_{i}) = \nu_{i}$, and let $n_{i}$ denote the smallest such integer. It follows that $f^{n_{i}}(A_{i})$ is a point-wise almost periodic correspondence on $A_{i}$.

If $f^{n_{i}}(\nu_{i}) = \nu_{i}$ and $f^{n_{i}}(\nu_{i}) = \nu_{i}$, then $f^{n_{i}}$ leaves every point of $A_{i}$ invariant by 6.4.1 and we put $n_{i} = n_{i}$.

If $f^{n_{i}}(\nu_{i}) = \nu_{i}$ and $f^{n_{i}}(\nu_{i}) = \nu_{i}$, there is a point $p_{i}$ invariant under $f^{n_{i}}$ by 4.4.1. There is a smallest integer $n_{i}$ (a multiple of $n_{i}$) such that $f^{n_{i}}(p_{i}) = p_{i}$. It follows easily that $f^{n_{i}}$ is a point-wise almost periodic correspondence on the entire arc $A_{i}$, $f^{n_{i}}(\nu_{i}) = \nu_{i}$ and $f^{n_{i}}(\nu_{i}) = \nu_{i}$. Then $f^{n_{i}}$ leaves every point of $A_{i}$ invariant by 6.4.1.

For each $A_{i}$ there exists then an integer $n_{i}$ such that $f^{n_{i}}$ leaves every point of $A_{i}$ invariant. Hence $f^{n_{i}}$, where $m = \sum n_{i}$, leaves every point of $X$ invariant, i.e. $f$ is periodic on $X$. The University of Michigan.