Or, je vais montrer qu'il n'existe aucun ensemble linéaire K de I-e catégorie, pas plus que deux suites infinies croissantes de nombres naturels $\{m_k\}$ et $\{n_k\}$, telles qu'on ait

(1) $$\lim_{k=\infty} f_{n_k}^{m_k}(x) = 0 \quad \text{pour} \quad x \; non \in K.$$

Supposons, en effet, qu'un tel ensemble K et deux suites $\{m_k\}$ et $\{n_k\}$ existent. Il existerait donc une décomposition

(2) $$K = N_1 + N_2 + \ldots$$

en ensembles N_i $(i=1,2,3,\ldots)$ non-denses. On aurait donc, pour un intervalle (fermé) d_1, l'égalité $d_1 N_1 = 0$. La suite d'indices $\{m_k\}$ étant croissante, il existe un nombre naturel k_1, tel que

(3) $$1/m_{k_1} < \bar{d}_1,$$

où $\bar{d}_1$ désigne la longueur de l'intervalle d_1. La fonction $f_n^m(x)$ prenant par définition la valeur 1 dans chacun des intervalles de longueur $1/m$, il existe d'après (3) un nombre $x_1 \in d_1$, tel que $f_{n_{k_1}}^{m_{k_1}}(x_1) = 1$. La fonction $f_{n_{k_1}}^{m_{k_1}}(x)$ étant continue, il existe un intervalle fermé $\delta_1 \subset d_1$, tel que $f_{n_{k_1}}^{m_{k_1}}(x) > 1/2$ pour $x \in \delta_1$.

Or, l'ensemble N_2 étant non-dense, il existe un intervalle $d_2 \subset \delta_1$ tel que $d_2 N_2 = 0$. Comme plus haut, on trouve un entier $k_2 > k_1$ tel que $1/m_{k_2} < \bar{d}_2$, ensuite un $x_2 \in d_2$ tel que $f_{n_{k_2}}^{m_{k_2}}(x_2) = 1$, et, enfin, un intervalle $\delta_2 \subset d_2$ tel que $f_{n_{k_2}}^{m_{k_2}}(x) > 1/2$ pour $x \in \delta_2$.

En raisonnant ainsi de suite, nous obtenons une suite infinie d'intervalles fermés

(4) $$d_1 \supset \delta_1 \supset d_2 \supset \delta_2 \supset d_3 \supset \ldots$$

et une suite infinie croissante d'indices $\{k_i\}$, telles que $d_i N_i = 0$ $(i=1,2,3,\ldots)$ et que

(5) $$f_{n_{k_i}}^{m_{k_i}}(x) > 1/2 \quad \text{pour} \quad x \in \delta_i \qquad (i=1,2,\ldots).$$

D'après (4), il existe un point x_0 tel que $x_0 \in d_i \delta_i$ pour $i=1,2,\ldots$ Comme $d_i N_i = 0$, on aurait donc d'après (2) $x_0 \; non \in K$ et d'après (5) $f_{n_{k_i}}^{m_{k_i}}(x_0) > 1/2$ pour $i=1,2,\ldots$, de sorte que la formule (1) ne serait pas vraie pour $x = x_0$.

Exceptional Sets.

By

H. Blumberg (Columbus, Ohio).

The following considerations originate from a trivial remark, called the *germ principle*, and analogous trivia, and lead by a line of reasoning that readily suggests itself to a variety of simple properties, mostly new, of the general set and the general (real) function. The results of the use of the germ principle are of a comprehensive scope, have numerous connections with the literature, and unify various results appearing in the literature as unrelated or remotely related.

Our considerations will first relate to the linear continuum. By an *interval property* P, we understand a property such that if I is a (closed) interval of the linear continuum, then either I^P (I has property P) or $I^{\bar{P}}$ (I hasn't property P, I has the contrary property $\bar{P}$). Since an interval $I=(a,b)$ has two ends, and the relations I^P, $I^{\bar{P}}$ are mutually exclusive, it follows that *if a is such that $(a,b)^P$, then b is not such that $(a,b)^{\bar{P}}$*. This form of an utter triviality is the germ of all that follows.

If P is a given interval property, we shall say that a is a *point of antisymmetry* (with respect to P) if, for all sufficiently small positive h, $(a,a+h)^P$, $(a,a+h)^{\bar{P}}$ respectively imply $(a,a-h)^{\bar{P}}$, $(a,a-h)^P$. a will be said to be a point of δ-antisymmetry if it is a point of antisymmetry and the relations defining antisymmetry hold for all $h<\delta$. Let T be a set of positive numbers with 0 as a limit point; and let S be the set of points a of antisymmetry such that, for sufficiently small positive h, $(a, a+h)^{P \text{ or } \overline{P}}$ according as h belongs or does not belong to T. We classify the points of S, defining S_n for positive, integral n as the set of points of S which are of $1/n$ — antisymmetry and such that, for all $h<1/n$, we have $(a, a+h)^{P \text{ or } \overline{P}}$ according as h belongs or does not belong to T. Using the germ principle of the present paper, we note that if a is a point of S_n, no point $a \pm h$ belongs to S_n if $h<1/n$. S_n is therefore an isolated set, and hence at most denumerable. It follows that S is at most denumerable, and we may state

Theorem 1. *If P is an interval property; T a set of positive numbers with with 0 as a limit point; and S the set of points a of antisymmetry such that, for sufficiently small positive h, $(a, a+h)^{P \text{ or } \overline{P}}$ according as h belongs or does not belong to T, then S is at most denumerable.*

As an illustration of the use of Theorem I, we employ it to derive a property of unconditioned real functions. If $f(x)$ is a given real function — unconditioned, but understood to be finite and one-valued — let $I=(a,b)$ be said to have property P if $q(a,b)=\frac{f(b)-f(a)}{b-a}>k$, where k is a given real number. Let T of Theorem I denote the set of positive rational numbers. Theorem I then tells us that the points a such that, for sufficiently small positive h, $q(a,a+h)>k$, $q(a,a-h)\leqslant k$ if h is rational, and $q(a,a+h)\leqslant k$, $q(a,a-h)>k$ if h is irrational constitute a set which is at most denumerable. *A fortiori*, the points a such that, for sufficiently small positive h, we have the four inequalities just stated with $<$, $>$ respectively substituted for $\leqslant$, $\geqslant$ constitute an at most denumerable set, which we denote by D_k. Let $D=\Sigma D_k$, k ranging over the set of rational numbers; D is then denumerable. Suppose now that a point a is of the following character: 1) All the curve points $(a \pm h, f(a \pm h))$, with h positive, rational and sufficiently small, can be enclosed in a sector of a circle of center $(a, f(a))$ and angle less than 180°; (2) all the curve points $(a \pm h, f(a \pm h))$ with h positive, irrational and sufficiently small, can be likewise enclosed in a sector of center $(a, f(a))$ and angle less than 180°; and (3) these sectors have no common interior or boundary points except for $(a, f(a))$. If a has properties (1), (2) and (3), we shall say, for short,

that f has a „double-way corner" at a. If f has a double-way corner at a, it is possible to choose a rational number k so as to have a satisfy the relations imposed upon the elements of D_k, or else have a satisfy the same relations with the inequality signs reversed, and then a will belong to a set D'_k, say, likewise denumerable. We may therefore conclude that *the points at which a given, arbitrary, real function has a double-way corner constitute a set which is at most denumerable*. This property of an arbitrary function obviously remains valid if instead of the set of rationals and the set of irrationals we take any two complementary sets of the real continuum each symmetric about the origin. In the particular case where one of these two sets is the null set, the property just obtained reduces to a known property of unconditioned functions, derived below by different reasoning.

If in Theorem I we particularize T to be the totality of positive numbers, we get

Corollary I. *If P is an interval property, and S the set of points a such that, for sufficiently small positive h, $(a, a+h)^{P}$, and $(a, a-h)^{\overline{P}}$, then S is at most denumerable.*

Let $f(I)=f(a,b)$ be a given (real) interval function; that is to say, with every interval $I=(a,b)$ — of the linear continuum or of a basal interval — there is associated a real number $f(I)=f(a,b)$. In terms of f and a given real number k, we define the interval property P_k by stipulating that I^{P_k} mean $f(I)>k$. Substituting P_k for P in Corollary I, we note that the set of points ξ such that, for sufficiently small positive h, $f(\xi, \xi+h)>k$ and $f(\xi, \xi-h)<k$ is at most denumerable. Denoting this set by D_k, and letting $D=\Sigma D_k$, where the summation is extended over all rational k, we conclude that D is at most denumerable. Now if $f_+ = \liminf_{h\to 0} f(\xi, \xi+h) > \limsup_{h\to 0} f(\xi, \xi-h) = f^-$, we can choose a rational number k such that, for all sufficiently small positive h, $f(\xi,\xi+h)>k$ and $f(\xi,\xi-h)<k$. ξ thus belongs to D_k. The points where $f_+ > f^-$ therefore constitute an at most denumerable set. Likewise the points where

$$f_- = \liminf_{h\to 0} f(\xi, \xi-h) > \limsup_{h\to 0} f(\xi, \xi+h) = f^+$$

constitute an at most denumerable set. We conclude that, except at the points of an at most denumerable set, we have everywhere $f^{-} \geqslant f_{+}$ and $f^{+} \geqslant f_{-}$. Calling the intervals (f_{-}, f^{-}), (f_{+}, f^{+}) the left, right limit interval of $f(I)$, we may state

Corollary II. *If $f(I)$ is an interval function, then at every point, with the possible exception of an at most denumerable set, the two limit intervals of $f(I)$ overlap or abut.*

Corollary II can be utilized to derive a variety of theorems on general (real) functions. To illustrate, let $f(x)$, then, be a given (one-valued, real) function. The difference quotient $q(a,b)=\dfrac{f(b)-f(a)}{b-a}$ is an associated interval function. Applying Corollary II, we get the result:

For every function $f(x)$, the upper right (left) derivate exceeds or equals the lower left (right) derivate at every point with the possible exception of an at most denumerable set[1]).

In defining antisymmetry at ξ we required that the intervals $(\xi, \xi-h)$, $(\xi, \xi+h)$ be „differently qualified“ — i. e., the one has property P, the other $\overline{P}$ — for *all* sufficiently small positive h. This type of antisymmetry may be regarded as of absolute character. We may pass to a relative notion by requiring such different qualification only for the h's of a given set H with reference to which the relative notion is to be valid. Let H, then, be a given set of positive numbers having 0 as limit point (the latter restriction giving the only case of interest). If P is a given interval property, we shall say that ξ is a point of antisymmetry of P relative to H if for all sufficiently small h belonging to H, the intervals $(\xi, \xi-h)$, $(\xi, \xi+h)$ are differently qualified. If H is the set of positive numbers — we then revert to absolute antisymmetry — there cannot be, according to Corollary I, more than $\aleph_0$ points a of such symmetry at which $(a, a+h)^{P}$ for sufficiently small h. If H is a given residual set, — with respect to the totality of positive numbers — let E be the set of points ξ such that $(\xi, \xi+h)^{P}$ and $(\xi, \xi-h)^{\overline{P}}$ if h belongs to H and is sufficiently small. Let E_n, where n is a positive integer, be the subset of E consisting of the points ξ for which h is small enough if $<1/n$. It follows from the germ principle that if ξ belongs to E_n, no point $\xi+h$, $\xi-h$, with $h<1/n$ and in H can belong to E_n. E_n is therefore such that every point of it can be enclosed in an interval with the part of E_n in it an exhaustible set. It follows that E_n is exhaustible, and therefore $E=\sum_{1}^{\infty} E_n$ is exhaustible. We thus have the result:

[1]) Due to W. Sierpiński, Bull. Acad. Sc. Cracovie (1912), p. 850, and G. C. Young, Acta Math. **37** (1914), p. 147.

If P is an interval property, H a set of positive numbers such that the positive numbers not belonging to H form an exhaustible set, and E the set of points ξ such that there exists an h_ξ with the property that if h belongs to H and is less than h_ξ we have $(\xi, \xi+h)^{P}$ and $(\xi, \xi-h)^{\overline{P}}$, then E is exhaustible.

By replacing P by $\overline{P}$, keeping H the same, we obtain another associated exhaustible set say E_1. If ξ belongs neither to E nor to E_1, there exists an infinitesimal number $h_\xi^{(n)}$ $(n=1,2,\ldots,\infty)$, belonging to H such that $(\xi, \xi+h_\xi^{(n)})$ and $(\xi, \xi-h_\xi^{(n)})$ either both have property P or both have property $\overline{P}$, that is, as we may say, these two intervals are „identically qualified“ (with respect to P); or also there exists on infinitesimal number $h_\xi^{(n)}$ belonging to H such that $(\xi, \xi+h_\xi^{(n)})^{P \text{ or } \overline{P}}$ according as n is odd or even. We may thus state the following theorem.

Theorem I_a. *If P is an interval property, and H a set of positive numbers residual with respect to the totality of positive numbers, then for the points ξ of a set residual with respect to the linear continuum we can find an infinitesimal number $h_\xi^{(n)}$ $(n=1,2,\ldots,\infty)$ belonging to H such that $(\xi, \xi+h_\xi^{(n)})$ and $(\xi, \xi-h_\xi^{(n)})$ are identically qualified for all the n, or such that $(\xi, \xi+h_\xi^{(n)})^{P \text{ or } \overline{P}}$ according as n is odd or even.*

If H is such that its complement with respect to the totality of positive numbers is of measure 0, we obtain, by analogous reasoning, the following

Theorem I_b. *If P is an interval property, and H a set of positive numbers whose complement with respect to the totality of positive numbers is of measure 0, then for the points ξ of a set whose complement in the linear continuum is of measure 0, we can find an infinitesimal number $h_\xi^{(n)}$ $(n=1,2,\ldots,\infty)$ belonging to H such that $(\xi, \xi+h_\xi^{(n)})$, and $(\xi, \xi-h_\xi^{(n)})$ are identically qualified for all the n, or such that $(\xi, \xi+h_\xi^{(n)})^{P \text{ or } \overline{P}}$ according as n is odd or even.*

Theorem I_b can be extended so as not to require that the complement of H, with respect to the totality of positive numbers, be of measure 0. For suppose H is such that its upper, interior metric density is positive at 0; that is to say, there exists a sequence of positive numbers h_n with 0 as limit such that the relative interior measure of H in $(0,h_n)$ exceeds a fixed positive number. We can then prove by means of the germ principle that if ξ is a point of E_n, where E_n has the readily understood meaning analogous to that of E_n in the argument preceding Theorem I_a, then the upper, interior metric density of the complement of E_n (with respect to the linear continuum) is positive at ξ. It follows that E_n is of measure 0, and therefore $E=\sum_1^\infty E_n$ is of measure 0. We thus have the following extension of Theorem I_b.

Theorem I_c. *If P is an interval property, and H a set of positive numbers which is of positive, upper, interior metric density at 0, then for the points ξ of a set whose complement in the linear continuum is of measure 0, we can find an infinitesimal number $h_\xi^{(n)}$ $(n=1,2,\ldots,\infty)$ belonging to H such that $(\xi,\xi+h_\xi^{(n)})$ and $(\xi,\xi-h_\xi^{(n)})$ are identically qualified for all the n, or such that $(\xi,\xi+h_\xi^{(n)})^{P \text{ or } \overline{P}}$ according as n is odd or even.*

Since an interval of the linear continuum may be represented by means of a point in the plane, an interval property may be represented by means of a planar point set. Our results on interval properties may thus be interpreted as giving information about planar point sets. It will be simpler, in passing from interval properties to planar point sets, to confine ourselves to intervals whose end points are positive, as we may, without change in generality. The linear interval will then be represented by means of the planar point (x,y); and the totality of intervals under consideration will be represented by means of the points in the first quadrant bounded by $x=0$, $y=x$. We denote this half quadrant by R. With a given interval property P, we associate the planar point set S consisting of the points of R that represent intervals having property P. In virtue of this association, we can rewrite Theorems I, I_a, I_b and I_c as theorems on arbitrary point sets in R. Theorem I thus becomes the following theorem — when we conjoin its validity for P with its validity when P is replaced by $\overline{P}$.

Theorem I'. *If S is a subset of R, and T a set of positive numbers having 0 as limit point, then, for all real numbers ξ not belonging to an exceptional set which is at most denumerable, either a) there exists a sequence $h_\xi^{(n)}$ $(n=1,2,\ldots,\infty)$ of positive numbers with 0 as limit such that for every n the two points $(\xi-h_\xi^{(n)},\xi)$ and $(\xi,\xi+h_\xi^{(n)})$ belong either both to S or both to $\overline{S}=R-S$; or b) there exists a sequence $h_\xi^{(n)}$ $(n=1,2,\ldots,\infty)$ of numbers with limit 0 belonging all to T or all to $\overline{T}$ (=complement of T with respect to the set positive numbers) such that $(\xi,\xi+h_\xi^{(n)})$ belongs to S or $\overline{S}$ according as n is odd or even; or c) there exists a sequence $h_\xi^{(n)}$ of numbers with limit 0 such that $h_\xi^{(n)}$ belongs to T or $\overline{T}$ according as n is odd or even, and $(\xi,\xi+h_\xi^{(n)})$ belongs to S for all n or to $\overline{S}$ for all n.*

Theorem I' states a relationship between an arbitrary subset S of R and the boundary $y=x$ of R in terms of the local behavior of S at a point of this boundary as the point is approached from R in the directions parallel to the coordinate axes. It is obvious, however, — as may be seen, for example, by means of transformation of coordinates — that the relationship in question does not depend on the fact that the boundaries of R make an angle of 45^0 with one another, or that the directions of approach to the general point (ξ,ξ) of one of these boundaries are taken respectively parallel and perpendicular to the other boundary. Theorem I', stripped of these obviously superfluous conditions, takes the following form.

Theorem II'. *Let S be a planar set; l a straight line; α and β two directions of approach to l from one of the two half planes into which l divides the plane; T a set of positive numbers having 0 as a limit point; and λ a positive number [2]). Then, for all real numbers ξ not belonging to an exceptional set which is at most denumerable, either a) there exists a sequence $h_\xi^{(n)}$ $(n=1,2,\ldots,\infty)$ of positive numbers with 0 as limit such that the two points $(\xi-h_\xi^{(n)},\xi)$ and $(\xi,\xi+\lambda h_\xi^{(n)})$ belong both to S or both to $\overline{S}$ for every n; or b) there exists a sequence of numbers $h_\xi^{(n)}$ $(n=1,2,\ldots,\infty)$ with limit 0 belonging all to T or all to $\overline{T}$ such that $(\xi,\xi+\lambda h_\xi^{(n)})$ belongs to S or $\overline{S}$ according as n is odd or even; or c) there exists a sequence of numbers $h_\xi^{(n)}$ with limit 0 such that $h_\xi^{(n)}$ belongs to T or $\overline{T}$ according as n is odd or even, and $(\xi,\xi+\lambda h_\xi^{(n)})$ belongs to S for all n or to $\overline{S}$ for all n.*

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[2]) λ represents the ratio of the units of length for the transformed axes, which have the directions α and β.

Theorem I″ readily admits of further generalizations in different directions. But it is not our present purpose to pursue such extensions.

If in Theorem I″ we let T be the set of all positive numbers, we obtain as an implication the following simple geometric property of the arbitrary point set.

Corollary III. *If S is a point set lying in the plane π; l a straight line in π; and α, β two directions of approach to l from one of the two half planes into which l divides π, then, except for an at most denumerable set, every point of l is approached in both directions either via S or in both directions via $\overline{S}(=\pi-S)$.*

Thus, in the sense specified, every point of l, with the exception of a set which is at most denumerable, bears the same relation, in respect to S, to both directions α and β.

We omit the statement of the theorems on general point sets derived from Theorems I_a and I_b by means of the representation of an interval property as a planar point set. But we shall state the theorem on point sets corresponding to theorem I_c.

Theorem I_c''. *If S is a point set lying in the plane π; l a straight line in π; α, β two directions of approach to l from one of the two half planes into which l divides π; H a set of positive numbers of positive, upper, interior metric density at 0; and λ a positive number; then for every point A of l not belonging to an exceptional set of measure 0, there exists an infinitesimal element $h_A^{(n)}$ $(n=1,2,\ldots,\infty)$ of H such that the two points $A-h_A^{(n)}\cdot\alpha$, $A-\lambda h_A^{(n)}\cdot\beta$ both belong either to S or to $\overline{S}$ $(=\pi-S)$ for all n, or such that $A-h_A^{(n)}\cdot\alpha$ belongs to S or $\overline{S}$ according as n is odd or even; here $A-t\cdot\delta$, where A is a point, t a real positive number, and δ a direction, means the point B at distance t from A such that BA has direction δ.*

In Theorem I_a, H is a fixed set independent of the point ξ. If we permit it to depend on ξ, — let us denote it in tbis case by H_ξ — it can be shown by means of the continuum hypothesis, that it is possible to define an interval property P such that eevry point ξ is a point of antisymmetry, with $(\xi,\xi+h)^P$ for sufficiently small h, on the understanding that the intervals $(\xi,\xi+h)$, with h not in H_ξ, are negligible, where H_ξ is the set of positive numbers minus an at most denumerable set variable with ξ. The following example shows this possibility. Let $x_1, x_2, \ldots, x_\omega, \ldots, x_\alpha, \ldots$, $\alpha<\Omega$, be a normal order of the continuum of type Ω, the ordinal initiating $\aleph_1$. We define the desired interval property by induction. We first prescribe that

$(x_1x_1+h)^P$, $(x_1,x_1-h)^{\overline{P}}$ for all positive h. Assuming that qualification relative to P has been made for all intervals with x_λ as end point if $\lambda<\alpha$, we require that $(x_\alpha,x_\alpha+h)^P$, $(x_\alpha,x_\alpha-h)^{\overline{P}}$ for all positive h except such as lead to conflict with previous prescription. If ξ is a real number, let H_ξ be the set of h's such that $(\xi,\xi+h)^P$ and $(\xi,\xi-h)^{\overline{P}}$. Since every proper initial segement of the Ω order is at most denumerable, it follows that H_ξ contains every positive number with $\aleph_0$ exceptions at most. Therefore, *if denumerable sets be deemed negligible, there exist interval properties P such that every point ξ is a point of antisymmetry* (and, indeed, such that $(\xi,\xi+h)^P$, $(\xi,\xi-h)^{\overline{P}}$ for all positive h with at most $\aleph_0$ exceptions)[3]).

We shall now show how, by use of the germ principle, we are readily led to a theorem on the general segment property, defined for all linear segments in the plane. Let P be such a property; that is to say, P is unrestricted except that, if I is any linear segment lying in the plane, either I^P or $I^{\overline{P}}$. If α and β are two given different directions in the plane, we let $S_{\alpha\beta h}$, where h is a positive number, be the set of points A of the plane such that 1) if B is a point at distance less than h from A and within the angle ($\leqslant 180^0$) of vertex A and α, β as sides, the segment AB has property P; and 2) if B, of distance less than h from A, is in the vertically opposite angle, the segment AB has property $\overline{P}$. It follows by means of the germ principle that if A belongs to $S_{\alpha\beta h}$, no point at distance $<h$ from A within either of the two vertically opposite angles of 1) and 2) can belong to $S_{\alpha\beta h}$. Every point of $S_{\alpha\beta h}$ is therefore the vertex of a double sector in which there are no points of $S_{\alpha\beta h}$. We may therefore conclude that $S_{\alpha\beta h}$ is a set which W. H. Young called „ridé“ [4]). We shall use the term *sparse* in place of *ridé*. Let

[3]) It may be readily shown with the aid of normal orders of the set of real numbers and of the set of perfect sets — without recourse the continuum hypothesis — that an interval property may be defined for which the set of points of antisymmetry is inexhaustible and non-measurable.

[4]) This term has been introduced by W. H. Young to designate what he regards as a particularly natural generalization of *denumerable* to the plane. See *La symétrie de structure des fonctions de variables réelles*, Bull. de Sci. math. sér. 2, **52** (1928) pp. 265—280. As will be seen later (paragraph following Theorem VI), this apparent naturalness is relative to the type of theorem to be extended from the straight line to the plane, and the manner in which such extension is envisaged. Thus, for example, the very theorem which W. H. Young takes as leading naturally to the ridé set as a generalization of the denumerable set can be extended to the plane (see paragraph and footnote following Theorem VI) in a manner no less natural without bringing in a new type of set.

us say that a direction is rational if its slope is rational. Letting α and β range independently over the set of rational directions, and h, independently of α and β, over the set of positive rational numbers, we let S= sum of all $S_{\alpha\beta h}$ obtained by such independent variation of α,β and h. Since the sum of $\aleph_0$ sparse sets is sparse — as follows immediately from the definition — S is a sparse set. If now A is vertex of two vertically opposite (equal) sectors such that if B is in one of them AB^P, and if in the other $AB^{\overline{P}}$, then A is vertex of two such sectors with rational sides and rational radius. A therefore belongs to S. We thus have the following

Theorem II. *If P is a a planar segment property, then all points A of the plane not belonging to an exceptional sparse set are such that, for every pair of vertically opposite sectors with vertex A, there exist two points B and C, one in each sector, such that either both AB and AC have property P, or they both have property $\overline{P}$.*

By means of this theorem, we can obtain a property of the general planar point set. For let S be a given planar set. Let us say that the segment AB has property P if A and B either both belong to S or both belong to $\overline{S}$ (=complement of S with respect to the plane), With this meaning for P, Theorem II yields the following

Corollary IV. *If S is a planar set, then all points A of the plane not belonging to an exceptional sparse set are such that for every pair of vertically opposite sectors having A as vertex there exist two points, one in each sector, either both belonging to S or both belonging to $\overline{S}$.*

We illustrate the use of Theorem II by employing it to derive a property of the general function as related to (Dini) partial derivatives. Let $f(x,y)$ be a given one-valued function. If $A=(x,y)$, we understand $f(A)$ to mean $f(x,y)$. If A and B are two points, we set $\dfrac{f(B)-f(A)}{\delta(A,B)}=Q(A,B)$, where $\delta(A,B)$ is the distance from A to B. We define as follows the segment property P_k in terms of f and the real number k: AB^{P_k} if $Q(A,B)>k$. According to Theorem II, there is associated with P_k an exceptional sparse set E_k. Let $E=\Sigma E_k$ as k ranges over the set of rational numbers; E is a sparse set. Let A be a point of the plane, vertex of the angle $\alpha\beta$ ($<180^0$) with sides the half lines of directions α,β. If $h>0$, let C be the sector of vertex A,

angle $\alpha\beta$, and radius h; and let $l_{\alpha\beta h}$, $u_{\alpha\beta h}$ respectively represent the greatest lower, least upper bound of the numbers $Q(A,B)$ as B ranges in C. Let $l_{\alpha\beta}, u_{\alpha\beta}=\lim\limits_{h\to 0} l_{\alpha\beta h}$, $\lim\limits_{h\to 0} u_{\alpha\beta h}$ respectively. The numbers $l_{\alpha\beta}, u_{\alpha\beta}$ may be regarded as the extreme partial derivatives of f at A within angle $\alpha\beta$. We call these respectively the lower, upper derivate of f at A in angle $\alpha\beta$. If $\overline{\alpha\beta}$ denotes the angle vertically opposite $\alpha\beta$, and $l_{\alpha\beta}>u_{\overline{\alpha\beta}}$, let r be a rational number between $u_{\overline{\alpha\beta}}$ and $l_{\alpha\beta}$. It is then possible to find an h so small that the sectors $C,\overline{C}$ of vertex A, radius h, and angles $\alpha\beta, \overline{\alpha\beta}$ respectively are such that if B is a point of $C,\overline{C}$ we respectively have $Q(A,B)>r$, $Q(A,B)<r$; accordingly A belongs to E, and we may state the following property of the general function.

Corollary V. *If $f(A)=f(x,y)$, where $A=(x,y)$, is a one-valued function, all points A not belonging to an exceptional, sparse set are such that for every pair of vertically opposite angles with vertex A the lower derivate of f at A in each of the angles is less than or equal to the upper derivate of f in the other.*

The argument made in deriving Corollary V shows that in the definition of P_k it is not essential for $Q(A,B)$ to signify difference quotient. The same argument applies just as well to every (real) segment function. Let, then, $Q(A,B)$ stand for any given (real) segment function whatsoever, i. e., if AB is a segment (in the plane), $Q(AB)$ is a real number. If a point A of the plane is vertex of the angle $\alpha\beta$, we understand by $Q_{\alpha\beta}$, $\overline{Q}_{\alpha\beta}$ — in analogy with the definitions for $l_{\alpha\beta}, u_{\alpha\beta}$ — the lower, upper limit of Q at A in the angle $\alpha\beta$. We may then state the following generalization of Corollary V.

Corollary VI. *If $Q(AB)$ is a given real segment function (in the plane), all points A not belonging to an exceptional, sparse set are such that for every pair of vertically opposite angles with vertex A the lower limit of Q at A in each of these angles is less than or equal to the upper limit of Q in the other.*

Examples of the application of Corollary VI are:

a) $Q(A,B)$=saltus of a given function $f(x,y)$ on the segment AB.

$Q_{\alpha\beta}$, $\overline{Q}_{\alpha\beta}$ represent, we may say, the limit inferior, limit superior, for rectilinear approach, of the saltus of f at A in angle $\alpha\beta$.

b) $Q(A,B)$=saltus of f on AB if sets of a given type are regarded as negligible, as when denumerable sets, (linear) exhaustible sets, or (linear) sets of measure 0 are taken as negligible.

c) $Q(A,B)$=relative measure of S on AB, S being a given planar set.

We shall now use (a variant of) the germ principle to derive certain properties of the general point set by reasoning from set character rather than segment property. Let S, then, be any planar set whatsoever, and A a point of the plane. We shall demand from A a certain double character, and it is this double character which, in virtue of the germ principle, will render A exceptional. We shall, namely, suppose first that A is vertex of an acute angle of sides α,β — we designate the respective directions of the sides also by α,β — and that within this angle there is an open, simple arc terminating in A, consisting exclusively of points of S, and containing a point at distance $>h$ from A, where h is a given positive number [5]). We suppose secondly that A is vertex of another, non-overlapping acute angle of sides γ,δ so related to α,β that by shifting the vertex of the second angle an arbitrarily small amount, keeping the direction of its sides unchanged, we can bring both of its sides into intersection with both α and β; moreover, within angle $\gamma\delta$ there is an open, simple arc terminating in A, consisting exclusively of points of $\bar{S}$, and containing a point at distance $>h$ from A. The set constituted by the points having the double character just described we designate by $E_{\alpha\beta\gamma\delta h}$. If A and B are two distinct points of $E_{\alpha\beta\gamma\delta h}$, the connected set for A in $\alpha\beta$ has no points in common with the connected set for B in $\gamma\delta$. Therefore, if A is a point of $E_{\alpha\beta\gamma\delta h}$, there exists a sector with vertex A containing no points of $E_{\alpha\beta\gamma\delta h}$. It follows that $E_{\alpha\beta\gamma\delta h}$ is a sparse set. Let $E=\Sigma E_{\alpha\beta\gamma\delta h}$ as the five subscripts range independently over all possible admissible rational numbers, a direction, as before, being, regarded as rational if it is of rational slope. E also is therefore a sparse set. Suppose now that the point A, whether or not belonging to S, is such that there exists an open simple arc C_1, terminating at A and of determinate direction at A, consisting exclusively of points of S; and that there exists another open simple arc C_2 terminating at A and of determinate direction at A, different from and not opposite to that of C_1, consisting exclusively of points of $\bar{S}$. If A has such a double character, we can choose $\alpha,\beta,\gamma,\delta,h$ rational so as to have A belong to $E_{\alpha\beta\gamma\delta h}$; hence A belongs to E. We may therefore state the following

[5]) A more general curve than simple arc can be here employed, but our present considerations lend no particular interest to such generalization.

Theorem III. *If S is a planar set, let E be the set of points A for which there are two open simple arcs with A as common end point, having determinate directions at A which are neither the same nor opposite, and consisting the one exclusively of points of S, and the other exclusively of points of $\bar{S}$. Then E is a sparse set.*

In other words, if A does not belong to E, and C_1, C_2 are two open simple arcs with A as common end point and of determinate directions at A which are neither the same nor opposite, then C_1 and C_2 both contain points of S or they both contain points of $\bar{S}$. Therefore, if S is a planar set, a point A of the plane, in relation to S, is necessarily of one or the other (or both) of the following types a) and b) — sparse sets being regarded as negligible: a) Every open simple arc terminating at A and of determinate direction at A contains points of S, exception being made of arcs having one particular direction δ_A at A — variable with A — or its opposite; b) every simple arc admissible in a) contains points of $\bar{S}$. Let us call the (entire) straight line through A — where it exists — of the particular direction δ_A the S-singular line through A; and the corresponding line for b) the $\bar{S}$-singular line through A. If A, then, is of type a), it is approachable *via* S along every simple arc having a determinate direction at A not tangent to the S-singular line at A (if the latter exists); and if A is of type b), the same kind of approach is possible with $\bar{S}$ substituted for S.

We may therefore state the following variant of Theorem III, which attests a remarkable symmetry in the structure of every planar set.

Theorem III'. *If S is a planar set, every point A not belonging to an exceptional sparse set is approachable via S along every simple arc of determinate direction at A not tangent to a singular line at A (which may or may not exist), or else A is similarly approachable via $\bar{S}$.*

In regard to converse questions relating to Theorem III′, we define, with the aid of the continuum hypothesis, a planar set S such that at every point of a set which is not sparse we have approach exclusively *via* S along one direction and exclusively via $\bar{S}$ along the opposite direction. For convenience, we confine ourselves to the unit square Q. Let $A_1, A_2, \ldots, A_\omega, \ldots, A_\alpha \ldots$, $\alpha<\Omega$, be a normal order of type Ω of the points of Q, where Ω is the ordinal initiating $\aleph_1$; and let $H_1, H_2, \ldots; H_\omega, \ldots, H_\alpha, \ldots$, $\alpha<\Omega$, be a normal order of the perfect subsets of Q of positive measure. Let B_1 be the first A_α in $H_1=H'$. Let H'' be the first H_α not containing B_1, and B_2 the first A_α in H_α of abscissa different from that of B_1. Assuming that B_λ has been defined for $\lambda<\beta$, let $H^{(\beta)}$ be the first H_α containing no B_λ for $\lambda<\beta$, and $B^{(\beta)}$ the first A_α in $H^{(\beta)}$ of abscissa different from all the abscissas of the B_λ, $\lambda<\beta$. Since there are at most $\aleph_0$ λ's less than β, $H^{(\beta)}$ exists; and since $H^{(\beta)}$ is of positive measure, it necessarily contains a point of abscissa different from that of B_λ for every $\lambda<\beta$, so that $B^{(\beta)}$ exists. Let E be the set of points $B^{(\beta)}$, β ranging over the totality of ordinals $<\Omega$. E contains at least one point from every H_α, $\alpha<\Omega$, and is therefore not of measure zero. We may now define the set S as follows, adjoining to the unit square Q the square Q_1 of vertices $(0,0), (1,0), (1,-1), (0,-1)$ and the square Q_2 of vertices $(0,1), (1,1), (1,2), (0,2)$, and letting $R=Q+Q_1+Q_2$: S consists of the points of R vertically above the points of E; and $\bar{S}=R-S$. It follows that if A is a point of E, it is approachable from above exclusively via S, from below exclusively via $\bar{S}$. Since a sparse set is necessarily of measure 0, and E is not of measure 0, we have an example of a set S such that at all the points of a set which is not sparse we have exclusive approach *via* S (*via* $\bar{S}$) along the negative (positive) direction of the Y-axis.

When we compare Theorem II with Corollary III, we note that the latter is of a less general character, in that it restricts the approach to two fixed directions. But by following a line of reasoning substantially like that for the derivation of Theorem III, we can secure a theorem similar to Theorem III and relating to the approach of a set to a straight line. If S is a given planar set, and l a given straight line, we consider, as before, the set of points A of l which have a certain double character, analogous to that described in the argument for Theorem III. We demand, namely, first that A be vertex of an acute angle of sides α, β — and in conformity with the hypothesis of Corollary III stipulate that the half lines α, β both lie on the same side of l — and that within this angle there is an open, simple arc terminating at A, of diameter greater than a given positive number h, and consisting exclusively of points of S. Secondly, A is to be vertex of a second acute angle $\gamma\delta$ lying on the same side of l as $\alpha\beta$, and so related to $\alpha\beta$ that if the vertex of $\gamma\delta$ is shifted on l to the appropriate side of A, its sides remaining unchanged in direction, the new sides will both intersect both α and β; moreover, within angle $\gamma\delta$ there is an open, simple arc terminating at A, of diameter $>h$, and consisting exclusively of points of $\bar{S}$. The set of points on l having the double character just described we denote by $E_{\alpha\beta\gamma\delta h}$. If A belongs to $E_{\alpha\beta\gamma\delta h}$, there are no points on l belonging to this set for a certain distance from A on one side of it. Therefore $E_{\alpha\beta\gamma\delta h}$ is at most denumerable. Let $E=\Sigma E_{\alpha\beta\gamma\delta h}$, the subscripts ranging independently over rational values admissible according to the defined double character of the points of $E_{\alpha\beta\gamma\delta h}$; hereby we may regard the side of an angle as rational if the angle it makes with l is a rational number of radians. E is then denumerable. Suppose now A is common terminal point of two open, simple arcs of determinate and distinct directions at A, composed, the one exclusively of points of S, the other exclusively of points of $\bar{S}$. A then belongs to some $E_{\alpha\beta\gamma\delta h}$, with subscripts all rational, and therefore to E. If A is not in E, and threre exists a simple, open arc C with determinate direction at A consisting exclusively of points of S, then A is necessarily approachable *via* S along every simple arc of determinate direction at A different from that of C. Similarly if C consists exclusively of points of $\bar{S}$. We thus have the following

Theorem IV. *If l is a given straight line in the plane; π — regarded as not including l — one of the half planes into which l divides the plane; and S a point set in π, then every point A of l, with the exception of an at most denumerable number, is such that every simple arc in π with determinate direction at A — with the possible exception of arcs having one particular direction δ_A, variable with A — contains points of S, or every such arc, with similar exception, contains points of $\pi-S$.*

If we confine ourselves to rectilinear approach, we have the following corollary of Theorem IV.

Corollary VII. *If l is a given straight line, π one of the (open) half planes into which l divides the plane, and S a point set in π, then every point A of l not belonging to an exceptional, at most denumerable set is approachable via S in every direction of approach to it from π, or A is so approachable via $\pi - S$.*

If $f(x,y)$ is a given real function, and k a given real number, let T_k be the set of points (x,y) such that $f(x,y)>k$. If l is a given straight line, and π one of the (open) half planes into which l divides the plane, let $T_k\pi = S_k$. Let E_k be the exceptional, at most denumerable set of Corollary VII when S of this corollary is identified with S_k, and let $E = \Sigma E_k$, k ranging over the set of rational numbers. If A is a point of l not belonging to E, it belongs to no E_k for rational k, and therefore either every segment in π terminating in A contains points of S_k, or every such segment contains points of $\pi - S_k$. Hence either $\limsup_{(x,y)\to A} \geqslant k$ on every straight line in π terminating in A, or $\liminf_{(x,y)\to A} \leqslant k$ on every such line. Consequently there is no pair of segments in π terminating in A such that $\limsup f(x,y)<k$ as (x,y) approaches A along the one, and $\liminf f(x,y)>k$ as (x,y) approaches A along the other. Since this holds for every rational k, it must be that for every pair of segments in π terminating in A, $\limsup f$ along the one is equal to or greater than $\liminf f$ along the other. We thus obtain the following corollary, a result derived by Mabel Schmeiser[6]) in a different way.

Corollary VIII. *If $f(x,y)$ is a given function, l a given straight line, and π one of the half planes into which l divides the plane, then for every point A of l not belonging to an exceptional, at most denumerable set, and every pair of half lines in π terminating in A, $\limsup f(x,y)$ as (x,y) approaches A along one of these half lines equals or exceeds $\liminf f(x,y)$ as (x,y) approaches A along the other half line.*

We now return to the use of the germ principle for the purpose of deriving additional theorems for the plane. We shall first derive a result for planar regions analogous to Corollary I for intervals of the linear continuum. Let P be a given property of planar, simply-connected, polygonal regions, i.e., simply-connected regions bounded by straight lines. We call such regions p-regions. Conforming to our procedure for applying the germ principle, we shall consider those points of the plane which have a certain double character with respect to P. Let α,β and γ,δ be 2 pairs of directions; $\alpha\beta$ the angle made by the first pair as α rotates counterclockwise to β; and $\gamma\delta$ the corresponding angle for the second pair of directions. We furthermore suppose that there is a direction ($\neq\alpha$ and $\pm\beta$) assumed by α as it rotates to β which is opposite to a direction ($\neq\gamma$ and $\neq\delta$) assumed by γ as it rotates to δ. In such a case, we say $\alpha\beta$ and $\gamma\delta$ „contain opposite directions". If P is a given property of p-regions, and $\alpha\beta, \gamma\delta$ two angles containing opposite directions, let E be the set of points A of the following double character: a) P is valid for all p-regions of diameter $<h_A$ — a positive number depending on A — having vertex A, $\alpha\beta$ as the angle at A, and lying wholly within $\alpha\beta$; and b) $\overline{P}$ is valid for all p-regions of diameter h_A, with vertex A, $\gamma\delta$ as angle at A, and lying wholly within $\gamma\delta$. Let E_r be the subset of E consisting of the points A for which h_A can be taken equal to the fixed positive number r. It is seen that if A belongs to E_r there exists a sector of a circle with vertex A containing no points of E_r. It follows that E_r is a sparse set. By employing a procedure utilized earlier, we can extend the argument for proving E sparse to show that the sum of all the E's obtained by taking $\alpha\beta$ and $\gamma\delta$ freely variable — in accordance with the specified restrictions — is still sparse. To this end, suppose that $\alpha_2\beta_1$ and $\gamma_2\delta_1$ are two angles — each less than 360^0 and generated counterclockwise — containing opposite directions; and $\alpha_1\beta_2, \gamma_1\delta_2$ ($\alpha_1, \beta_1, \gamma_1, \delta_1 \neq \alpha_2, \beta_2, \gamma_2, \delta_2$ respectively) two angles — less than 360^0, generated counterclockwise — respectively containing $\alpha_2\beta_1, \gamma_2\delta_1$. Let A be a point of the following double character: a) There exists an angle $\alpha\beta$ containing $\alpha_2\beta_1$ and contained in $\alpha_1\beta_2$ such that P is valid for all p-regions of sufficiently small diameter with vertex A and $\alpha\beta$ as angle at A and lying wholly within $\alpha\beta$; b) there exists an angle $\gamma\delta$ containing $\gamma_2\delta_1$ and contained in $\gamma_1\delta_2$ such that $\overline{P}$ is valid for all p-regions of sufficiently small diameter with vertex A and $\gamma\delta$ as angle at A and lying wholly within $\gamma\delta$. We denote by $E_{\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\delta_1\delta_2}$ the set of points A just described for which the respective validity of P and $\overline{P}$ applies to the p-regions as characterized which have diameter $<r$, where r is a fixed, positive number. It is seen that if A belongs to $E_{\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\delta_1\delta_2 r}$, there is a sector of a circle with vertex A containing no points of this set,

[6]) Fund. Math. **22** (1934) p. 72.

which is consequently sparse. Therefore the sum of all $E_{\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\delta_1\delta_2 r}$ for the admissible rational values of the 9 subscripts is likewise sparse. We call this sum E, discarding the earlier meaning of this letter. Suppose now A is a point for which there exist two angles $\alpha\beta$ and $\gamma\delta$ containing opposite directions such that P is valid for all p-regions with vertex A, $\alpha\beta$ angle at A, and lying wholly within $\alpha\beta$, and $\bar{P}$ similarly valid for $\gamma\delta$. It is then possible to choose rational $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, r$ so that A belongs to $E_{\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2\delta_1\delta_2 r}$. A is therefore element of E, and we may conclude that the totality of points A for which there exist two angles α,β and γ,δ of the described character is a sparse set. We may therefore state the following

Theorem V. *If P is a property p-regions (i.e., simply-connected, polygonal, planar regions), every point A of the plane not belonging to an exceptional, sparse set is such that, for every pair of angles $\alpha\beta$ and $\gamma\delta$ with vertex A containing opposite directions, either there is a p-region having property P of infinitesimal diameter with vertex A, angle $\alpha\beta$ at A, and lying wholly within $\alpha\beta$, and also such a p-region for $\gamma\delta$, or else there exist two such p-regions having property $\bar{P}$.*

If, in particular, we let $\gamma\delta = \beta\alpha$ we obtain the following

Corollary IX. *If P is a p-region property, every point A of the plane not belonging to an exceptional sparse set is such that, for every angle $\alpha\beta$ with vertex A, either there are two p-regions of infinitesimal diameter of property P with vertex A, the one having angle $\alpha\beta$ at A and lying wholly within $\alpha\beta$ and the other having angle $\beta\alpha$ at A and lying wholly within $\beta\alpha$, or else there are two such p-regions of property $\bar{P}$.*

Commonly a region property P is ascending or descending; that is to say, if R_1 and R_2 are two p-regions, R_1 has property P, and $R_1 \subset R_2$, $R_1 \supset R_2$ respectively, then R_2 has property P. For example, if S is a given planar point set, and R^P (R has property P) means that RS is at most denumerable, or non-dense, or exhaustible, or of measure 0, P is descending. If P is descending, ascending, $\bar{P}$ is respectively ascending, descending — the two properties are thus reducible the one to the other. Let P be a given descending p-region property, α and β two directions such that angle $\alpha\beta < 180^0$, and r a given positive number. Let E_r be the set of points A of the following double character: Every p-region of diameter $<r$ with A as vertex and $\alpha\beta$ as angle at A lying wholly within $\alpha A\beta$ has property $\bar{P}$, whereas every p-region so related to $\beta\alpha$ has property P. Since P is descending, every p-region lying in $\beta A\alpha$ whose points are all at distance $<r$ from A has property P, and therefore no point within $\beta A\alpha$ at distance $<r$ from A belongs to E_r. Moreover, if B is in $\alpha\beta$ or on its boundary and at distance $<r$ from A, there exists a p-region R, of diameter $<r$, with vertex B and angle $\beta\alpha$ at B which lies within $\beta B\alpha$ and contains A as interior point. Since P is descending, and there are p-regions of infinitesimal diameter with A as vertex having property $\bar{P}$, every p-region containing A in its interior has property $\bar{P}$. R therefore has property $\bar{P}$, so B cannot belong to E_r. The set E_r has thus been proven isolated, and it is therefore denumerable. It follows that if $E = \Sigma E_r$, r ranging over the set of positive rational numbers, E also is denumerable. By availing ourselves of a type of argument already employed a number of times, we can extend this result and show that the sum of all the E's obtained for variable α, β and r — and r no longer restricted to be rational — is likewise denumerable. We are thus led to the following

Theorem VI. *If P is a descending p-region property, A a point of the plane not belonging to an exceptional denumerable set, and $\alpha\beta$ an angle $<180^0$, then there is either a p-region of infinitesimal diameter, vertex A, angle $\alpha A\beta$, and lying in $\alpha A\beta$ which has property P, or else a p-region of infinitesimal diameter, vertex A, angle $\beta A\alpha$ and lying in $\beta A\alpha$ which has property $\bar{P}$.*

Suppose S is a given planar set, and we call a point A of the plane a „corner point'' if there is a sector of a circle with vertex A and angle $\alpha\beta > 180^0$ containing no points of S, whereas every circle sector of vertex A and angle $\beta\alpha$ contains in its interior points of S. Let the p-region property P be defined by the property that the p-region R has property P when and only when R contains no points of S its interior. Since P is descending, we may apply Theorem VI, and we obtain the result that the set of corner points associated with a planar set is at most denumerable. This well known result, whose origin, as it seems, is not subsumed in the literature under simpler, more general considerations, appears here as a ready consequence of the application of our germ principle. Our derivation, too, removes the apparent mystery of the condition $\alpha\beta > 180^0$. It is the descending character of P which explains the non-symmetry. Theorem VI, together with its implication for functions [7]), may be regarded as an extension to the plane of properties of the straight line whose extension to the plane seemed to W. H. Young to require the introduction of the sparse set §. But it is only one type of extension which suggests this introduction; another type, as represented by Theorem VI, permits the property of denumerability to go over unchanged from straight line to plane, without recourse to a new notion.

We add some remarks about the genesis of descending properties. We have employed the p-region as a convenient type of set, but in relation to the remarks we are now about to make it will be better to discard the p-region and take in its place the general open set, with which we shall use the word „region" as synonymous. Let Q be a given region property. With Q we associate the region property P defined as follows: G^P when and only when G contains no sub-region of property Q. It follows that P is descending. Conversely, if P is a given descending region property, let $Q = \overline{P}$. Then G^P when and only when G contains no sub-region of property Q.

With every region property we have thus associated a descending region property, the defined association. being such that every descending region property is the associated property of some region property.

This association indicates the scope of applicability of Theorem VI. We shall mention a number of examples of descending region properties P thus associated with properties Q. With reference to a given planar set S:

a) Let G^Q if GS is non-denumerable; G^P then means that GS consists of at most a denumerable number of points.

b) G^Q if S is dense in G; G^P then means that S is nowhere dense in G.

c) G^Q if GS is of measure 0; G^P means that S is of positive exterior measure in every sub-region of G.

[7]) See Theorem VIII. § l. c., p. 276 ff.

These illustrative properties P, as all descending region properties, yield, according to Theorem VI, properties of the general point set in terms of its manner of approach to points of the plane.

With reference to a given function $f(x,y)$:

a) G^Q if f is of one sign $(+, -$ or $0)$ in G; G^P if f is double-signed in every sub-region of G.

b) G^Q if f is non-measurable in G; G^P if f is measurable in G.

Every descending region property, like a) and b) yields a property of the general function.

Examples of descending region properties may also be readily given for sequences of sets and for sequences of functions.

In reference to the scope of application of Theorem VI, it may be noted, too, that the number of descending region properties is the same as the number of region properties altogether, namely 2^c. That there are in all 2^c region properties follows from the fact that there are in all c regions and that a region property is determined when and only when it is determined for every region whether it has or hasn't the property. That the number of descending region properties is at least 2^c can be seen as follows. Let T be a set of c distinct regions such that no region of T contains another region of T. To define the region property P, let us invest each element of T, arbitrarily and independently, with the property P or $\overline{P}$; and if G is not an element of T, let G^P or $G^{\overline{P}}$ according as G does not contain or contains as sub-region an element of T of property $\overline{P}$. P is descending, and since there are 2^c possible distributions of P and $\overline{P}$ among the elements of T, there are 2^c distinct properties P as defined.

It may be of interest to note that if Q is a point (instead of region) property, and the region property P is defined by the requirement that G^P mean that G contains no points of property Q, then P is not only descending but also additive, the latter term signifying that sums of regions of property P are also of property P. Conversely, suppose P is a descending, additive region property. If G is a region of property $\overline{P}$, it must contain a point A such that every region containing A has property $\overline{P}$. For if this were not so, we could enclose every point of G in a region of property P.

On account of the additivity of P, the sum of such enclosing regions would be of property P, and therefore, because P is descending, G itself would be of property P. Let S be the set of points A of the plane such that every region containing a point of A has property $\bar{P}$. Let us say that the points of S have property Q, the points of $\bar{S}$, property $\bar{Q}$. Q is then a point property such that if G contains a point of property Q we have $G^{\bar{P}}$, and every region of property $\bar{P}$ contains a point of property Q. We may therefore state the following theorem, whose converse is obvious.

Theorem VII. *With every descending, additive, region property P, we may associate a point set S such that the region G is of property $\bar{P}$ or P according as G countains or does not contain a point of S.*

If S is a given set, and P the (descending, additive) region property such that $G^{\bar{P}}$ or G^{P} according as G contains or does not contain points of S, we may adjoin to S any number of its limit points without thereby changing the property P thus associated with S. Since, according to Theorem VII, all descending, additive region properties are obtainable by means of such association, there are no more descending, additive region properties than closed point sets. Since two different closed point sets S yield two different aassociated region properties P, we conclude that *there are in all $\mathfrak{c}$ descending, additive region properties* [8]).

We have so far dealt mainly with properties of sets, but have a number of times derived from these set properties — or from interval properties — various results for the general function. The theorems on region properties can similarly yield properties of the general function. The procedure suggested for passing from the types of theorem on the general set treated in the present paper to theorems on the general function can be readily described systematically. We shall do so for (real) functions $f(x,y)$ of two (real) variables. Let r_1, r_2 $(r_1<r_2)$ be a pair of rational numbers, and $S_{r_1 r_2}$ the set of points (x,y) such that $r_1<f(x,y)<r_2$.

[8]) If we say that P is *denumerably additive* — as distinguished from (unqualifiedly) additive — when the sum of $\aleph_0$ regions of property P is of property P, it follows that every denumerably additive region property is (unqualifiedly) additive, since the sum of any number of regions is the sum of a selected denumerably infinite number of them.

The theorems of the present paper on the general set S associate with S an exceptional set E, which from one point of view or another (cardinal number, density, measure) may be regarded as meager in elements. A theorem $\mathfrak{T}$ on the general set of the kind dealt with permits us, then, to associate with $S_{r_1 r_2}$ an exceptional set $E_{r_1 r_2}$. Let us call the particular type of exceptional set referred to in the formulation of $\mathfrak{T}$ a τ-set. The τ-sets we have considered in the present paper have all been such that the sum of $\aleph_0$ τ-sets is again a τ-set — we may, indeed, deem this property as requisite for duly registering the meagerness of a τ-set. If this property is assumed, the set $E=\Sigma E_{r_1 r_2}$, r_1 and r_2 $(r_1<r_2)$ ranging independently over the set of rational numbers, is a τ-set. Again, the τ's of the present paper have been such that no τ-set is identical with the entire plane of points, and this property, too, would be naturally included in an abstract postulational definition of a τ-set. If this second property of τ-sets is presupposed, we have $\bar{E}\neq 0$. If A is a point of $\bar{E}$, then for every pair of rationals r_1, r_2 $(r_1<r_2)$ the relationship of $S_{r_1 r_2}$ to A is „regular" in the sense that A is not in the exceptional set $E_{r_1 r_2}$ associated with $S_{r_1 r_2}$. It accords with the character of the τ-set to assume that a subset of a τ-set is a τ-set, and that therefore if the relationship of S to A is regular, the relationship to A of every superset of S is also regular. With such an additional assumption for τ, we may conclude that if the relationship of $S_{r_1 r_2}$ to A is regular for every pair of rationals r_1, r_2 $(r_1<r_2)$, then the relationship of S_{kl} to A is regular for every pair of real numbers k, l $(k<l)$. We shall say that the approach of the surface $z=f(x,y)$ is "regular" at the point (ξ,η,ζ) if S_{kl} is regular at (ξ,η) for every pair of real numbers k,l for which $k<\zeta<l$. Under the assumptions we have made for τ, we may deduce from a theorem $\mathfrak{T}$ for the general set the following property of the general function: If (ξ,η,ζ) is a point of space such that (ξ,η) does not belong to an exceptional τ-set, the approach of the surface $z=f(x,y)$ is regular at (ξ,η,ζ). Summarizing, we formulate this result in the following

Theorem VIII. *Let τ be a type of (planar) set such that*

a) the entire plane of points is not a τ-set;

b) a subset of a τ-set is a τ-set;

c) the sum of $\aleph_0$ τ-sets is a τ-set.

Let $\mathfrak{T}$ be a theorem which asserts that for every planar set S there exists a τ-set E such that if A is a point of the plane not belonging to E the set S is "ϱ-regular" at A[9]*). Then $\mathfrak{T}$ implies the following property of the general function $f(x,y)$: For all points (ξ,η) of the plane not belonging to an exceptional τ-set, and for every real number ζ, the approach of the surface $z=f(x,y)$ is ϱ-regular at (ξ,η,ζ), in the sense that if k and l are two real numbers such that $k<\zeta<l$, the set $S_{kl}=S_{k<f(x,y)<l}$ is ϱ-regular at (ξ,η). If $\mathfrak{T}'$ is a theorem asserting the same as $\mathfrak{T}$ except that in its formulation A is restricted to belong to S, then $\mathfrak{T}'$ implies the property of the general function which is the same as the one implied by $\mathfrak{T}$ except that (ξ,η,ζ) is to be restricted to belong to the surface $z=f(x,y)$.*

The proof of the last part of Theorem VIII, which refers to $\mathfrak{T}'$, is parallel to that referring to $\mathfrak{T}$. As remarked, conditions a), b) and c) are valid for all the types of exceptional set we have dealt with. Following are examples of the application of Theorem VIII.

1. Let $\mathfrak{T}$ be Theorem III'. In this case, Theorem VIII permits us to conclude that if $f(x,y)$ is a given function, there exists a sparse set E such that if B: (ξ,η,ζ) is a point of space and A: (ξ,η) is not in E, the surface $z=f(x,y)$ is *ϱ-regular* at B in the sense that for every pair of real numbers k,l, with $k<\zeta<l$, the set $S_{kl}=S(k<f(x,y)<l)$ is ϱ-regular at (ξ,η); and this latter regularity means that either (a) for every simple arc C of the xy-plane through A with a determinate direction at A which is different from each of two opposite (singular) directions, A is approachable *via* CS_{kl}, or else (b) for every curve C of type described in (a), A is approachable *via* $C\bar{S}_{kl}$. Now if a) holds for every k and l, the singular directions must stay the same as k and l vary, because $S_{k'l'}\subset S_{kl}$ if $k\leqslant k'<l'\leqslant l$. In this case, there is consequently at most one singular pair of opposite directions, and if C is a simple arc in the xy-plane through A with a determinate direction at A different from the singular directions, the point (ξ,η,ζ) is approachable by the surface $z=f(x,y)$ *via* C. If a) does not hold for all k,l, there exists a pair of numbers k,l such that if C is a simple arc through A with a determinate direction at A different from each of the two opposite singular directions, there is a sequence of points (x_n,y_n) on C with A as limit such that for every n, $f(x_n,y_n)\leqslant k$ or $\geqslant l$. Hence every pair of numbers k',l'

[9]) That is to say, S has a specified relationship to A, which, in view of the applications of Theorem VIII is termed ϱ-regularity.

such that $k\leqslant k'<l'\leqslant l$ also has the property just stated for k,l. If a) does not hold for all k,l — a) and b) may, of course, hold simultaneously — we shall accordingly say that the point (ξ,η,ζ) is approached by the (space) complement of the surface $z=f(x,y)$ *via* every simple arc C of the xy-plane through A which has a determinate direction at A different from each of two opposite singular directions. By means of Theorem VIII, we have thus been led from Theorem III' to a property of the general function. It will be observed that the argument we have used to arrive at this conclusion is essentially valid for the case where $f(x,y)$ is not restricted to be one-valued; moreover, the functional values of f need not be defined outside an arbitrarily chosen planar set. The greater freedom thus accorded the functions $f(x,y)$ by the removal of these restrictions especially conforms to the nature of the conclusion we have reached for $f(x,y)$, a conclusion which pertains rather to a spatial point set than to a "surface" $z=f(x,y)$. We may, then state the following

Theorem IX. *Let $f(x,y)$ be a real function defined at the points of an arbitrarily chosen planar point set, and assuming, at the points where it is defined, one or more values — the number of these values ranging independently of (x,y) from 1 to c. Then for all points B: (ξ,η,ζ) of space such that A: (ξ,η) does not belong to an exceptional sparse set of the (x,y) plane, either B is approached by the "surface" $z=f(x,y)$ via every simple arc C of the xy-plane through A which has a determinate direction at A different from each of two opposite singular directions — which, however, need not exist — or else B is approached via every such curve C by the space complement of the surface $z=f(x,y)$.*

Similarly, we derive, for example, from Corollary VII the following theorem on the general function, where again we drop the restrictions on $f(x,y)$ that it be one-valued and defined for the entire plane.

Theorem X. *Let $f(x,y)$ be a one- or many-valued, real function defined in a arbitrary point set, l a straight line, and π one of the half planes into which l divides the plane. Then there exists an exceptional, at most denumerable subset E of l such that if B: (ξ,η,ζ) is a space point with A: (ξ,η) in l but not in E, either B is approached by the surface $z=f(x,y)$ via every direction of approach to A from π, or B is so approached by the space complement of $z=f(x,y)$.*

Without stopping to formulate the theorems on the general function derivable, by means of Theorem VIII, from other results we have obtained, we shall mention some examples of such theorems on the general function derivable, by means of Theorem VIII, from theorems we have as yet not alluded to. If S is a planar set, let us say that A is *exhaustibly approachable* by S if there exists an open set G with A as boundary point such that GS is exhaustible. If A is exhaustibly approachable by S, every neighborhood of A contains an open subset whose common part with S is exhaustible. It follows that the points of S which are exhaustibly approachable by S constitute an exhaustible set. From this property of the general set, we can obtain, by means of Theorem VIII, a property of the general function. If $f(x,y)$ is a given function, we shall say that the point (ξ,η,ζ) is a point of *symmertic inexhaustible approach* of the surface $z=f(x,y)$ if for every pair of real numbers k,l such that $k<\zeta<l$ and every open set G of the xy-plane having (ξ,η) as boundary point, the set GS_{kl} is inexhaustible. We may then state the following

Theorem XI. *All the surface points $(\xi,\eta,f(\xi,\eta))$ of the surface $z=f(x,y)$, where $f(x,y)$ is a real, one — or many — valued function defined for an arbitrary point set of the xy-plane, are points of symmetrical, inexhaustible approach of the surface, with the exception of those for which (ξ,η) belongs to a fixed exhaustible set*[10].

If S is a planar set, we have seen that the points of S which are exhaustibly approachable by S constitute an exhaustible set. Analogous to this descriptive property is the metric property that the points A of S at which the lower (exterior) metric density of S is less than 1 constitute a set of measure 0. If a point A of S is of the sort just described, we shall say that it is a point of *deficient metric clustering* of S. We can then say that *the points of deficient metric clustering of S which are exhaustibly approachable by S constitute an exhaustible set of measure* 0. We shall say[11] that the surface $z=f(x,y)$ is *quasi-continous* at the point (ξ,η) of the (x,y) plane if for every pair k,l of real numbers such that $k<f(\zeta,\eta)<l$ the set S_{kl} has (exterior) metric density 1 at (ξ,η). With the aid of Theorem VIII, we may then state the following

[10]) This theorem, stated for one-valued functions, is contained in the author' paper, *New properties of all real functions*, Trans. Am. Math. Soc. **24**, (1922).

[11]) Cf. l. c., p. 126.

Theorem XII. *Let $f(x,y)$ be a real, one- or many-valued function defined on an arbitrary point set of the xy-plane. Then if the points (ξ,η) of the xy-plane belonging to an exceptional, exhaustible set of measure 0 are excluded, every point (ξ,η) is either a point of quasi-continuity of f — for all values of $f(x,\eta)$ — or it is such that $(\xi,\eta,f(\xi,\eta))$ — for all values of $f(\xi,\eta)$ — is a point of symmetric, inexhaustible approach of the surface $z=f(x,y)$.*

Similar results are obtainable by means of considerations of cardinal number. If the descriptive part is dropped from Theorem XII, we obtain the result that *every function is quasi-continuous almost everywhere*[12]. This implies that a measurable function is metrically continous (=approximately continous, according to Denjoy) almost everywhere. This latter result may also be obtained by the use of Theorem VIII if it is suitably modified — as may be done in an obvious way — to make it applicable to the passage from the general measurable set to the general measurable function, instead of that from the general set to the general function. Also such modification of Theorem VIII can be generalized.

By bringing in the idea of the *negligible* set, we can derive additional results. If S is a given planar set, let us consider, for example, those points A of the plane which are such that if BA is an open segment of a straight line of sufficiently small length and of the given direction α, it consists exclusively of points of S, whereas if BA is an open, straight line segment of sufficiently small length and of given direction β, it consists, except possibly for $\aleph_0$ points, exclusively of points of $\bar{S}$. Let E be the totality of such points A associated with S, α and β being regarded as fixed; and let E_r be the subset of E consisting of the points A for which (the positive number) r is sufficient as length of BA in the definitional requirements for A. It follows that if A is a point of E_r, there exists a parallelogram, with A as vertex, no point of which belongs to E_r, exception being taken to an at most denumerable number of straight lines parallel to one of the sides of the parallelogram. E_r is therefore sparse, and consequently $E(=\Sigma E_r$ for rational $r)$ is also sparse. We may thus state the following result:

[12]) l. c., Theorem IX.

If S is a planar set, and α and β two given directions, the points of the plane approachable in direction α exclusively via S, and in direction β, if denumerable sets are regarded as negligible, exclusively via $\bar{S}$ constitute a sparse set.

By means of the argument for Theorem III, we were able to remove the restriction of approach along two fixed directions. But a similar argument would fail when applied to the result just obtained pertaining to the negligible denumerable set.

In a manner similar to that employed for deriving this latter result, we may obtain an analogous result by taking a set of exterior metric density 0 as negligible. If S is a planar set, let E be the set of points A of the plane such that if we approach A in the given direction α we eventually have just points of S, wheras if we approach A along the straight line l of given direction β we eventually have just points of $\bar{S}$ provided a set on l of (linear) exterior metric density 0 at A may be neglected. Let E_r be the subset of points A of E such that if B $(\neq A)$ is a point at distance $<r$ from A with BA of direction α, B belongs to S. It follows that if A belongs to E_r, there exists a parallelogram $ABCD$ such that if exception be made of the points on a set of lines parallel to AD which intersect AB in a point set of exterior metric density 0 at A, no point of the interior of the parallelogram $ABCD$ is an element of E_r. Consequently E_r is of measure 0, and therefore $E = \Sigma E_r$ for (positive) rational r is of measure 0. We thus have the following result:

If S is a planar set, and α and β two directions, the set of points A of the plane approachable in direction α exclusively via S, and in direction β, provided a linear set of exterior metric density 0 at A may be neglected, exclusively via $\bar{S}$ is of measure 0.

Analogous results are obtainable in case exhaustible sets or sets of other types are regarded as negligible.

By applying the idea of the negligible set, we may obtain similar results on the approach of a given set to a given straight line. But we shall enter into no further such details.

It is obvious, too, that the results of the present paper lend themselves, in the main, to extension to n-dimensional space. We shall not attempt a comprehensive dicsussion of such possible extension, but shall give one illustration. Let S be a given set in 3-dimensional space. Let A be a point of space of the following double character:

a) The points within an (open) planar angle $\alpha A \beta$ of vertex A which are sufficiently near A all belong to S;

b) the points sufficiently near A on an open half line of direction γ not contained in $\alpha\beta$ (i. e., not contained in a plane containing α and β) all belong to $\bar{S}$.

Let E be the set of points A having properties a) and b). It may be shown by means of an analogue of the germ principle that if A is a point of E, α, β and γ remaining fixed, such that properties a) and b) hold for all the points in question whose distance from A is less than a fixed positive number h there is a triangular pyramid of vertex A no interior point of which belongs to E. E is therefore a (spatial) sparse set [13]). Moreover, by an argument like that already employed a number of times, it may be shown — with the aid of *rational* directions — that the result that E is sparse remains true if α, β and γ are permitted to vary in accordance with the condition that $\alpha\beta$ does not contain γ. In this way we arrive at the following property of the general, spatial point set.

Theorem XIII. *If S is a spatial point set, every point A of space not belonging to an exceptional sparse set is appoachable via S along every linear direction, or it is approachable via $\bar{S}$ along the interior of every planar angle of vertex A, exception being made of planar angles lying in the planes of an axial pencil with a fixed line through A as axis.*

The considerations and results of the present paper are also largely extensible to point transformations. We add a few details. Suppose, for concreteness, we have a general point transformation from the xy-plane to the uv-plane; that is to say, with every point A: (x,y) of the xy-plane there is associated a point B: (u,v) of the uv-plane, the function $B = f(A)$ expressing the transformation being conditioned only in that every point A of the xy-plane has one and only one mate B of the uv-plane. In analogy with the sets $S_{k<f(x)<l}$ which we have employed in reference to a function $y = f(x)$ of a single variable, we may now employ the sets $S_{\substack{u_1<u<u_2 \\ v_1<v<v_2}} = S_{u_1,u_2;v_1,v_2}$ constituted by the A's of the xy-plane such that the corresponding

[13]) It may be readily shown that a set, such as E, having a *pyramidal void* at each of its points is the sum of $\aleph_0$ sets E_n each of which has the property that on every straight line parallel to a fixed line l_n the points of E_n are at least at distance k_n apart, where k_n is a fixed number for fixed n. The use of the term *sparse* is thus justified for E.

B's have coordinates satysfying the inequalities $u_1 < u < u_2$, $v_1 < v < v_2$. To the set $S_{u_1,u_2;v_1,v_2}$, we may apply the theorems on general sets which we have proved, and may thus conclude that, except for the points of a certain, say τ-exceptional set $E_{u_2,u_2;v_1,v_2}$, the set $S_{u_1,u_2;v_1,v_2}$ is in a certain specific sense "regular", say, at every point of $S_{u_1,u_2;v_1,v_2}$. Letting u_1, u_2, v_1, v_2 vary independently over the set of rational numbers, we obtain $\aleph_0$ sets $E_{u_1,u_2;v_1,v_2}$ whose sum E is again τ-exceptional. The points A of the xy-plane not belonging to E will be such that every $S_{u_1,u_2;v_1,v_2}$ containing A, with u_1, u_2, v_1, v_2 all rational, will show a certain regular character at A; and on account of the nature of the exceptional sets employed, this property will remain valid when the condition that u_1, u_2, v_1, v_2 be rational is dropped. A will thus be a point of a certain specific regularity with reference to the point transformation $B = f(A)$. An example of a theorem obtainable by means of the indicated line of reasoning is the following: If $B = f(A)$, where $A = (x, y)$ and $B = (u, v)$, is a point transformation subject only to the condition that to every A there corresponds one and only one B, every point A of the xy-plane, with the exception of a sparse set, is such that either (a) every straight line through A contains a sequence of points A_n having A as limit such that $\lim f(A_n) = B$; or (b) there exists a positive constant k such that every straight line through A contains a sequence of points A_n having A as limit and the property that the distance between $f(A_n)$ and B is more than k for every n.

An analogous result can be proved for point transformations from m-space to n-space. It may be shown, too, that the condition that $f(A)$ be unique is not essential in the argument.

Über projektive Funktionen [1]).

Von

Hans Fried (Wien).

1. Mit P_0 und C_0 werden die Borelschen Mengen, mit P_n (wo n immer eine natürliche Zahl bedeutet) die stetigen Bilder der Mengen C_{n-1} und mit C_n die Komplemente der Mengen P_n bezeichnet. Mit B_n werden die Mengen bezeichnet, die sowohl Mengen P_n als auch Mengen C_n sind.

Es werden reelle Funktionen betrachtet, die in dem Baireschen Nullraum definiert sind und auch die Werte $+\infty$ und $-\infty$ annehmen können.

Eine Funktion $F(x)$ wird als eine Funktion P_n (bzw. C_n, bzw. B_n) bezeichnet, wenn die Menge $\underset{x}{E}[F(x) > c]$ für jedes c eine Menge P_n (bzw. C_n, bzw. B_n) ist.

Eine Funktion $F(x)$, für die es eine Darstellung

$$F(x) = \sup_t \{\psi[\varphi^{-1}(x)]\} \tag{1}$$

gibt, bei der $x = \varphi(t)$ eine stetige Abbildung des Baireschen Nullraumes auf sich und $\psi(t)$ eine Bairesche Funktion ist, wird als Funktion g^1 bezeichnet; eine Funktion $F(x)$, für die es eine Darstellung

$$F(x) = \inf\{\psi[\varphi^{-1}(x)]\} \tag{2}$$

gibt, bei der $x = \varphi(t)$ eine stetige Abbildung des Baireschen Nullraumes auf sich und $\psi(t)$ eine Bairesche Funktion ist, wird als Funktion g_1 bezeichnet. Die Funktionen g^n und g_n werden durch vollständige Induktion definiert. Mit g^n wird eine Funktion $F(x)$ bezeichnet, für die es eine Darstellung (1) gibt, bei der $x = \varphi(t)$ eine stetige Abbildung des Baireschen Nullraumes auf sich und

[1]) Die Ergebnisse dieser Arbeit habe ich auf dem III. Polnischen Mathematischen Kongress in Warschau 1937 (vgl. Annales Soc. Polon. Math. **16**, 1938, S. 191) mitgeteilt.