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A generalized theorem on oscillating functions.

By

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A one-valued continuous function of a real variable which oscillates everywhere in a given interval I, repeats, according to Koenig [1]), at least one of its values an infinite number of times in I. We generalize this theorem by showing that it suffices to assume that the function oscillates everywhere in any perfect subset K or I [2]) in order to reach the same conclusion about the existence of infinitely many times repeated functional values in I.

The application of this result enables us to offer a straightforward treatment, based on elementary point set theory, of the following problem: Let $x(t)$ and $y(t)$ be one-valued continuous functions in a given interval in which the derivative of $y(t)$ with respect to $x(t)$ vanishes everywhere in the interval; in other words, let $\lim \frac{y(t)-y(t_0)}{x(t)-x(t_0)}=0$ whenever t tends toward t_0 by a sequence of values other than those for which $y(t)-y(t_0)=x(t)-x(t_0)=0$. It is required to show that $y(t)$ is constant throughout the interval [3]).

1. *Lemma.* *Let $x(t)$ be a one-valued continuous function in a closed interval I and let $x(t)$ oscillate everywhere in a perfect subset K of I. Then $x(t)$ repeats at least one of its values an infinite number of times in I.*

[1]) See A. Schoenflies, *Bericht über der Mengenlehre*, 1900; p. 160.

[2]) In other words, there exists no open interval of I, having points in common with K, in which the functional values of K never increase or never decrease.

[3]) See K. Petrovsky, *Rec. Math. Soc. Math. Moscou* **41** (1934), 48-58. Also S. Saks, *Theory of the Integral*, Monografie Matematyczne **7** (1937), p. 275, and R. Caccioppoli, *Sul lemma fondamentale del calcolo integrale*, Atti Mem. Accad. Sci. Padova **50** (1934), 93-98.

We notice that unless the complement of K in I is zero, it consists of open intervals, whose end-points are in K, no two intervals having an end-point in common. We denote by I_i' any such interval in which $x(t)$ assumes the same value at both its end-points, and by I any of the remaining intervals.

We choose a closed interval J of the real line and select in J a set of points $\{z_N\}$ similar to set of the intervals $\bar{I}_i'$ [1]) in I in their natural order arrangement. It is possible then to set up an order-preserving correpondence S which carries in one-to-one way each interval $\bar{I}_i'$ into the corresponding z_N and, likewise, every remaining point of I in one-to-one way into a point z of the complement of $\{z_N\}$ in J. Under S every interval I_i will go over into an open interval δ_i and the set K into a perfect set T of J. (In case no intervals I_i' are present in the complement of K in I, or in case this complement is zero, we simply choose for J the interval I and for S the identical transformation).

We define in J a function $x_1(z)$ by assuming that, for any point of the set $\{z_N\}$, $x_1(z_N)$ is equal to the value of $x(t)$ at either end-point of the corresponding $\bar{I}_i'$; for any point z of T which is not in $\{z_N\}$, and thus corresponds to a unique t, $x_1(z)$ is equal to $x(t)$; finally, on any interval δ_i, $x_1(z)$ is assumed to vary linearly between the two values fixed at its end-points in accordance with the foregoing rule. It is clear that the function $x_1(z)$ thus defined is one-valued and continuous; furthermore, since $x(t)$ assumes on any interval I_i' every value which is taken on by $x_1(z)$ on the corresponding δ_i, it follows that if $x_1(z)$ has values repeated an infinite number of times in J, the same conclusion must hold for $x(t)$ in I. We shall show that $x_1(z)$ actually possesses this property.

We notice that $x_1(z)$ must oscillate everywhere in T if $x(t)$ oscillates in K. Hence every point α of T must be either a proper maximum of $x_1(z)$ or a limit point of such proper maxima; otherwise in any neighborhood of α there would be available an open interval in which the functional values of T would never increase or never decrease. We denote by M the set of the proper maxima of $x_1(z)$ in J. Because $x_1(z)$ varies linearly on every interval δ_i, no point of M belongs to a $\delta_i \cdot M$, therefore, is a proper subset of T lying everywhere dense in T and hence is dense in itself.

[1]) As usual, $\bar{I}_i'$ denotes the closure of I_i'.

With every point μ of M we associate two intervals $\theta_l = (\mu - m_l, \mu]$ and $\theta_r = [\mu, \mu + m_r)$ such that their sum θ becomes the largest interval containing μ for every point z of which, with the exception of μ, $x_1(z) < x_1(\mu)$.

We observe that both θ_l and θ_r belonging to a point μ of M must have points in common with M besides μ. This is obvious if μ is not an end-point of a δ_i, for in this case, in any neighborhood of μ, there are available points of T, and hence points of M, lying both to the right or left of μ. If, however, μ is an end-point of a δ_i, say, a right-hand point, then θ_r will certainly have points in common with M besides μ; as regards θ_l we notice that is must, in this case, contain the left-end point of δ_i in its interior and the conclusion is the same.

We are in a position now, following Koenig's line of argument, to complete the proof of the lemma. We choose, namely in J a point μ of M. We denote by θ' any one of its intervals θ_l and θ_r which contains the absolute minimum value of $x_1(z)$ in θ. The other part of θ we denote by θ_1'. We choose then in the interior of θ_1' a point μ_1 of M. It is clear that there must be available in θ' a point μ_1', such that $x(\mu_1) = x(\mu_1')$. We repeat the same process with μ', Its interval θ_1 is thus sub-divided in two intervals θ'' and θ_1'' and again we fix in the intervals θ', θ'' two points μ_2', μ_2'', respectively, such that $x_1(\mu_2) = x_1(\mu_2') = x_1(\mu_2'')$. We keep on repeating this process indefinitely and obtain in this way a sequence of intervals $\theta', \theta'', \theta''', \ldots, \theta^{(\lambda)}, \ldots$ It is clear that no two of these intervals, end-points included, can have points in common. Let $\mu_1', \mu_2', \mu_3', \ldots$ be the points determined by the described process in θ'. Let μ_ω' be one of their limit points. We are able to choose in each of the remaining intervals $\theta'', \theta''', \ldots, \theta^{(\lambda)}, \ldots$ limit points $\mu_\omega'', \mu_\omega''', \ldots, \mu_\omega^{(\lambda)}, \ldots$ such that

$$x_1(\mu_\omega') = x_1(\mu_\omega'') = \ldots = x_1(\mu_\omega^{(\lambda)}) = \ldots$$

This proves our lemma.

2. *Theorem.* *Let $x(t)$ and $y(t)$ be one-valued continuous functions in a closed interval I. Then if the derivative of $y(t)$ with respect to $x(t)$ vanishes everywhere in I, $y(t)$ is constant throughout I.*

We denote by C the curve $y = y(t)$, $x = x(t)$. Let $y(t)$ not be constant everywhere in I. We show that under this assumption C cannot be a simple curve.

Suppose, indeed, that C has no multiple points. This implies, of course, that for any sequence of points $t_1, t_2, \ldots, t_n, \ldots$ converging toward a point t in I, the relation $y(t)-y(t_i)=x(t)-x(t_i)=0$ is never satisfied, and hence the relation

$$\lim_{n=\infty}\frac{y(t_n)-y(t)}{x(t_n)-x(t)}=0 \tag{1}$$

holds for any converging sequence in I without exceptions. We conclude that $x(t)$ can, therefore, repeat none of its values in points of an infinite subset of I, for at a limit point of such points (1) could not hold.

We denote by Q the set of those points of I every one of which can be covered by an open interval in which $y(t)$ remains constant. Q is obviously an open set. Because $y(t)$ is continuous and not constant, the complement of Q in I is a perfect set K distinct from zero. In any neighborhood of a point a of K the function $y(t)$ is never constant. We shall show that this conclusion leads to a contradiction.

The components of Q are intervals of constancy of $y(t)$. We denote, as in the proof of the lemma, by I'_i any of these intervals in which $x(t)$ assumes the same value at both its end-points and by I_i any one of the remaining intervals. We introduce again the interval J and determine in the latter by means of the correspondence S the open intervals δ_i and the perfect set T. By using the device given in the proof of the lemma, we finally carry both functions $x(t)$ and $y(t)$ in I into functions $x_1(z)$ and $y_1(z)$ in J. We recall that $x_1(z)$ can repeat none of its values an infinite number of times in J because $x(t)$ does not possess this property in I. We observe also that $y_1(z)$ is never constant in any neighborhood of a point of T.

Let $z_1, z_2, \ldots, z_n, \ldots$ be any sequence of points of J with z as a limit point. We show that

$$\lim_{n=\infty}\frac{y_1(z_n)-y_1(z)}{x_1(z_n)-x_1(z)}=0. \tag{2}$$

This relation is certainly satisfied whenever z is an inner point of a δ_i, for the numerators in (2) vanish then for sufficiently large values of n. Let, therefore, z be a point of T. Without loss of generality we may assume that the z_i's approach z from one side only, say, from the left. Let, accordingly, $z_1<z_2<\ldots<z_n<\ldots<z$.

It is always possible to choose in I a corresponding sequence $t_1<t_2<\ldots<t_n<\ldots$ such that $y_1(z_i)=y(t_i)$ and $x_1(z_i)=x(t_i)$, for every i. Whenever, namely, z_i is in T and is the transform of a unique point of I under S, we choose this latter point as t_i; if, however, z_i is the transform of a whole intervalle I'_i, we take for t_i any one of its end-points; finally, if z is a point of a δ_i, we select t_i from among those, always available, points of the corresponding I_i for which $x_1(z_i)=x(t_i)$. The sequence of the t_i's obviously converges toward a point t. Because of continuity, $x(t)=x_1(z)$ and $y(t)=y_1(z)$; hence

$$\lim_{n=\infty}\frac{y_1(z_n)-y_1(z)}{x_1(z_n)-x_1(z)}=\lim_{n=\infty}\frac{y(t_n)-y(t)}{x(t_n)-x(t)}=0.$$

Since $x_1(z)$ repeats none of its values an infinite number of times in J, it follows from our lemma that $x_1(z)$ does not oscillate everywhere in T. In other words, there exists in J an open interval η which has points in common with T and in which the functional values of T never increase or never decrease. Because the complement of the intersection $D(\eta, T)$ in η consists of intervals δ_i, in every one of which $x_1(z)$ varies linearly, we conclude that $x_1(z)$ is strictly monotone in η. In virtue of (2), $y_1(z)$ is, therefore, constant in η. This contradicts the above reached conclusion that $y_1(z)$ is never constant in any neighborhood of a point belonging to T. Hence if $y(t)$ is not constant the curve C is not simple.

We must, therefore, assume that C has multiple points if $y(t)$ is not constant. We shall show that this assumption ir also untenable.

Let, indeed, $A(x_1, y)$ and $B(x_2, y_2)$ be two points of the curve C, with $y_1 \neq y_2$. It is known that there exists then a simple curve $C_1 \subset C$ which joins the points A and B. The curve C_1 may be taken to be parametrically represented by two one-valued continuous functions $x=\overline{x}(T)$ and $y-\overline{y}(T)$ where T varies in the interval $[0,1]$. Let $T_1, T_2, \ldots, T_n, \ldots$ be a converging sequence of points, with T as a limit point, and let σ be an accumulation point of the sequence of ratios $\dfrac{\overline{y}(T_i)-\overline{y}(T)}{\overline{x}(T_i)-\overline{y}(T)}$, $i=1,2,\ldots,n,\ldots$ There exists then a subsequence $T'_1, T'_2, \ldots, T'_n, \ldots$ of the given sequence for which,

$$\lim_{n=\infty}\frac{\overline{y}(T'_n)-\overline{y}(T)}{\overline{x}(T'_n)-\overline{y}(T)}=\sigma.$$

With each point T_i' in the interval $[0,1]$ we associate a point t_i in I for which $x(t_i)=\overline{x}(T_i')$ and $y(t_i)=\overline{y}(T_i')$. Let t be an accumulation point of the t_i's and let the properly choosen sub-sequence $t_1', t_2, \dots, t_n', \dots$ of the t_i's, associated with the sub-sequence $T_1'', T_2'', \dots, T_n'', \dots$ of the T_i''s, converge towards t. It is clear that $x(t)=\overline{x}(T)$ and $y(t)=\overline{y}(T)$. We have further:

$$\sigma=\lim_{n=\infty}\frac{\overline{y}(T_n')-\overline{y}(T)}{\overline{x}(T_n')-\overline{x}(T)}=\lim_{n=\infty}\frac{\overline{y}(T_n'')-\overline{y}(T)}{\overline{x}(T_n'')-\overline{x}(T)}=\lim_{n=\infty}\frac{y(t_n')-y(t)}{x(t_n')-x(t)}=0.$$

Hence the sequence $\dfrac{\overline{y}(T_i)-\overline{y}(T)}{\overline{x}(T_i)-\overline{x}(T)}$, $i=1,2,\dots,n,\dots$ always converges toward zero. As we have shown above, this conclusion conflicts with the assumption that C_1 is a simple curve. Hence $y(t)$ must be constant throughout I.

3. In conclusion we shall state without proof one more result related to the problem discussed in our lemma. We notice that the proper maxima of the function $x_1(z)$, introduced in the process of the proof, form a set dense in itself. It can be shown that, in general, whenever the proper maxima of a one-valued continuous function in a given interval form a set dense in itself the function must have at least one value repeated an infinite number of times in the interval.

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Sur les courbes ε-déformables en arcs simples.

Par

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Le but de cette Note est la caractérisation intrinsèque des courbes planes qui, pour tout $\varepsilon>0$, se laissent ε-déformer en un arc simple. La famille de toutes ces courbes sera désignée par (A); je les appelle aussi *apparentées avec l'arc simple*.

Je démontre que la famille (A) coïncide avec celle des courbes planes K qui ne coupent pas le plan et qui jouissent en chacun de leurs points de la propriété suivante [1]):

(p_3) *pour chaque système de 3 sous-continus de K qui contiennent le point donné, l'un d'eux fait partie de la somme de deux autres.*

Termes et notations. Je désigne, pour chaque couple de points x,y d'un espace métrique R, par $\varrho(x,y)$ la distance entre ces points et par $\widehat{xy}$ un arc simple aux extrémités x et y, contenu dans R.

Etant donnés dans R deux ensembles quelconques A et B, je pose

$$\varrho(A,B)=\inf_{x\in A,\, y\in B}\varrho(x,y);$$

le diamètre de A sera désigné par $d(A)$.

J'appelle *dendrite finie* une dendrite (c. à d. continu localement connexe ne contenant aucune courbe fermée) qui est somme d'un nombre fini d'arcs simples.

Une transformation continue f d'un ensemble $A\subset R$ en un autre $B\subset R$ est dite une *ε-déformation*, lorsqu'on a

$$\sup_{x\in A}\varrho\bigl(x,f(x)\bigr)\leqslant\varepsilon.$$

Si une telle déformation existe pour tout $\varepsilon>0$, A est dit *ε-déformable* en B.

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[1]) L'équivalence en question a été signalée dans mon travail *O pokrewieństwie kontynuów*, Wiadomości Matematyczne **43** (1936), p. 1-57 (en polonais) qui en renferme une ébauche de la démonstration.