Travaux cités.

[1] S. Bergmann. *Über ausgezeichnete Randflächen in der Theorie der Funktionen von zwei komplexen Veränderlichen.* Math. Ann. **104** (1931), p. 611–636.

[2] — *Zur Veranschaulichung der Kreiskörper und Bereiche mit ausgezeichneter Randfläche.* Jahresbericht der Deutschen Math. Verein. **44** (1933), p. 238–252.

[3] — *Über eine in gewissen Bereichen mit Maximumfläche gültige Integraldarstellung der Funktionen zweier komplexer Variabler.* II. Math. Zeit. **39** (1934), p. 605–608.

[4] G. H. Hardy et J. E. Littlewood. *A maximal theorem with function-theoretic applications.* Acta Math. **54** (1930), p. 81–116.

[5] B. Jessen, J. Marcinkiewicz et A. Zygmund. *Note on the differentiability of multiple integrals.* Fund. Math. **25** (1935), p. 217–234.

[6] O. D. Kellog. *Harmonic functions and Green integral.* Trans. Americ. Math. Soc. **13** (1912), p. 109–132.

[7] L. Lichtenstein. *Über die konforme Abbildung ebener analytischer Gebiete mit Ecken.* Journal für Math. **140** (1911), p. 100–119.

[8] M. Riesz. *Sur les fonctions conjuguées.* Math. Zeit. **27** (1927), p. 218–244.

[9] S. Warschawski. *Über einen Satz von O. D. Kellog.* Nachrichten der Ges. Wiss. Göttingen (1932), p. 73–86.

[10] A. Zygmund. *Trigonometrical Series.* Monografie Matematyczne **5**, Warszawa-Lwów 1939.

On transformations having periodic properties[1].

By

W. L. Ayres (Ann Arbor, Mich., U. S. A.).

1. The purpose of this paper is to study the relations between properties of four types of periodic and semi-periodic transformations. It is shown that these types are identical only for quite special spaces such as linear graphs and dendrites, but that they give always identical properties relative to the cyclic elements of a Peano space. These properties are studied in detail. The principal results are that the set of cyclic elements which are invariant in the large form a non-vacuous Peano space, and that the components of the space minus this invariant set are permuted among themselves in a definite manner.

2. We consider a metric space X and a single-valued transformation (function) f of X into itself, i. e. for each $x \epsilon X$, $f(x) \epsilon X$ and there is an $x_1 \epsilon X$ such that $f(x_1)=x$. We denote by $f^2(x)$ the point $f(f(x))$, and by f^n the result of repeating f n times.

Property P_1. f is said to be *periodic* if there is a positive integer n such that $f^n = I$, where I is the identity transformation.

Property P_2. f is said to be *point-wise periodic* if for each $x \epsilon X$ there is a positive integer $n=n(x)$ such that $f^n(x)=x$.

Property P_3. f is said to be *almost periodic* if for each $\varepsilon>0$ there exists a positive integer $n=n(\varepsilon)$ such that $\varrho(x, f^n(x))<\varepsilon$ for every $x \epsilon X$.

Property P_4. f is said to be *point-wise almost periodic* if for each $x \epsilon X$ and any $\varepsilon>0$ there is a positive integer $n=n(x,\varepsilon)$ such that $\varrho(x, f^n(x))<\varepsilon$.

[1]) Presented to the American Mathematical Society Dec. 30, 1936.

Directly from the definitions we have

2.1. ***Theorem.*** *If f has Property P_1, it has Properties P_2, P_3, and P_4. If f has Property P_2 or P_3, it has Property P_4.*

In section 6 we shall show that these are the only relations between these properties except for special spaces X.

2.2. ***Theorem.*** *If f has any one of the Properties P_i there exists an infinite set of integers n satisfying the conditions of P_i.*

3. ***Theorem.*** *If f has Property P_1, P_2, or P_3, then f^{-1}, the inverse of f, is single-valued.*

From 2.1 we need consider P_2 and P_3 only. Suppose f has property P_2 and there exist points x and y such that $f(x)=f(y)$ and $x\neq y$. Then $f^n(x)=f^n(y)$ for every n. By definition there exist integers r and s such that $f^r(x)=x$ and $f^s(y)=y$. But $x=f^{rs}(x)=f^{rs}(y)=y$.

Suppose f has Property P_3 and there exist points x and y such that $f(x)=f(y)$ and $x\neq y$. Then $f^n(x)=f^n(y)$ for every n. Let $\varepsilon=\varrho(x,y)/3$. By definition there exists an integer n such that $\varrho(x,f^n(x))<\varepsilon$ for every $x\in X$. Then

$$3\varepsilon=\varrho(x,y)\leqslant\varrho(x,f^n(x))+\varrho(f^n(y),y)<2\varepsilon.$$

3.1. A point-wise almost periodic transformation P_4 may fail to have a unique inverse as may be seen in the following

Example. Let X consist of the points $\{x_i\}$ and $\{y_j\}$ for all integers i and all negative integers j. Now arrange the symbols $\{x_i\}$ and $\{y_j\}$ $(i\leqslant 0, j\leqslant 0)$ in a single infinite sequence $z_1,z_2,z_3,\ldots$ such that

(a) each symbol x_i and y_j appears infinitely many times,
(b) infinitely many of the spaces z_i are left blank,
(c) z_1 is not blank.

The points x_i and y_j for $i\leqslant 0$ and $j<0$ are subject to no restrictions whatever except that they be distinct points. Now let x_1 be any point distinct from all the previously defined points such that $\varrho(x_1,z_1)<1$. Place the symbol x_1 in the first place z_l which is blank and in alternate blanks from there on. Then x_1 appears in the z_i sequence infinitely many times and still leaves infinitely many blanks. In general let x_n be any point distinct from all previously defined points that $\varrho(x_n,z_n)<1/n$ and place the symbol x_n in the first and alternate places still blank in the z_i sequence. We define f as follows:

$$f(x_i)=x_{i+1},\qquad f(y_j)=y_{j+1},\qquad f(y_{-1})=x_0.$$

Then f is point-wise almost periodic, does not have a unique inverse since $f(x_{-1})=f(y_{-1})=x_0$, and is neither periodic, point-wise periodic nor almost periodic.

3.2. Due to this example we define

Property P_4'. f is said to be a *point-wise almost periodic correspondence* if f has Property P_4 and a single-valued inverse.

3.3. ***Theorem.*** *If X is compact and f is continuous and has any one of the Properties P_1, P_2, P_3, P_4', then f is a homeomorphism.*

4. In this section we wish to consider the role of the continuous transformations f with one of the periodic properties in the Whyburn cyclic element theory[1]. Each of the four properties P_1, P_2, P_3, P_4' give precisely the same results here so the proofs will be carried out with Property P_4' and the other results follow from 2.1. Hereafter in section 4 we assume X is a compact Peano space and f is continuous and has Property P_4'.

Directly from 3.3 we have

4.1. ***Theorem.*** *If C is a cyclic element of X, then $f(C)$ is a cyclic element of X and there is a cyclic element C' such that $f(C')=C$.*

4.2. ***Theorem.*** *If C_3 is a cyclic element of the cyclic chain* Chain(C_1,C_2) *between the cyclic elements C_1 and C_2, then $f(C_3)$ is a cyclic element of* Chain$(f(C_1),f(C_2))$.

4.3. ***Theorem.*** *If C_1 and C_2 are invariant cyclic elements, under f, then the* Chain(C_1,C_2) *is invariant under f.*

By invariant cyclic elements we mean $f(C)=C$ only and individual points may vary. Similarly by invariant chain A we mean $f(A)=A$ only and individual cyclic elements may vary. However we shall show later (4.5) that if the end-elements of a chain are invariant, then every cyclic element of the chain is invariant.

[1]) We presuppose a knowledge of the cyclic element theory. See C. Kuratowski and G. T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math. **16** (1930), pp. 305-31 and bibliography of earlier papers found therein.

4.4. ***Theorem.*** *There exists a cyclic element C which is invariant under f.*

Since f is a homeomorphism by 3.3, X contains a cyclic element C such that $f(C) \subset C$ [1]). Then $f(C)=C$ by 4.1.

4.4.1. ***Corollary.*** *If X is a dendrite, then f leaves at least one point invariant.*

4.5. ***Theorem.*** *If the cyclic elements C_1 and C_2 of X are invariant under f, then every cyclic element of the* Chain(C_1, C_2) *is invariant under f.*

Suppose $f(C_3)=C_4 \neq C_3$. Then $C_1 \neq C_3 \neq C_2$. By 4.2 C_4 belongs to Chain(C_1, C_2) and we have the order $C_1C_3C_4C_2$ or $C_1C_4C_3C_2$ and we assume the former [2]). Since order is preserved under homeomorphisms we have the order $C_1C_3C_4C_5C_2$ where $C_5=f^2(C_3)$, and $C_1C_nC_{n+1}C_2$ where $C_n=f^{n-3}(C_3)$. There is a point y such that $X-y=X_1+X_2$, where X_1 and X_2 are mutually separated and $X_1 \supset C_1+C_3$, $X_2 \supset C_2+C_5$. Let $x \in C_3$ and let ε be any positive number less than $\varrho(x, X_2)$. Then $f^n(x) \subset X_2$ for $n \geqslant 2$ and thus $\varrho(x, f^n(x)) > \varepsilon$ for all $n \geqslant 2$.

4.5.1. ***Corollary.*** *If X is a dendrite and x and y are points of X which are invariant under f, then every point of the arc xy of X is invariant under f.*

4.6. ***Theorem.*** *The set of invariant cyclic elements under f forms a Peano space I_f.*

By 4.5 I_f is a connected collection of cyclic elements of X and it remains to prove that I_f is closed. Let $x \in \bar{I}_f - I_f$. From the cyclic element theory x is a cut point or end point of X, and, in either case, is a limit point of cut points which are cyclic element of I_f. Since these cyclic elements are points and invariant, we have x invariant from the continuity of f.

4.6.1. ***Corollary.*** *If X is a dendrite, the set of points invariant under f forms a non-vacuous sub-dendrite.*

---

1) W. L. Ayres, *Some generalizations of the Scherrer Fixed-point Theorem*, Fund. Math. **16** (1930), p. 333.

2) If C_3 and C_4 are distinct cyclic elements of Chain(C_1, C_2), then either Chain$(C_1, C_3) \subset$ Chain(C_1, C_4) or Chain$(C_1, C_3) \supset$ Chain(C_1, C_4). The former defines order $C_1C_3C_4C_2$ and the latter order $C_1C_4C_3C_2$.

4.7. ***Theorem.*** *If C_1 and C_2 are distinct cyclic elements of X and $f(C_1)=C_2$, then the* Chain(C_1, C_2) *contains one and only one cyclic element which is invariant under f.*

Since the invariant cyclic elements form a closed set I_f by 4.6, there exists an invariant cyclic element F such that the Chain(C_1, F) contains no other invariant cyclic element. By 4.2 Chain(C_1, F) is carried by f into a cyclic chain from $f(C_1)=C_2$ to $f(F)=F$. In the order from F to C_1 let J_1 be the last cyclic element these chains have in common. If $J_1=F$, the sum of the two chains is a chain from C_1 to C_2 and our theorem is proved.

We shall show that $J_1 \neq F$ leads to a contradiction. If $J_1 \neq F$, then Chain(C_i, J_1) + Chain(J_1, F) = Chain(C_i, F), Chain(C_1, J_1) + Chain(C_2, J_1) = Chain(C_1, C_2), and no two of the chains Chain(C_1, J_1), Chain(C_2, J_1), Chain(J_1, F) have any common point save J_1. By 4.2 $f(J_1)=J_2$ is a cyclic element of Chain(C_2, F), and $J_1 \neq J_2$ since J_1 is not invariant under f. Let $C_n=f(C_{n-1})$ and $J_n=f(J_{n-1})$. By 3.3 Chain(C_3, J_2), Chain(C_2, J_2), Chain(J_2, F) are three chains such that any two have just J_2 in common.

Case I. We have the order $FJ_2J_1C_2$ on Chain(F, C_2). Then Chain$(J_2, F) \subset$ Chain(J_1, F), Chain$(J_2, C_2) \supset$ Chain(J_1, C_2) and Chain$(C_3, J_2) \cdot$ Chain$(C_2, F) = J_2$. Further from Chain$(J_2, F) \subset$ Chain(J_1, F), we have Chain$(J_3, F) \subset$ Chain(J_2, F) and so on. Then $C_1, C_2, C_3, \ldots$ all belong to distinct components of $X-$Chain(J_1, F). Let $x \in C_1$. It follows from the local connectivity of X that there exists a number ε such that $\varrho(x, y) < \varepsilon$ implies that y belongs to the component of $X-$Chain(J_1, F) containing C_1. Then as $f^n(x)$ belongs to a different component of $X-$Chain(J_1, F) for every n, f cannot be point-wise almost periodic.

Case II. We have the order $FJ_1J_2C_2$ on Chain(F, C_2). Then Chain$(C_3, J_2) \cdot$ Chain$(F, C_2) = J_2$. Also $J_2 \subset$ Chain(J_1, C_2) implies $J_3 \subset$ Chain(J_2, C_3). Continuing this we see that for every $n>2$ J_n and C_n belong to the component of $X-$Chain(F, C_2) which contains C_3. As this component is different from that containing C_1 we may show that f is not point-wise almost periodic as in Case I.

We have seen now that Chain(C_1, C_2) contains one invariant cyclic element F and Chain(C_1, C_2) = Chain(C_1, F) + Chain(C_2, F), Chain$(C_1, F) \cdot$ Chain$(C_2, F) = F$. Now suppose that Chain(C_1, C_2) contains an invariant cyclic element $F' \neq F$. If $F' \subset$ Chain(C_1, F), then

$f(F')\subset \text{Chain}(C_2,F)$ by 4.2 and $f(F')\neq F'$. If $F'\subset \text{Chain}(C_2,F)$, then $f(F')$ is not contained in $\text{Chain}(C_2,F)$ and $f(F')\neq F'$. This follows from the fact that by 4.2 each cyclic element of $\text{Chain}(C_2,F)$ is the image of a cyclic element of $\text{Chain}(C_1,F)$ under f and could not also be the image of F' by 3.3.

4.7.1. ***Corollary.*** *If X is a dendrite and $x \epsilon X$, then the arc of X from x to $f(x)$ contains one and only one point which is invariant under f.*

4.7.2. ***Corollary.*** *If C_1 and C_2 are distinct cyclic elements of $X-I_f$ and $f(C_1)=C_2$, then C_1 and C_2 belong to different components H_1 and H_2 of $X-I_f$, $f(H_1)=H_2$, and H_1 and H_2 have their limit points in the same cyclic element of I_f* [1]).

5. From 4.7.2 we see that f produces a certain permutation among those components of $X-I_f$ whose limit points belong to the same cyclic element of I_f. From this it might be supposed that if f were periodic there would be a direct relation between the number of such components and the period of f. That this is false may be seen as follows: Let X consist of the intervals from $(0,1)$ to $(0,-1)$, from $(1,1)$ to $(-1,1)$, and from $(1,-1)$ to $(-1,-1)$. If p is the point (x,y) let $f(p)$ be defined as the point $(x,-y)$. Also let us define a transformation f' as follows: If p is the point (x,y), let $f(p)$ be the point $(-x,-y)$ if $y\geqslant 0$ and $(x,-y)$ if $y<0$. Both the transformations f and f' are periodic; and $I_f=I_{f'}=(0,0)$ so that the number of components of both $X-I_f$ and $X-I_{f'}$ is two. But the period of f is two and the period of f' is four. Also X may be modified by adding more "arms" at $(0,1)$ and $(0,-1)$ so that the number of components of $X-I_f$ remains two but the period of f may be made as high as we please.

6. In section 4 we found that relative to the cyclic elements all four periodic properties play the same role. In the present section we wish to see that the transformations are really different and find conditions on X under which they are the same.

6.1. There exist almost periodic transformations (and thus point-wise almost periodic) which are neither periodic nor point-wise periodic.

[1]) See the closely related results of L. Whyburn, *Rotation groups about a set of fixed points*, Fund. Math. **28** (1937), pp. 124–130.

Let X be a circle and let f be a rotation of X through an angle of $i\cdot 2\pi$ radians, where i is an irrational number between 0 and 1. The transformation f^n is then a rotation of $ni\cdot 2\pi$ radians and f is neither periodic nor point-wise periodic. But f is almost periodic since for any $\varepsilon>0$ there exist integers m and n so that $|mi-n|<\varepsilon$.

6.2. There exist point-wise periodic transformations (and thus point-wise almost periodic) which are neither periodic nor almost periodic [1]).

For each integer $n>1$ let X_n consist of n triangular 2-cells T_{ni}, each having the interval from $(1,0,0)$ to $(-1,0,0)$ as one side and the third vertex on the ellipse $x^2+n^2y^2=1$, $z=1/n$. Let these vertices be arranged on the ellipse so as to divide the ellipse into arcs of equal length, so that the vertices are in the order $T_{n1}, T_{n2}, \ldots, T_{nn}, T_{n1}$ around the ellipse, and so that no triangle of X_n intersects any triangle of X_m $(m<n)$ except in the base-line $y=z=0$. Let $X=\sum_{n=2}^{\infty}X_n$. Let f be defined on each T_{ni} as the collineation carrying T_{ni} into T_{ni+1} (mod n) and keeping the z-coordinate of each point and the points of the base-line invariant. Then f is point-wise periodic on X, periodic of period n on X_n; and thus clearly not periodic on X.

Now let n be any positive integer and $0<\varepsilon<1$. Then if x is the ellipse-vertex of that triangle $T_{2n,i}$ $(0<i\leqslant 2n)$ which is nearest the point $(-1,0,0)$, $f^n(x)$ will be the ellipse-vertex of that triangle $T_{2n,j}$ which is nearest $(1,0,0)$. Hence $\varrho(x,f^n(x))>\varepsilon$ and f is not almost periodic.

6.3. Even in case X is a dendrite, the transformations may not be equivalent if there are an infinite number of non-invariant end points. For consider the following example: For each integer $n>1$ let X_n consist of n intervals of length $1/n$ radiating out from the point $(0,0)$, dividing the plane into equal angles, and so chosen that X_i and X_j have only the point $(0,0)$ in common for $i\neq j$. Let $X=\sum X_n$ and let f be a rotation of angle $2\pi/n$ on each X_n. Then f is point-wise periodic but not periodic.

[1]) This is a simplification of an example due to my student, Mr. Ralph Phillips. My earlier example was for a non-Peanian space X and raised the question of the existence of an example in Peano space, which problem Mr. Phillips solved.

In the above example X has a point of increasing order and $X-I_f$ has infinitely many components. However the example may be modified so that X has points of finite order only and $X-I$ has just two components.

6.4. ***Theorem.*** *If X is a dendrite, f is a continuous point-wise almost periodic correspondence, and only a finite number of end points of X are non-invariant under f, then f is periodic.*

Let $e_1, e_2, \ldots, e_k$ denote the non-invariant end points. For each i let ε_i denote the smallest of the numbers $\varrho(e_i, e_j)$, $i \neq j$. By hypothesis there exists an integer n_i such that $\varrho(e_i, f^{n_i}(e_i)) < \varepsilon_i$. Now $I_f \subset I_{f^{n_i}}$ and $f^{n_i}(e_i)$ is an end point of X by 3.3. Hence $f^{n_i}(e_i) = e_i$. Then if $m = \prod_{i=1}^{k} n_i$, the transformation f^m leaves all end points of X invariant and thus is the identity transformation by 4.5.1.

6.4.1. ***Corollary.*** *If f is a continuous point-wise almost periodic correspondence on a set X and xy is an arc of X such that $f(\mathrm{arc}\, xy) = \mathrm{arc}\, xy$, $f(x) = x$, and $f(y) = y$, then $f(z) = z$ for every point z of the arc xy.*

6.5.1. ***Lemma.*** *If f is a continuous point-wise almost periodic correspondence on a simple closed curve X, then if f leaves one point of X invariant it leaves every point of X invariant.*

This follows from the fact that order is preserved on an arc under a homeomorphism.

6.5.2. ***Theorem.*** *If X is a finite linear graph containing at least one branch point*[1]) *and f is a continuous point-wise almost periodic correspondence on X, then f is periodic on X.*

Let $\{A_i\}$ be the maximal free arcs of X (we allow the possibility of the two end points of A_i being identical making A_i a simple closed curve which is a node or end element of X) and $\{v_i\}$ be the set of points which are end points of the arcs of $\{A_i\}$. As the arcs A_i are maximal free arcs, the set $\sum v_i$ is the set of end points and branch points of X and $f(\sum v_i) = \sum v_i$ since f is a homeomorphism.

[1]) This condition is necessary to prevent X being a simple closed curve. See example 6.1.

Let A_i be any one of the free arcs which is a simple closed curve and let v_i be its single end point. Since $X \neq A_i$, v_i is a branch point of X. Let p_i be an interior point of A_i and $\varepsilon_i = \varrho(p_i, X - A_i + v_i)$. Then for the integer n such that $\varrho(p_i, f^n(p_i)) < \varepsilon_i$,

(a) $f^n(A_i) = A_i$,

(b) $f^n(v_i) = v_i$.

Let n_i denote the smallest integer for which $f^n(A_i) = A_i$. Now for each interior point x of A_i and each n for which $f^n(x) \varepsilon A_i$, we have $f^n(A_i) = A_i$ and it follows that n is a multiple of n_i so that f^{n_i} is a point-wise almost periodic correspondence on A_i. As v_i is invariant under f^{n_i}, every point of A_i is invariant under f^{n_i} by 6.5.1.

Now consider any A_i which is a true arc and let v_s and v_t, denote its end points. As above there is an integer n such that, $f^n(A_i) = A_i$, $f^n(v_s + v_t) = v_s + v_t$; and let n_i' denote the smallest such integer. It follows that $f_1 = f^{n_i'}$ is a point-wise almost periodic correspondence on A_i. If $f_1(v_s) = v_s$ and $f_1(v_t) = v_t$, then f_1 leaves every point of A_i invariant by 6.4.1 and we put $n_i = n_i'$.

If $f_1(v_s) = v_t$ and $f_1(v_t) = v_s$, there is a point p_i invariant under f_1 by 4.4.1. There is a smallest integer n_i (a multiple of n_i') such that $f^{n_i}(\mathrm{arc}\, v_s p_i) = \mathrm{arc}\, v_s p_i$. It follows easily that f^{n_i} is a point-wise almost periodic correspondence on the entire arc A_i, $f^{n_i}(v_s) = v_s$ and $f^{n_i}(v_t) = v_t$. Then f^{n_i} leaves every point of A_i invariant by 6.4.1.

For each A_i there exists then an integer n_i such that f^{n_i} leaves every point of A_i invariant. Hence f^m, where $m = \prod n_i$, leaves every point of X invariant, i. e. f is periodic on X.

The University of Michigan.