The preceding results have an amusing formal consequence, valid for any ring $\mathfrak{R}$, with a unit element 1. Let $\eta_1, \eta_2, \dots$ be an infinite sequence of elements in $\mathfrak{R}$, with repetitions allowed. Then $\eta'_q = \eta''_q$, where η'_q and η''_q are defined by the recurrence formulae, $\eta'_0 = \eta''_0 = 1$ and

$$\eta'_q = -(\eta'_{q-1}\eta_1 + \eta'_{q-2}\eta_2 + \dots + \eta_q),$$
$$\eta''_q = -(\eta_1\eta''_{q-1} + \eta_2\eta''_{q-2} + \dots + \eta_q).$$

This is true for any ring since it is true for the ring which is freely generated by $\eta_0 = 1, \eta_1, \eta_2, \dots$, with infinite sums allowed, provided no product $\pm\eta_{m_1}\dots\eta_{m_n}$ is repeated infinitely many times. For if a degree, given by $\delta(\pm\eta_{m_1}\dots\eta_{m_n}) = m_1 + \dots + m_n$, is assigned to each product, only a finite number of terms in such a sum can have the same degree. It follows from induction on q that η'_q and η''_q are homogeneous of degree q and, as before, that $\eta'\eta = \eta\eta'' = 1$, where

$$\eta = 1 + \eta_1 + \eta_2 + \dots, \qquad \eta' = 1 + \eta'_1 + \eta'_2 + \dots, \qquad \eta'' = 1 + \eta''_1 + \eta''_2 + \dots$$

Therefore $\eta' = \eta''$, whence $\eta'_q = \eta''_q$.

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On the relation between the fundamental group of a space and the higher homotopy groups.

By

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1. $\mathcal{Y}$ will denote a separable, connected metric space locally connected in dimensions $0, 1, \dots, n$ [1]). Given a compact metric space $\mathcal{X}$, the continuous functions $f(\mathcal{X}) \subset \mathcal{Y}$ with the distance formula

$$|f_0 - f_1| = \sup_{x \in \mathcal{X}} |f_0(x) - f_1(x)|$$

form a metric space $\mathcal{Y}^{\mathcal{X}}$.

Given two points $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ the equation $f(x_0) = y_0$ defines a closed subset $\mathcal{Y}^{\mathcal{X}}(x_0, y_0)$ of $\mathcal{Y}^{\mathcal{X}}$.

I will denote the closed interval $[0,1]$ by $\mathcal{J}$ and $\mathcal{X} \times \mathcal{J}$ will stand for the cartesian product of $\mathcal{X}$ and $\mathcal{J}$. Two functions $f_0, f_1 \in Y^X$ will be called *homotopic* if there is a function $g \in \mathcal{Y}^{\mathcal{X} \times \mathcal{J}}$ such that

$$f_0(x) = g(x, 0), \quad f_1(x) = g(x, 1) \quad \text{for all } x \in \mathcal{X}.$$

If also

$$g(x_0, t) = y_0 \quad \text{for all } t \in \mathcal{J},$$

we say that $f_0, f_1 \in \mathcal{Y}^{\mathcal{X}}(x_0, y_0)$ are *homotopic rel.* (x_0, y_0).

2. Let $\mathcal{X}$ be a polyhedron and X a subpolyhedron of $\mathcal{X}$. It is well known that $T = \mathcal{X} \times (0) + X \times \mathcal{J}$ is a retract of $\mathcal{X} \times \mathcal{J}$ and therefore that

(2.1) *Every* $f \in \mathcal{Y}^T$ *has an extension* $f' \in \mathcal{Y}^{\mathcal{X} \times \mathcal{J}}$ [2]).

It follows immediately from (2.1) that

(2.2) *Given two homotopic functions* $f_0, f_1 \in \mathcal{Y}^X$ *and an extension* $f'_0 \in \mathcal{Y}^{\mathcal{X}}$ *of* f_0, *there is an extension* $f'_1 \in \mathcal{Y}^{\mathcal{X}}$ *of* f_1 *homotopic to* f'_0 [2]).

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[1]) C. Kuratowski, Fund. Math. **24** (1935), p. 269.

[2]) See for instance P. Alexandroff und H. Hopf, *Topologie I*, Berlin 1935, p. 501.

3. Let S^n $(n>0)$ be the n-dimensional sphere. It follows [3]) from our hypothesis on $\mathcal{Y}$ that $\mathcal{Y}^{S^n}$ and $\mathcal{Y}^{S^n}(x_0,y_0)$ where $x_0 \epsilon S^n$ are locally connected in dimension 0 and therefore that the classes of homotopy of $\mathcal{Y}^{S^n}$ (or the classes of homotopy rel. (x_0,y_0) in $\mathcal{Y}^{S^n}(x_0,y_0)$) coincide with the components of $\mathcal{Y}^{S^n}$ (or of $\mathcal{Y}^{S^n}(x_0,y_0)$) [4]).

(3.1) *Every component of* $\mathcal{Y}^{S^n}$ *contains at least one component of* $\mathcal{Y}^{S^n}(x_0,y_0)$.

Proof. Let $f_0' \epsilon \mathcal{Y}^{S^n}$. Put $\mathcal{X}=S^n$, $X=(x_0)$, $f_0(x_0)=f_0'(x)$, $f_1(x_0)=y_0$. Since is an arc joining $f_0(x_0)$ and y_0 in $\mathcal{Y}$ therefore $f_0,f_1 \epsilon \mathcal{Y}^X$ are homotopic. It follows from (2.2) that there is a function $f_1' \epsilon \mathcal{Y}^{S^n}(x_0,y_0)$ homotopic to f_0'.

4. $\mathcal{Y}$ will be called *simple in dimension n*, or shorter *n-simple*, if every component of $\mathcal{Y}^{S^n}$ contains exactly one component of $\mathcal{Y}^{S^n}(x_0,y_0)$, where $x_0 \epsilon S^n$.

This definition is obviously independent of the choice of $x_0 \epsilon S^n$. It will follow later (theorem (5.1)) that it does not depend on the choice of $y_0 \epsilon \mathcal{Y}$ either.

The components of $\mathcal{Y}^{S^n}(x_0,y_0)$ are the elements of the n-th homotopy group $\pi_n(\mathcal{Y})$ of $\mathcal{Y}$ [5]). Therefore if $\mathcal{Y}$ is n-simple we may consider the components of $\mathcal{Y}^{S^n}$ as the elements of $\pi_n(\mathcal{Y})$. Of course

(4.1) *If* $\pi_n(\mathcal{Y})=0$, *then* $\mathcal{Y}$ *is n-simple.*

For $\mathcal{Y}^{S^n}(x_0,y_0)$ is then connected.

5. Let S^1 be the set of complex numbers z such that $|z|=1$. Consider in $S^n\times S^1$ the set $M^n=S^n\times(1)+(x_0)\times S^1$.

(5.1) ***Theorem.*** $\mathcal{Y}$ *is n-simple if and only if every* $g\epsilon\mathcal{Y}^{M^n}$ *has an extension* $g'\epsilon\mathcal{Y}^{S^n\times S^1}$.

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[3]) C. Kuratowski, loc. cit. p. 285.

[4]) This is the only place where we use that $\mathcal{Y}$ is locally connected in dimensions $0,1,\dots,n$. The hypothesis that every two points in $\mathcal{Y}$ can be connected by an arc is quite sufficient if we agree to consider all the time homotopy classes and homotopy classes rel. (x_0,y_0) instead of components of $\mathcal{Y}^{S^n}$ and of $\mathcal{Y}^{S^n}(x_0,y_0)$. In the later part of the paper where we consider the "universal covering space" of $\mathcal{Y}$ the hypothesis that $\mathcal{Y}$ is locally connected in dimensions 0 and 1 is needed.

[5]) W. Hurewicz, Proceed. Akad. Amsterdam **38** (1935), p. 113.

It follows from (2.2) that we may admit $g(x_0,1)=y^0$.

Consider in $S^n\times\mathcal{J}$ the set $N^n=S^n\times(0)+S^n\times(1)+(x_0)\times\mathcal{J}$. Theorem (5.1) is the obviously equivalent with

(5.2) $\mathcal{Y}$ *is n-simple if and only if every* $g\epsilon\mathcal{Y}^{N^n}$ *such that*

$$g(x,0)=g(x,1) \quad \text{for each} \quad x\epsilon S^n,$$
$$g(x_0,0)=g(x_0,1)=y_0,$$

has an extension $g'\epsilon\mathcal{Y}^{S^n\times\mathcal{J}}$.

Necessity. Putting $\mathcal{X}=S^n$ and $X=(x_0)$ we obtain from (2.1) a mapping $g_1\epsilon\mathcal{Y}^{S^n\times\mathcal{J}}$ such that

$$g_1(x,t)=g(x,t) \quad \text{for} \quad (x,t)\epsilon T=S^n\times(0)+(x_0)\times\mathcal{J}.$$

The two mappings $g_1(x,1)$ and $g_1(x,0)=g(x,0)=g(x,1)$ are then homotopic and since $g_1(x_0,1)=g(x_0,1)=y_0$ and $\mathcal{Y}$ is n-simple they are also homotopic rel. (x_0,y_0). It follows that $g\epsilon\mathcal{Y}^{N^n}$ and $g_1\epsilon\mathcal{Y}^{N^n}$ are homotopic and by (2.2) there is an extension $g'\epsilon\mathcal{Y}^{S^n\times\mathcal{J}}$ of g.

Sufficiency. Let $f_0,f_1\epsilon\mathcal{Y}^{S^n}(x_0,y_0)$ be homotopic. Then there is a map $g\epsilon\mathcal{Y}^{S^n\times\mathcal{J}}$ such that

$$f_0(x)=g(x,0), \qquad f_1(x)=g(x,1) \qquad \text{for each} \quad x\epsilon S^n.$$

Let $g_1\epsilon\mathcal{Y}^{N^n}$ be the map given by

$$g_1(x,0)=g_1(x,1)=f_1(x) \qquad \text{for} \quad x\epsilon S^n,$$
$$g_1(x_0,t)=g(x_0,t) \qquad \text{for} \quad t\epsilon\mathcal{J}.$$

By hypothesis there exists an extension $g_1'\epsilon\mathcal{Y}^{S^n\times\mathcal{J}}$ of g_1. Putting

$$g_2(x,t)=\begin{cases} g(x,2t) & \text{for} \quad x\epsilon S^n \quad \text{and} \quad 0\leqslant t\leqslant\frac{1}{2} \\ g_1'(x,2-2t) & \text{for} \quad x\epsilon S^n \quad \text{and} \quad \frac{1}{2}\leqslant t\leqslant 1\end{cases}$$

we obtain a function $g_2\epsilon\mathcal{Y}^{S^n\times\mathcal{J}}$ such that

$$g_2(x,0)=f_0(x), \quad g_2(x,1)=f_1(x) \qquad \text{for} \quad x\epsilon S^n,$$
$$g_2(x_0,t)=g_2(x_0,1-t) \qquad \text{for} \quad t\epsilon\mathcal{J}.$$

This function considered only on N^n is homotopic to $g_3\epsilon\mathcal{Y}^{N^n}$, defined as follows

$$g_3(x,0)=f_0(x), \qquad g_3(x,1)=f_1(x) \qquad \text{for} \quad x\epsilon S^n,$$
$$g_3(x_0,t)=y_0 \qquad \text{for} \quad t\epsilon\mathcal{J}.$$

According to (2.2) there is an extension $g_3'\epsilon\mathcal{Y}^{S^n\times\mathcal{J}}$ of g_3 and therefore f_0 and f_1 are homotopic rel. (x_0,y_0).

6. *Theorem. If $\pi_1(\mathcal{Y})=0$, $\mathcal{Y}$ is n-simple for $n=1,2,\ldots$*

Proof. Let $f_0, f_1 \epsilon \mathcal{Y}^{S^n}(x_0,y_0)$ be homotopic. Then there is a map $g \epsilon \mathcal{Y}^{S^n\times\mathcal{J}}$ such that

$$f_0(x)=g(x,0), \qquad f_1(x)=g(x,1) \qquad \text{for each} \quad x \epsilon S^n.$$

Since $\pi_1(\mathcal{Y})=0$, the map $g \epsilon \mathcal{Y}^{N^n}$ is homotopic to $g_1 \epsilon \mathcal{Y}^{N^n}$ given by $g_1(x,0)=f_0(x)$, $g_1(x,1)=f_1(x)$, $g_1(x_0,t)=y_0$ and the theorem follows from (2.2).

7. Let us consider an arbitrary $g \epsilon \mathcal{Y}^{S^n\times\mathcal{J}}$ such that

(7.1) $$g(x_0,0)=g(x_0,1)=y_0.$$

The functions $f_0(x)=g(x,0)$ and $f_1(x)=g(x,1)$ belong to $\mathcal{Y}^{S^n}(x_0,y_0)$ and therefore they define two elements α_0 and α_1 of the n-th homotopy group $\pi_n(\mathcal{Y})$ of $\mathcal{Y}$. The function $g(x_0,t)$ determines because of (7.1) a uniquely defined element w of the fundamental group $\pi_1(\mathcal{Y})$ of $\mathcal{Y}$. We define

(7.2) $$\alpha_0=w(\alpha_1).$$

It will be proved later that α_0 is defined uniquely by $w \epsilon \pi_1(\mathcal{Y})$ and $\alpha_1 \epsilon \pi_n(\mathcal{Y})$ independently of the choice of g [6]).

8. The case $n=1$. Cutting $S^1\times\mathcal{J}$ along $(1)\times\mathcal{J}$ we obtain a squaire with its four vertices mapped by g into y_0 and its four edges representing the elements $\alpha_0, w, \alpha_1^{-1}, w^{-1}$ of $\pi_1(Y)$. It follows that (7.2) is equivalent with

(8.1) $$\alpha_0=w\alpha_1w^{-1}.$$

So the operator (7.2) is simply the inner automorphism of $\pi_1(\mathcal{Y})$ induced by w.

9. The case $n>1$. Let $\tilde{\mathcal{Y}}$ be the universal covering space of $\mathcal{Y}$ [7]), $u(\tilde{\mathcal{Y}})=\mathcal{Y}$ the function "projecting" $\tilde{\mathcal{Y}}$ on $\mathcal{Y}$, $\tilde{y}_0 \epsilon \tilde{\mathcal{Y}}$ a point such that $u(\tilde{y}_0)=y_0$. To each element $w \epsilon \pi_1(\mathcal{Y})$ corresponds a homeomorphism $\tilde{w}(\tilde{\mathcal{Y}})=\tilde{\mathcal{Y}}$ such that $u[\tilde{w}(\tilde{y})]=u(\tilde{y})$ and $\widetilde{w_2w_1}(\tilde{y})=\tilde{w}_2[\tilde{w}_1(\tilde{y})]$ for all $\tilde{y} \epsilon \tilde{\mathcal{Y}}$. For every function $\tilde{f} \epsilon \tilde{\mathcal{Y}}^{\mathcal{J}}$ such that $\tilde{f}(0)=\tilde{y}_0$, $\tilde{f}(1)=w(\tilde{y}_0)$ the function $f=u\tilde{f}$ represents the element w of $\pi_1(\mathcal{Y})$. Further we have

(9.1) *Given: a connected polyhedron Q such that $\pi_1(Q)=0$, a point $q_0 \epsilon Q$ and a function $f \epsilon \mathcal{Y}^Q(q_0,y_0)$, there is one and only one $\tilde{f} \epsilon \tilde{\mathcal{Y}}^Q(q_0,\tilde{y}_0)$ such that $u\tilde{f}=f$.*

Applying (9.1) for $Q=S^n$ we see that the relation between f and $\tilde{f}$ is a homeomorphism of $\mathcal{Y}^{S^n}(x_0,y_0)$ and $\tilde{\mathcal{Y}}^{S^n}(x_0,\tilde{y}_0)$ which establishes a (1-1)-isomorphism of the groups $\pi_n(\mathcal{Y})$ and $\pi_n(\tilde{\mathcal{Y}})$ [8]). The element of $\pi_n(\tilde{\mathcal{Y}})$ corresponding to $\alpha \epsilon \pi_n(\mathcal{Y})$ will be denoted by $\tilde{\alpha}$.

It follows from $\pi_1(\tilde{\mathcal{Y}})=0$ and from (6.1) that $\tilde{\mathcal{Y}}$ is n-simple. Therefore the elements of $\pi_n(\tilde{\mathcal{Y}})$ can be identified with the components of the space $\tilde{\mathcal{Y}}^{S^n}$. It follows that for each $w \epsilon \pi_1(\mathcal{Y})$ the homeomorphism $\tilde{w}(\tilde{\mathcal{Y}})=\tilde{\mathcal{Y}}$ defines a (1-1)-isomorphism

$$\tilde{w}[\pi_n(\tilde{\mathcal{Y}})]=\pi_n(\tilde{\mathcal{Y}}).$$

We are going to prove now that (7.2) is equivalent to

(9.2) $$\tilde{\alpha}_0=\tilde{w}(\tilde{\alpha}_1).$$

(7.2)$\rightarrow$(9.2). Applying (9.1) we obtain three functions $\tilde{g} \epsilon \tilde{\mathcal{Y}}^{S^n\times\mathcal{J}}$, $\tilde{f}_0, \tilde{f}_1 \epsilon \tilde{\mathcal{Y}}^{S^n}$ such that

$$\tilde{g}(x_0,0)=\tilde{y}_0, \qquad u\tilde{g}=g,$$
$$\tilde{f}_i(x_0)=\tilde{y}_0, \qquad u\tilde{f}_i=f, \qquad \tilde{f}_i \epsilon \alpha_i \qquad (i=0,1).$$

Since the mapping $g(x_0,t)$ represents the element $w \epsilon \pi_1(Y)$ we have $\tilde{g}(x_0,1)=\tilde{w}(\tilde{y}_0)$. It follows from (9.1) that

(9.3) $$\tilde{g}(x,0)=\tilde{f}_0(x), \quad \tilde{g}(x,1)=\tilde{w}[\tilde{f}_1(x)] \qquad \text{for all } x \epsilon S^n,$$

$\tilde{f}_0$ and $\tilde{w}\tilde{f}_1$ are thus homotopic, which implies (9.2).

(9.2)$\rightarrow$(7.2). Let $f_i \epsilon \mathcal{Y}^{S^n}(x_0,y_0)$, $f_i \epsilon \alpha_i$ $i=0,1$. It follows from (9.2) that $\tilde{f}_0$ and $\tilde{w}\tilde{f}_i$ are homotopic, so let $\tilde{g} \epsilon \tilde{\mathcal{Y}}^{S^n\times\mathcal{J}}$ be such that (9.3) hold. Putting $g=u\tilde{g}$ we verify immediately (7.2).

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[6]) The transformation $w(\alpha)$ has been introduced by J. H. C. Whitehead in a paper which will soon appear in the Proc. London Math. Soc.

[7]) We assume that the reader is aquainted with the covering spaces although the complete theory is published only in the case when $\mathcal{Y}$ is a polyhedron (Seifert-Threlfall, *Lehrbuch der Topologie*, Leipzig-Berlin 1934, Chapter 8). In particular we are not proving (9.1).

[8]) W. Hurewicz, loc. cit.

10. From (8.1) and (9.2) we deduce the following properties of the operator $w(\alpha)$:

(10.1) *$w(\alpha)$ is a (1-1)-isomorphism transforming $\pi_n(\mathcal{Y})$ into itself,*

(10.2) $w_2[w_1(\alpha)]=w_2w_1(\alpha)$, $1(\alpha)=\alpha$.

We see that $\pi_1(\mathcal{Y})$ is a group of operators for the group $\pi_n(\mathcal{Y})$ with the unit element as a unit operator.

Let $c_n(\mathcal{Y})$ be the set of all α such that

(10.3) $$\alpha=w(\alpha)$$

for all w, and $z_n(\mathcal{Y})$ the set of all w such that (10.3) holds for all α.

(10.4) *$c_n(\mathcal{Y})$ is a subgroup of $\pi_n(\mathcal{Y})$,*

(10.5) *$z_n(\mathcal{Y})$ is a self-conjugate subgroup of $\pi_1(\mathcal{Y})$,*

(10.6) *If $\pi_n(\mathcal{Y})$ has no elements of finite order, then $c_n(\mathcal{Y})$ is a subgroup of $\pi_n(\mathcal{Y})$ with division.*

(10.4) and (10.5) follow immediately from (10.1) and (10.2). In order to prove (10.6) let $n\alpha\,\epsilon\, c_n(\mathcal{Y})$, $n\neq 0$. We have then $n\alpha=w(n\alpha)=nw(\alpha)$ and therefore $\alpha=w(\alpha)$.

In the case $n=1$ it follows from (8.1) that (10.3) is equivalent with $\alpha w=w\alpha$ and therefore

(10.7) *The group $c_1(\mathcal{Y})=z_1(\mathcal{Y})$ is the centrum of the group $\pi_1(\mathcal{Y})$.*

11. We return now to the notations used in **7**. The functions $g(x,0)$ and $g(x,1)$ being homotopic, the corresponding elements α_0 and α_1, which are components of $\mathcal{Y}^{S^n}(x_0,y_0)$, belong to same component of $\mathcal{Y}^{S^n}$.

On the other hand, given $f_0, f_1\,\epsilon\,\mathcal{Y}^{S^n}(x_0,y_0)$ which are homotopic, there is an $g\,\epsilon\,\mathcal{Y}^{S^n\times\mathcal{J}}$ such that

$$f_0(x)=g(x,0) \quad f_1(x)=g(x,1) \quad \text{for all} \quad x\,\epsilon\, S_n,$$

and therefore there is a $w\,\epsilon\,\pi_1(\mathcal{Y})$ such that (7.2), where $f_i\,\epsilon\,\alpha_i$ $(i=0,1)$. Hence we obtain

(11.1) ***Theorem.*** *Two elements $\alpha_0,\alpha_1\,\epsilon\,\pi_1(\mathcal{Y})$ (considered as components of $\mathcal{Y}^{S^n}(x_0,y_0)$) are contained in one component of $\mathcal{Y}^{S^n}$ if and only if there is a $w\,\epsilon\,\pi_1(\mathcal{Y})$ such that $\alpha_0=w(\alpha_1)$ [9].*

[9]) For the case $n=1$ see Seifert-Threlfall, loc. cit., p. 176.

Therefore we have the theorems:

(11.2) ***Theorem.*** *The following conditions are equivalent:*

(a) *$\mathcal{Y}$ is n-simple,*

(b) *$\alpha=w(\alpha)$ for every $\alpha\,\epsilon\,\pi_n(\mathcal{Y})$ and $w\,\epsilon\,\pi_1(\mathcal{Y})$,*

(c) *$c_n(\mathcal{Y})=\pi_n(\mathcal{Y})$,*

(d) *$z_n(\mathcal{Y})=\pi_1(\mathcal{Y})$.*

(11.3) ***Theorem.*** *$\mathcal{Y}$ is 1-simple if and only if the group $\pi_1(\mathcal{Y})$ is abelian.*

12. From now on we are going to admit that

(12.1) *$\mathcal{Y}$ is a n-dimensional $(n>1)$ finite or infinite connected polyhedron.*

Fixing a simplicial division P^n of $\mathcal{Y}$ we obtain a corresponding division $\widetilde{P}^n$ of $\widetilde{\mathcal{Y}}$ such that the mapping $u(\widetilde{\mathcal{Y}})=\mathcal{Y}$ and the homeomorphisms $\widetilde{w}(\widetilde{\mathcal{Y}})=\widetilde{\mathcal{Y}}$ are simplicial.

Let $B^n(\mathcal{Y})$ $[B^n(\widetilde{\mathcal{Y}})]$ be the group of all n-dimensional finite cycles $\gamma^n[\widetilde{\gamma}^n]$ in $P^n[\widetilde{P}^n]$ with integer coefficients. To each $f\,\epsilon\,\mathcal{Y}^{S^n}$ $[f\,\epsilon\,\widetilde{\mathcal{Y}}^{S^n}]$ corresponds a unique cycle $h(f)\,\epsilon\, B^n(\mathcal{Y})$ $[h(f)\,\epsilon\, B^n(\widetilde{\mathcal{Y}})]$. If f_0 and f_1 are homotopic we have $h(f_0)=h(f_1)$ and therefore we obtain a homomorphism $h[\pi_n(\mathcal{Y})]\subset B^n(Y)$ $[h[\pi_n(\widetilde{\mathcal{Y}})]\subset B^n(\widetilde{\mathcal{Y}})]$. We verify easily that

(12.2) $$h(\alpha)=u[h(\widetilde{\alpha})],$$

(12.3) $$h[\widetilde{w}(\widetilde{\alpha})]=\widetilde{w}[h(\widetilde{\alpha})],$$

and it follows from (11.1) that

(12.4) $$h(\alpha)=h[w(\alpha)].$$

Let $\widetilde{\gamma}^n\,\epsilon\, B^n(\widetilde{\mathcal{Y}})$ and let Q be a finite subpolyhedron of $\widetilde{\mathcal{Y}}$ containing $\widetilde{\gamma}^n$. Since the inequality $Q\cdot\widetilde{w}(Q)\neq 0$ has [10]) only a finite set of solutions $w\,\epsilon\,\pi_1(\mathcal{Y})$ it follows that

(12.5) *Given $\widetilde{\gamma}^n\,\epsilon\, B^n(\widetilde{\mathcal{Y}})$, $\widetilde{\gamma}^n\neq 0$, the equation $\widetilde{\gamma}^n=w(\widetilde{\gamma}^n)$ has only a finite set of solutions $w\,\epsilon\,\pi_1(\mathcal{Y})$.*

[10]) Cf. S. Eilenberg, Fund. Math. **28** (1937), p. 236.

13. In this section we assume that

(13.1) $$\pi_i(\mathcal{Y})=0 \quad \text{for} \quad 1<i<n.$$

This condition is always satisfied if $n=2$. It follows that

(13.2) $$\pi_i(\widetilde{\mathcal{Y}})=0 \quad \textit{for} \quad i=1,2,\ldots,n-1.$$

and therefore by a theorem of Hurewicz[11]):

(13.3) *$h(\widetilde{\alpha})$ is a (1—1)-isomorphism of the groups $\pi_n(\mathcal{Y})$ and $B^n(\widetilde{\mathcal{Y}})$.*

It follows from (13.3), (12.3) and (12.5) that

(13.4) *Given $\alpha\epsilon\pi_n(\mathcal{Y})$, $\alpha\neq 0$, the equation $\alpha=w(\alpha)$ has only a finite set of solutions.*

This implies:

(13.5) *If $c_n(\mathcal{Y})\neq 0$, the group $\pi_1(\mathcal{Y})$ is finite,*

(13.6) *If $\pi_n(\mathcal{Y})\neq 0$, the group $z_n(\mathcal{Y})$ is finite,*

(13.7). **Theorem.** *If $\mathcal{Y}$ is n-simple and $\pi_n(\mathcal{Y})\neq 0$ the group $\pi_1(\mathcal{Y})$ is finite.*

Given a natural m let $mB^n(\mathcal{Y})$ be the subgroup of $B^n(\mathcal{Y})$ of all the cycles of the form $m\gamma^n$.

(13.8) **Theorem.** *If the group $\pi_1(\mathcal{Y})$ consists of $m<\infty$ elements, the homomorphism h transforms $c_n(\mathcal{Y})$ isomorphically into the group $mB^n(\mathcal{Y})$.*

Proof. To each simplex Δ in $\mathcal{Y}$ there are in $\widetilde{\mathcal{Y}}$ exactly m simplexes $\Delta_1,\Delta_2,\ldots,\Delta_m$ such that $u(\Delta_i)=\Delta$ for $i=1,2,\ldots,m$. Given $\gamma^n\epsilon mB^n(Y)$, define $\Lambda(\gamma^n)$ by taking each Δ_i with the coeficient of Δ in γ^n divided by m. Let Γ be the subgroup of $B^n(\widetilde{\mathcal{Y}})$ composed of all the cycles $\widetilde{\gamma}^n$ such that $\widetilde{\gamma}^n=w(\widetilde{\gamma}^n)$ for all $w\epsilon\pi_1(\mathcal{Y})$. We verify easily that $\Lambda[mB^n(\mathcal{Y})]=\Gamma$ is a (1-1)-isomorphism and that $u(\Gamma)=mB^n(\mathcal{Y})$ is the isomorphism inverse to Λ. On the other hand $h(\widetilde{\alpha})$ is a (1-1)-isomorphism between $c_n(\mathcal{Y})$ and Γ, so that $u[h(\widetilde{\alpha})]$ is a (1-1)-isomorphism transforming $c_n(Y)$ into $mB^n(Y)$. Using (12.2) we obtain (13.8).

[11]) Proceed. Akad. Amsterdam **38** (1935), p. 522.

14. Theorem. *If $\mathcal{Y}$ satisfies (12.1) and (13.1), then $\mathcal{Y}$ is n-simple if and only if h transforms $\pi_n(\mathcal{Y})$ isomorphically into some subgroup of $B^n(\widetilde{\mathcal{Y}})$.*

Proof: The necessity of the condition follows from (13.7) if $\pi_1(\mathcal{Y})$ is infinite and from (11.2) and (13.8) if $\pi_1(Y)$ is finite. Sufficiency follows immediately from (12.4).

15. As an application let us consider the case when $\mathcal{Y}=M^n$ is a n-dimensional simplicial manifold such that $\widetilde{\mathcal{Y}}$ is the n-dimensional sphere $S^n (n>1)$. Condition (13.2) being satisfied, (13.1) and obviously (12.1) are also satisfied. The group $\pi_1(M^n)$ is finite. The group $\pi_n(M^n)$, being isomorphic to $B^n(S^n)$, is cyclic infinite.

Each homeomorphism $\widetilde{w}(S^n)=S^n$ has a degree $\varepsilon_w=\pm 1$. Since $\widetilde{w}(\widetilde{\gamma}^n)=\varepsilon_w\widetilde{\gamma}^n$ for each n-dimensional cycle $\widetilde{\gamma}^n$ in S^n, it follows from (13.3) and (12.3) that

(15.1) $$w(\alpha)=\varepsilon_w\,\alpha.$$

We distinguish two cases:

1⁰ M^n is orientable i.e. $\varepsilon_w=1$ for all $w\epsilon\pi_1(M^n)$. By (15.1) M^n is n-simple, $\pi_n(M^n)=c_n(M^n)$ and $\pi_1(M^n)=z_n(M^n)$. The n-dimensional real projective space for $n=2k+1$ and all the n-dimensional lens-speces[12]) are contained in this case.

2⁰ M^n is not orientable, i. e. $\varepsilon_w=-1$ for some $w\epsilon\pi_1(M^n)$. By (15.1) M^n is not n-simple. Since $\pi_n(M^n)$ is cyclic infinite it follows from (11.2) and (10.6) that $c_n(M^n)=0$. The group $z_n(M^n)$ consists of all $w\epsilon\pi_1(M^n)$ such that $\varepsilon_w=1$. This case contains the n-dimensional real projective space for $n=2k$.

[12]) Cf. Seifert-Threlfall, loc. cit., p. 210.