On Well-ordered Subsets of any Set.

By

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In § 1 of this paper I shall give, without the help of the axiom of choice (i. e. multiplicative axiom), certain results which are closely connected with two celebrated theorems of the general theory of sets: namely, the theorem of Cantor, according to which the system of all subsets of any set is of greater potency (i. e. cardinal number) than the set itself; and the theorem, also formulated by Cantor and rigorously proved by Zermelo (with the help of the axiom of choice), according to which every set can be well-ordered [1]). On the basis of the results given in § 1 and in connexion with certain considerations presented in my earlier publications, I shall give in § 2 two new equivalent formulations of the axiom of choice and some observations concerning the axiomatic introduction of inaccessible cardinal numbers.

I should like to mention that Corollary 9 in § 1 being chronologically the first among the results which will be given here was obtained and was kindly communicated to me by S. Leśniewski as early as 1929.

The present work is based upon the axiom-system of Zermelo, excluding the axiom of choice and Fraenkel's axiom of replacement („Ersetzungsaxiom") [2]). The results can however be carried over *mutatis mutandis* to other systems of the theory of sets, e. g. to the theory of classes of the *Principia Mathematica* of Whitehead and Russell [3]). The customary notation and terminology of set theory will be employed.

[1]) For these two theorems compare e. g. A. Schoenflies, *Entwicklung der Mengenlehre und ihre Anwendungen*, Leipzig and Berlin 1913, p. 63 seq. and 170 seqq.; and F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1917, p. 56 and 133 seqq.

[2]) See, for example, A. Fraenkel, *Einleitung in die Mengenlehre*, Berlin 1928, p. 268 seqq.

[3]) 2nd edit., Cambridge 1925–1927.

§ 1. A system of sets S is, as usual, called *hereditary* if all the subsets of every set X belonging to S also belong to S [1]).

Theorem 1. *Let M be any set and S an hereditary system of subsets of M. If there is a function f which correlates in one-one fashion every set $X \epsilon S$ with an element $f(X) \epsilon M$, then there is also a function g which correlates (not necessarily in one-one fashion) every set $X \epsilon S$ with one element $g(X) \epsilon M - X$.*

The proof depends on the same idea as the proof of the above mentioned theorem of Cantor on the system of all subsets of a set. If, namely, f is a one-one transformation of the system S into a set $N \subset M$, and if the transformation inverse to f is, as usual, denoted by f^{-1}, we put:

$$g(X) = f\left(\underset{y}{E}\,[y \,\epsilon\, X \cdot N - f^{-1}(y)]\right)$$

for every set $X \epsilon S$ and then show (by an indirect proof) that the function g thus defined has the properties required by theorem 1.

From this, if we pass to the complementary sets, we obtain at once:

Corollary 2. *Let M be any set and S a system of subsets of this set which contains as elements for every set $X \epsilon S$ also every set Y such that $X \subset Y \subset M$. If S has the same potency as a subset of M, then there exists a selective function in S, i. e. a function h which correlates every set $X \epsilon S$ with one and only one element $h(X) \epsilon X$.*

Theorem 3. *Let M be any set and S an hereditary system of subsets of M. If there is a function g which correlates one element $g(X) \epsilon M - X$ with every set $X \epsilon S$, then there also exists a subset N of M which can be well-ordered and does not belong to S.*

The proof follows the same lines as the (second) Zermelo's proof of the theorem on the well-ordering. For if g is a function with the properties given in Theorem 2, we denote by K the product of all systems of sets X which satisfy the following conditions:

(i) *if* $Y \epsilon X \cdot S$, *then* $Y + \{g(Y)\} \epsilon X$;

(ii) *if* $Y \subset X$, *then* $\sum_{Z \epsilon Y} Z \,\epsilon\, X$.

[1]) See, for example, my paper in Fund. Math. **16** (1930), p. 227.

Further we define N as the sum of all sets $Z \epsilon \boldsymbol{K}$. It can be easily shown that this set N has the required properties; in particular, N is well-ordered by the relation $\prec$ which subsists between two elements x and y of N if, and only if, there is a set $Z \epsilon \boldsymbol{K}$ such that $x \epsilon Z$ and $y \, non \epsilon Z$.

Corollary 4. *Let M be any set and $\boldsymbol{S}$ an hereditary system of subsets of M. In order that there should be a subset N of M which can be well-ordered and does not belong to $\boldsymbol{S}$, it is necessary and sufficient that there exists a function g which correlates one and only one element $g(X) \epsilon M - X$ with every set $X \epsilon \boldsymbol{S}$.*

To prove this, it is only necessary to show how the existence of the function g can be inferred from the existence of the set N (since the inverse implication has already been established in Theorem 3). For this purpose it suffices to consider a particular well-ordering of the elements of N and to correlate with every set $X \epsilon \boldsymbol{S}$ as $g(X)$ the first element (in the given order) $y \epsilon N$ which does not belong to X: were there no such element y, then N would be a subset of X and must therefore belong to $\boldsymbol{S}$, since $\boldsymbol{S}$ is hereditary.

If we apply Corollary 4 to the system $\boldsymbol{S}$ of all proper subsets of a set M and then pass to the complementary sets, we reach the conclusion that the possibility of well-ordering any set M is equivalent to the existence of a selective function in the system of non-void subsets of M. This well-known theorem (which, in its essentials, is contained in Zermelo's proof of the theorem on the well-ordering) [1]) thus represents a special case of Corollary 4.

From Theorems 1 and 3 we obtain at once:

Theorem 5. *Let M be any set and $\boldsymbol{S}$ an hereditary system of subsets of M. If $\boldsymbol{S}$ has the same potency as a subset of M, then there is a subset N of M which can be well-ordered and does not belong to $\boldsymbol{S}$.*

Corollary 6. *Let M be any set, $\mathfrak{n}$ any cardinal number and $\boldsymbol{S}$ the system of sets $X \subset M$ for which:*

(i) $\overline{\overline{X}} \leqslant \mathfrak{n}$, *or* (ii) $\overline{\overline{X}} < \mathfrak{n}$, *or* (iii) $\overline{\overline{X}}\, non \geqslant \mathfrak{n}$, *or* (iv) $\overline{\overline{X}}\, non > \mathfrak{n}$.

If M and $\boldsymbol{S}$ have the same potency, then there is a subset N of M which can be well-ordered and is such that:

(I) $\overline{\overline{N}}\, non \leqslant \mathfrak{n}$, *or* (II) $\overline{\overline{N}}\, non < \mathfrak{n}$, *or* (III) $\overline{\overline{N}} \geqslant \mathfrak{n}$, *or* (IV) $\overline{\overline{N}} > \mathfrak{n}$.

[1]) Compare A. Fraenkel, op. cit., p. 299 seq.

This corollary is an immediate consequence of Theorem 5.

Corollary 7. *Let M be any set and $\boldsymbol{S}$ the system of sets $X \subset M$ with the potency $\overline{\overline{X}} < \overline{\overline{M}}$. If M and $\boldsymbol{S}$ have the same potency, then they can both be well-ordered.*

In order to prove this we apply Corollary 6 in the case (ii), (II) replacing $\mathfrak{n}$ by $\overline{\overline{M}}$. According to the corollary there exists a set $N \subset M$ which can be well-ordered, the potency of which is not less than that of M. The sets M and N thus have the same potency; consequently M (and also $\boldsymbol{S}$) can be well-ordered. Q. E. D.

It is to be noted that the proof of Corollary 7 (in contrast to the proofs of theorems and corollaries 1 to 6) is not "effective": even if a one-one transformation of M into $\boldsymbol{S}$ is known, we can in general see no possibility of defining a relation by which the well-ordering of M is established.

Closely related to Corollary 6 is the following:

Theorem 8. *Let M be a non-void set, $\mathfrak{n}$ any cardinal number $\geqslant \aleph_0$ and $\boldsymbol{S}$ the system of sets $X \subset M$ with the potency $\overline{\overline{X}} = \mathfrak{n}$. If M and $\boldsymbol{S}$ are of equal potency then there exists a subset N of M which can be well-ordered and is such that $\overline{\overline{N}}\, non \leqslant \mathfrak{n}$.*

The proof of this theorem will not be given here. It is more complicated than the proof of Corollary 6 (and than that of the proceding theorems) and is based on certain results of mine on the arithmetic of the cardinal numbers which have so far only been published in part [1]).

Putting in Theorem 8 or in Corollary 6 (i), (I) $\mathfrak{n} = \aleph_0$ we obtain at once:

Corollary 9. *If the system $\boldsymbol{S}$ of all denumerable (or of all at most denumerable) sets of real numbers has the potency $\overline{\overline{\boldsymbol{S}}} = 2^{\aleph_0}$, then there is a set N of real numbers with the potency $\overline{\overline{N}} = \aleph_1$.*

In conclusion of this paragraph we give:

Theorem 10. *If M is any set and $\boldsymbol{S}$ the system of all subsets of M which can be well-ordered, then M has a smaller potency than $\boldsymbol{S}$.*

[1]) See the joint report by A. Lindenbaum and myself, C. R. Soc. Sc. Vars. **19** (1926), p. 299 seqq.

We must in fact have $\overline{\overline{M}} \leqslant \overline{\overline{S}}$, since M is of equal potency with certain sub-systems of S, e.g. with the system of those $X \subset M$ which consist of exactly one element. On the other hand S is evidently hereditary; for if it were the case that $\overline{\overline{S}} \leqslant \overline{\overline{M}}$, then by Theorem 5 there would be a set $N \subset M$ which would be well-ordered but did not belong to S, and this would contradict the hypothesis of the corollary. Consequently we must have $\overline{\overline{M}} < \overline{\overline{S}}$. Q. E. D.

Theorem 10 can be regarded as a strengthening of the theorem of Cantor mentioned at the beginning.

§ 2. With the help of the results reached in § 1 we shall now show that the axiom of choice is equivalent to each of the following two statements $\mathcal{A}_1$ and $\mathcal{A}_2$:

$\mathcal{A}_1$. *If* $\mathfrak{m}$ *is any infinite cardinal number,* $\mathfrak{n}$ *a cardinal number such that* $\mathfrak{n} \leqslant \mathfrak{m}$, M *is a set with the potency* $\mathfrak{m}$ *and* S *the system of sets* $X \subset M$ *such that* $\overline{\overline{X}}$ *non* $> \mathfrak{n}$, *then* S *has the potency* $\mathfrak{m}^{\mathfrak{n}}$.

$\mathcal{A}_2$. *For every set* N *there exists a set* M *which has the same potency as the system* S *of all those sets* $X \subset M$ *which contain no subset that has the same potency as* N.

The proof consists of three parts:

(1) *The axiom of choice implies* $\mathcal{A}_1$.

In order to show this, it will be remembered that the theorem of comparability of cardinal numbers is a consequence of the axiom of choice. The system S of $\mathcal{A}_1$ is accordingly identical with the system of sets $X \subset M$ whose potencies are $\leqslant \mathfrak{n}$; but with the help of the axiom of choice it can be proved without difficulty that the latter system has the potency $\mathfrak{m}^{\mathfrak{n}}$ [1]). In this way $\mathcal{A}_1$ is obtained.

(2) $\mathcal{A}_1$ *implies* $\mathcal{A}_2$.

Let N be any set, and let $\mathfrak{n}$ denote the potency of N. Disregarding the trivial case when $\mathfrak{n} \leqslant 1$ (in this case it suffices to put $M = N$ in order to fulfill the conditions of $\mathcal{A}_2$) we let $\mathfrak{m} = 2^{\mathfrak{n}^{\aleph_0}}$ and consider any set M with the potency $\mathfrak{m}$. The system T of the sets $X \subset M$ which contain no subset of the same potency as N is evidently a part of the system $S = \underset{X}{E}[X \subset M \text{ and } \overline{\overline{X}} \text{ non} > \mathfrak{n}]$ and therefore has, according to $\mathcal{A}_1$, a potency $\leqslant \mathfrak{m}^{\mathfrak{n}}$; moreover we have $\mathfrak{m}^{\mathfrak{n}} = (2^{\mathfrak{n}^{\aleph_0}})^{\mathfrak{n}} = 2^{\mathfrak{n}^{\aleph_0} \cdot \mathfrak{n}} = 2^{\mathfrak{n}^{\aleph_0}} = \mathfrak{m}$. On the other hand there certainly exist sub-systems of T which have the potency $\mathfrak{m}$, for example the system $U = \underset{X}{E}[X \subset M \text{ and } \overline{\overline{X}} = 1]$. Consequently, we have $\mathfrak{m} \leqslant \overline{\overline{T}} \leqslant \mathfrak{m}^{\mathfrak{n}} = \mathfrak{m}$, whence $\overline{\overline{T}} = \mathfrak{m}$. The set M is thus of equal potency with T and consequently satisfies the conditions of $\mathcal{A}_2$.

(3) $\mathcal{A}_2$ *implies the axiom of choice.*

Again we consider any set N. According to $\mathcal{A}_2$ there exists a set M which has the same potency as the system T of sets $X \subset M$ such that $\overline{\overline{X}}$ *non* $\geqslant \overline{\overline{N}}$. We now apply Corollary 6 in the case (iii), (III) replacing $\mathfrak{n}$ by N and $\overline{\overline{S}}$ by T. In accordance with this corollary there exists a subset N_1 of M which can be well-ordered and has the potency $\geqslant \overline{\overline{N}}$; hence the set N can also be well-ordered. We have therefore deduced the theorem on the well-ordering from $\mathcal{A}_2$; and from this theorem, as is well known, the axiom of choice can easily be derived [1]).

By (1), (2) and (3) the axiom of choice is equivalent to each of the statements $\mathcal{A}_1$ and $\mathcal{A}_2$.

This result represents an advance on an analogous result which was given in my paper: *Eine äquivalente Formulierung des Auswahlaxioms* [2]). For there I have shown that this axiom is equivalent to a statement $\mathcal{S}$ which is closely connected with $\mathcal{A}_2$, and which states that for every set N there exists a set M which is not merely of the same potency as, but is identical with, the system of sets S described in $\mathcal{A}_2$. $\mathcal{S}$ is logically stronger than $\mathcal{A}_2$. Whilst the derivation of the statement $\mathcal{A}_2$ from the axiom of choice can be carried out wholly within the framework of Zermelo's axiom-system, in the derivation of the statement $\mathcal{S}$ Fraenkel's axiom of replacement is used, and it can be shown that the use of this axiom

[1]) See W. Sierpiński, *Zarys teorii mnogości* (*Fundamentals of Set-theory*, polish), 1st Part, 3rd edit., Warszawa 1928, p. 251 seqq.; A. Tarski, Fund. Math. **16** (1930), p. 195 seq.

[1]) Cf. A. Fraenkel, op. cit., p. 299 seq.

[2]) Fund. Math. **30** (1938), p. 197 seqq.

is essential. On the other hand it is clear that the statement $\mathcal{S}$ within a system of set-theory which involves the theory of types (e. g. the theory of classes of *Principia Mathematica*) becomes quite meaningless. In contrast to this the statement $\mathcal{A}_2$ in such a case does not become meaningless at all and still has the axiom of choice as a consequence.

Analogous remarks apply also to a question which was discussed in my paper: *Über unerreichbare Kardinalzahlen* [1]). In this paper I have defined the concept of inaccessible cardinal number (in the narrower sense), and I have formulated an axiom $\mathcal{A}$ which guarantees the existence of inaccessible numbers as large as we please; on the basis of the usual axioms of Zermelo the existence of such numbers, apart from the two smallest of them, 2 and $\aleph_0$, cannot be established at all. The axiom referred to is as follows:

$\mathcal{A}$. *For every set N there exists a system M of sets which satisfies the following conditions*:

(i) $N \in M$;

(ii) *if* $X \in M$ *and* $Y \subset X$, *then* $Y \in M$;

(iii) *if* $X \in M$ *and* Z *is the system of all subsets of* X, *then* $Z \in M$;

(iv) *if* $X \subset M$ *and* X *and* M *do not have the same potency, then* $X \in M$.

Axiom $\mathcal{A}$ was suggested by a very special characterization of the inaccessible numbers (or sets) which was given in Theorem 21 of my last-mentioned paper. In this connexion the axiom may seem somewhat strange and artificial (it is in any case only significant if the system of set-theory in question does not depend upon the theory of types).

In my paper on inaccessibles cardinal numbers the logical power of Axiom $\mathcal{A}$ was emphasized. If it is inclused in the Zermelo's or Zermelo-Fraenkel's axiom-system this axiom brings with it a great simplification and reduction of the system; and, be it noted, the axiom of choice then becomes a provable theorem. It might be supposed that Axiom $\mathcal{A}$ owes its great deductive power not so much to its content as to those peculiarities of its formulation which were pointed out above. But with the help of the results of §1 it can be shown that this supposition would not be quite

[1]) Fund. Math. **30** (1938), p. 68–89, in particular p. 87 seqq.

correct: $\mathcal{A}$ can be replaced by an axiom $\mathcal{B}$ which serves about the same purpose, for it assures the existence of inaccessible numbers as large as we please and has the axiom of choice as a consequence, but is based on a much more "natural" characterization of the inaccessible numbers (and, in particular, does not violate the theory of types). This axiom is as follows:

$\mathcal{B}$. *For every set N there exists a set M with the following properties*:

(i) *N has the same potency as a subset of M*;

(ii) *the system of subsets of M which do not have the same potency as M has the same potency as M.*

(iii) *there exists no set P such that the system of all subsets of P has the same potency as M.*

In order to derive the axiom of choice from $\mathcal{B}$ we proceed in a manner analogous to that followed in the case of the derivation sketched above of the same axiom from statement $\mathcal{A}_2$ (although with the difference that we use Corollary 7 instead of Corollary 6). Having derived the axiom of choice we can regard Theorem 17 of my paper cited above on inaccessible cardinal numbers as proved. According to this theorem the potency of a non-void set M is an infinite and inaccessible cardinal number if, and only if, M satisfies the conditions (ii) and (iii) in $\mathcal{B}$. From this it is seen that the existence of inaccessible cardinal numbers as large as we please is really guaranteed by Axiom $\mathcal{B}$.

In conclusion it may be mentioned that Axiom $\mathcal{B}$ seems to be logically weaker than $\mathcal{A}$. In any case $\mathcal{A}$ has not so far been derived from $\mathcal{B}$ without the help of Fraenkel's axiom of replacement [1]), whilst the inverse implication can be derived from Zermelo's axioms quite easily.

[1]) The derivation of $\mathcal{A}$ from $\mathcal{B}$ on the basis of the axiom of replacement can be constructed after the manner of the proof of Lemma 18 in my paper in Fund. Math. **30** (1938), p. 77 seqq.