The preceding results have an amusing formal consequence, valid for any ring \( R \), with a unit element 1. Let \( \eta_1, \eta_2, \ldots \) be an infinite sequence of elements in \( R \), with repetitions allowed. Then \( \eta'_n = \eta''_n \) where \( \eta'_n \) and \( \eta''_n \) are defined by the recurrence formulae,

\[
\eta'_n = -\left( \eta'_{n-1} + \eta'_{n-2} + \ldots + \eta_1 \right), \\
\eta''_n = -\left( \eta''_{n-1} + \eta''_{n-2} + \ldots + \eta_1 \right).
\]

This is true for any ring since it is true for the ring which is freely generated by \( \eta_1, \eta_2, \ldots \) with infinite sums allowed, provided no product \( \pm \eta_{m_1} \ldots \eta_{m_k} \) is repeated infinitely many times. For if a degree, given by \( \delta(\pm \eta_{m_1} \ldots \eta_{m_k}) = m_1 + \ldots + m_k \), is assigned to each product, only a finite number of terms in such a sum can have the same degree. It follows from induction on \( g \) that \( \eta'_n \) and \( \eta''_n \) are homogeneous of degree \( g \) and, as before, that \( \eta''_n = \eta''_n \), where

\[
\eta'_n = \eta''_n, \quad \eta'_n = \eta'_n.
\]

Therefore \( \eta''_n = \eta''_n \), whence \( \eta''_n = \eta''_n \).


On the relation between the fundamental group of a space and the higher homotopy groups.

By

Samuel Eilenberg (Warszawa).

1. \( Y \) will denote a separable, connected metric space locally connected in dimensions \( 0,1,\ldots, n \). Given a compact metric space \( X \), the continuous functions \( f(X) \subset Y \) with the distance formula

\[
|f(x) - f(y)| = \sup_{x \in X} |f(x) - f(y)|
\]

form a metric space \( Y^X \).

Given two points \( x_0 \in X \) and \( y_0 \in Y \) the equation \( f(x_0) = y_0 \) defines a closed subset \( Y^x(x_0, y_0) \) of \( Y^X \).

I will denote the closed interval \([0,1]\) by \( I \) and \( X \times I \) will stand for the cartesian product of \( X \) and \( I \). Two functions \( f_0, f_1 \in Y^X \) will be called homotopic if there is a function \( g \in Y^{X \times I} \) such that

\[
\begin{align*}
  f_0(x) &= g(x, 0), \\
  f_1(x) &= g(x, 1)
\end{align*}
\]

for all \( x \in X \).

If also

\[
g(x_0, t) = y_0 \quad \text{for all} \quad t \in I,
\]

we say that \( f_0, f_1 \in Y^x(x_0, y_0) \) are homotopic rel. \( (x_0, y_0) \).

2. Let \( X \) be a polyhedron and \( X \) a subpolyhedron of \( X \). It is well known that \( T = X \cup (0) + X \times I \) is a retract of \( X \times I \) and therefore that

\[
(2.1) \quad \text{Every} \quad f \in Y^I \quad \text{has an extension} \quad f^* \in Y^{X \times I}.
\]

It follows immediately from (2.1) that

\[
(2.2) \quad \text{Given two homotopic functions} \quad f_0, f_1 \in Y^X \quad \text{and an extension} \quad f^* \in Y^{X} \quad \text{of} \quad f_0, \quad \text{there is an extension} \quad f^* \in Y^{X} \quad \text{of} \quad f_1 \quad \text{homotopic to} \quad f_0.
\]

---


Higher homotopy groups

It follows from (2.2) that we may admit \( g(x_0) = y_0 \).

Consider in \( S^0 \times T \) the set \( X = S^0 \times (0) + S^0 \times (1) \) and \( Y \).

Theorem (5.1) is the obviously equivalent with

\( Y \) is n-simple if and only if every \( g \in Y^{n+1} \) such that

\[
g(x_0) = g(x,1) \quad \text{for each} \quad x \in S^0,
\]

\[
g(x_0) = g(x_0,1) = y_0,
\]

has an extension \( g \cdot Y^{n+1} \times T \).

Necessity. Putting \( X = S^0 \) and \( Y = (x_0) \) we obtain from (2.1) a mapping \( g \in Y^{n+1} \times T \) such that

\[
g(x_0, t) = g(x, t) \quad \text{for} \quad (x, t) \in S^0 \times (0) + (x_0) \times T.
\]

The two mappings \( g(x_0,1) \) and \( g(x,0) = g(x_0,0) = g(x,1) \) are then homotopic and since \( g(x_0,1) = g(x_0,1) = y_0 \) and \( Y \) is n-simple they are also homotopic rel. \((x_0, y_0)\). It follows that \( g \in Y^{n+1} \) and \( g \in Y^{n+1} \) are homotopic and by (2.2) there is an extension \( g \cdot Y^{n+1} \times T \) of \( g \).

Sufficiency. Let \( f_0, f_1 \in Y^{n+1} \times T \) be homotopic. Then there is a map \( g \in Y^{n+1} \times T \) such that

\[
f_0(x) = g(x,0), \quad f_1(x) = g(x,1) \quad \text{for each} \quad x \in S^0.
\]

Let \( g \in Y^{n+1} \) be the map given by

\[
g(x_0) = g(x_0,1) = f_1(x) \quad \text{for} \quad x \in S^0,
\]

\[
g(x_0, t) = g(x_0, t) \quad \text{for} \quad t \in T.
\]

By hypothesis there exists an extension \( g \cdot Y^{n+1} \) of \( g \).

Putting

\[
g_2(x, t) = \begin{cases} g(x, 2t) & \text{for} \quad x \in S^0 \quad \text{and} \quad 0 \leq t \leq \frac{1}{2} \\ g(x, 2-2t) & \text{for} \quad x \in S^0 \quad \text{and} \quad \frac{1}{2} \leq t \leq 1 \end{cases}
\]

we obtain a function \( g_2 \in Y^{n+1} \times T \) such that

\[
g_2(x_0) = f_0(x), \quad g_2(x_0,1) = f_1(x) \quad \text{for} \quad x \in S^0,
\]

\[
g_2(x_0, t) = g_2(x_0, 1-t) \quad \text{for} \quad t \in T.
\]

This function considered only on \( S^0 \) is homotopic to \( g_2 \), defined as follows

\[
g(x_0) = f_0(x), \quad g(x_0,1) = f_1(x) \quad \text{for} \quad x \in S^0,
\]

\[
g(x_0, t) = y_0 \quad \text{for} \quad t \in T.
\]

According to (2.2) there is an extension \( g_3 \in Y^{n+1} \times T \) of \( g_2 \) and therefore \( f_0 \) and \( f_1 \) are homotopic rel. \((x_0, y_0)\).
6. Theorem. If \( \pi_n(Y) = 0 \), \( Y \) is \( n \)-simple for \( n = 1, 2, \ldots \).

Proof. Let \( f_0, f_1 : Y^{on}(x_0, y_0) \) be homotopic. Then there is a map \( g \in Y^{on} \) such that
\[
f_0(x) = g(x, 0), \quad f_1(x) = g(x, 1)
\]
for each \( x \in Y^{on} \).

Since \( \pi_n(Y) = 0 \), the map \( g \in Y^{on} \) is homotopic to \( g_1 \in Y^{on} \) given by \( g_1(x, 0) = f_0(x) \), \( g_1(x, 1) = f_1(x) \), \( g_1(x_0, t) = y_0 \) and the theorem follows from (2.2).

7. Let us consider an arbitrary \( g \in Y^{on} \times Y \) such that
\[
g(x_0, 0) = g(x_0, 1) = y_0.
\]

The functions \( f_0(x) = g(x, 0) \) and \( f_1(x) = g(x, 1) \) belong to \( Y^{on}(x_0, y_0) \) and therefore they define two elements \( a_0 \) and \( a_1 \) of the \( n \)-th homotopy group \( \pi_n(Y) \) of \( Y \). The function \( g(x, t) \) determines because of (7.1) a uniquely defined element \( w \) of the fundamental group \( \pi_1(Y) \) of \( Y \).

We define
\[
a_0 = wa(1) = wa_1.
\]

It will be proved later that \( a_0 \) is defined uniquely by \( w \in \pi_1(Y) \) and \( a_1 \in \pi_n(Y) \) independently of the choice of \( g \).

8. The case \( n = 1 \). Cutting \( S^1 \times T \) along \((1,1) \times T \) we obtain a square with its four vertices mapped by \( g \) into \( y_0 \) and its four edges representing the elements \( a_0, a_1, a_1^{-1}, a_0^{-1} \) of \( \pi_1(Y) \). It follows that (7.2) is equivalent with
\[
a_0 = wa_0 a_1 w^{-1}.
\]

So the operator (7.2) is simply the inner automorphism of \( \pi_1(Y) \) induced by \( w \).

9. The case \( n > 1 \). Let \( Y \) be the universal covering space of \( Y^{on} \), \( w(Y) = Y \) the function "projecting" \( Y \) on \( Y \), \( Y^{on} \) a point such that \( w(y_0) = y_0 \). To each element \( w \in \pi_n(Y) \) corresponds a homeomorphism \( \tilde{w}(Y) = Y \) such that \( w_0 \tilde{w}(Y) = \tilde{w} \). For every function \( f \in Y^{on} \) such that \( \tilde{f}(0) = \tilde{y}_0, \tilde{f}(1) = w(\tilde{y}_0) \) the function \( f = \tilde{w} \) represents the element \( w \) of \( \pi_n(Y) \). Further we have
\[
(9.1) \quad \text{Given: a connected polyhedron } Q \text{ such that } \pi_1(Q) = 0, \text{ a point } q_0 \in Q \text{ and a function } f : y \in Y^{qn}(q_0, y_0), \text{ there is one and only one } f \in Y^{qn}(q_0, y_0) \text{ such that } u^{-1} = f.
\]

Applying (9.1) for \( Q = S^0 \) we see that the relation between \( f \) and \( \tilde{f} \) is a homeomorphism of \( Y^{on}(x_0, y_0) \) and \( Y^{on}(x_0, y_0) \) which establishes a \((1,1)\)-isomorphism of the groups \( \pi_n(Y) \) and \( \pi_n(Y) \).

The element of \( \pi_n(Y) \) corresponding to \( a \in \pi_n(Y) \) will be denoted by \( \tilde{a} \).

It follows from \( \pi_1(Y) = 0 \) and from (6.1) that \( Y \) is \( n \)-simple. Therefore the elements of \( \pi_n(Y) \) can be identified with the components of the space \( Y^{on} \). It follows that for each \( w \in \pi_n(Y) \) the homeomorphism \( \tilde{w} : Y^{on} = Y \) defines a \((1,1)\)-isomorphism
\[
\tilde{w} : \pi_n(Y) = \pi_n(Y).
\]

We are going to prove now that (7.2) is equivalent to
\[
(9.2) \quad \tilde{a}_0 = \tilde{w}(\tilde{a}_1).
\]

\[
(7.2) \rightarrow (9.2). \quad \text{Applying (9.1) we obtain three functions } \tilde{Y}^{on} \times Y, \tilde{f}_0, \tilde{f}_1 \in \tilde{Y}^{on} \times Y \text{ such that }
\]
\[
\tilde{Y}(x_0, 0) = \tilde{y}_0, \quad \tilde{Y}(x_0, 1) = w(\tilde{y}_0), \quad \tilde{f}_0(x) = \tilde{y}_0, \quad \tilde{f}_1(x) = \tilde{y}_0, \quad \tilde{f}_1 \in \pi_n(Y) \quad (i = 0, 1).
\]

Since the mapping \( g(x, t) \) represents the element \( w \in \pi_1(Y) \) we have \( \tilde{w}(x_0, 1) = \tilde{w}(\tilde{y}_0) \). It follows from (9.1) that
\[
\tilde{Y}(x_0, 0) = \tilde{f}_0(x) \quad \text{for all } x \in Y^{on},
\]
\[
\tilde{Y}(x_0, 1) = \tilde{f}_1(x)
\]
\( \tilde{f}_0 \) and \( \tilde{f}_1 \) are thus homotopic, which implies (9.2).

\[
(9.2) \rightarrow (7.2). \quad \text{Let } f_0 \in \tilde{Y}^{on}(x_0, y_0), \text{ } f_0 \in \pi_n(Y), \text{ } i = 0, 1. \text{ It follows from (9.2) that } \tilde{f}_0 \text{ and } \tilde{f}_1 \text{ are homotopic, so let } \tilde{Y} = \tilde{Y}^{on} \times Y \text{ be such that (9.3) hold. Putting } g = w \tilde{Y} \text{ we verify immediately (7.2).}
\]

---

4) The transformation \( w(a) \) has been introduced by J. H. C. Whitehead in a paper which will soon appear in the Proc. London Math. Soc.

7) We assume that the reader is acquainted with the covering spaces although the complete theory is published only in the case when \( Y \) is a polyhedron (Seifert-Threlfall, Lehrbuch der Topologie, Leipzig-Berlin 1934, Chapter 8). In particular we are not proving (9.1).

8) W. Hurewicz, loc. cit.
10. From (8.1) and (9.2) we deduce the following properties of the operator \( w(a) \):

(10.1) \( w(a) \) is a \((1,1)-\)isomorphism transforming \( \pi_\alpha(Y) \) into itself,

(10.2) \( w_2w_1(a)=w_2w_1(a), \quad 1(a)=a. \)

We see that \( \pi_\alpha(Y) \) is a group of operators for the group \( \pi_\alpha(Y) \) with the unit element as a unit operator.

Let \( c_\alpha(Y) \) be the set of all \( a \) such that

(10.3) \[ a = w(a) \]

for all \( w, \) and \( x_\alpha(Y) \) the set of all \( w \) such that (10.3) holds for all \( a. \)

(10.4) \( c_\alpha(Y) \) is a subgroup of \( \pi_\alpha(Y) \),

(10.5) \( \pi_\alpha(Y) \) is a self-conjugate subgroup of \( \pi_\alpha(Y) \),

(10.6) If \( \pi_\alpha(Y) \) has no elements of finite order, then \( c_\alpha(Y) \) is a subgroup of \( \pi_\alpha(Y) \) with division.

(10.4) and (10.5) follow immediately from (10.1) and (10.2).

In order to prove (10.6) let \( n a \in c_\alpha(Y), \quad n \neq 0. \) We have then

\[ na = w(na) = w(na) \]

and therefore \( a = w(a) \).

In the case \( n = 1 \) it follows from (8.1) that (10.3) is equivalent with \( w = w(a) \) and therefore

(10.7) \[ \text{The group } c_\alpha(Y) = x_\alpha(Y) \text{ is the centrum of the group } \pi_\alpha(Y). \]

11. We return now to the notations used in 7. The functions \( g(x,0) \) and \( g(x,1) \) being homotopic, the corresponding elements \( a_0 \) and \( a_1 \), which are components of \( Y^{\omega^\alpha}(x_\alpha, y_\alpha) \), belong to same component of \( Y^{\omega^\alpha}. \)

On the other hand, given \( f_0, f_1: Y^{\omega^\alpha}(x_\alpha, y_\alpha) \) which are homotopic, there is an \( g: \omega^\alpha \times Y^{\omega^\alpha} \) such that

\[ f_0(x) = g(x,0), \quad f_1(x) = g(x,1) \quad \text{for all } x \in S_\alpha, \]

and therefore there is a \( w \in \pi_\alpha(Y) \) such that (7.2), where \( f_i \in a_0 \) \((i = 0, 1)\). Hence we obtain

(11.1) \[ \text{Theorem. Two elements } a_0, a_1 \in \pi_\alpha(Y) \text{ (considered as components of } Y^{\omega^\alpha}(x_\alpha, y_\alpha) \text{) are contained in one component of } Y^{\omega^\alpha}, \text{ if and only if there is a } w \in \pi_\alpha(Y) \text{ such that } a_0 = w(a_1). \]

1) For the case \( n = 1 \) see Seifert-Threlfall, loc. cit., p. 176.

Higher homotopy groups

Therefore we have the theorems:

11.2) \[ \textbf{Theorem. The following conditions are equivalent:} \]

(a) \( Y \) is \( n \)-simple,

(b) \( a = w(a) \) for every \( a \in \pi_\alpha(Y) \) and \( w \in \pi_\alpha(Y) \),

(c) \( c_\alpha(Y) = \pi_\alpha(Y) \),

(d) \( \pi_\alpha(Y) = \pi_\alpha(Y) \).

11.3) \[ \textbf{Theorem. } Y \text{ is } 1 \text{-simple if and only if the group } \pi_1(Y) \text{ is abelian.} \]

12. From now on we are going to admit that

12.1) \( Y \) is a \( n \)-dimensional \((n > 1)\) finite or infinite connected polyhedron.

Fixing a simplicial division \( P^n \) of \( Y \) we obtain a corresponding division \( P^n \) of \( Y \) such that the mapping \( u(Y) = Y \) and the homeomorphisms \( \varphi(Y) = Y \) are simplicial.

Let \( B^n(Y) \) \([B^n(Y)]\) be the group of all \( n \)-dimensional finite cycles \( \nu^\alpha(Y) \) in \( P^n \) \([P^n]\) with integer coefficients. To each \( f_\alpha: Y^{\omega^\alpha} \rightarrow Y^{\omega^\alpha} \) corresponds a unique cycle \( h(f_\alpha): B^n(Y) \rightarrow B^n(Y) \). If \( f_\alpha \) and \( f_\beta \) are homotopic we have \( h(f_\alpha) = h(f_\beta) \) and therefore we obtain a homomorphism \( h(\pi_\alpha(Y)) \rightarrow h(\pi_\alpha(Y)) \). We verify easily that

(12.2) \[ h(a) = w(\varphi(a)), \]

(12.3) \[ h(\varphi(\varphi)) = \varphi(h(\varphi)), \]

and it follows from (11.1) that

(12.4) \[ h(a) = w(a). \]

Let \( \tilde{\gamma}^\alpha \in B^n(Y) \) and let \( Q \) be a finite subpolyhedron of \( Y \) containing \( \tilde{\gamma}^\alpha \). Since the inequality \( Q \cdot \tilde{\gamma}(Q) = 0 \) has only a finite set of solutions \( w \in \pi_\alpha(Y) \) it follows that

(12.5) \[ \text{Given } \tilde{\gamma}^\alpha \in B^n(Y), \quad \tilde{\gamma} = 0, \text{ the equation } \tilde{\gamma}^\alpha = w(\tilde{\gamma}^\alpha) \text{ has only a finite set of solutions } w \in \pi_\alpha(Y). \]

13. In this section we assume that
(13.1) \( \pi_i(\mathcal{Y}) = 0 \) for \( 1 < i < n \).

This condition is always satisfied if \( n = 2 \). It follows that
(13.2) \( \pi_i(\mathcal{Y}) = 0 \) for \( i = 1, 2, \ldots, n-1 \).

and therefore by a theorem of Hurewicz:\[11]\):
(13.3) \( h(\alpha) \) is a \((1-1)\)-isomorphism of the groups \( \pi_n(\mathcal{Y}) \) and \( B^n(\mathcal{Y}) \).

It follows from (13.3), (12.8) and (12.5) that
(13.4) Given \( \alpha \in \pi_n(\mathcal{Y}) \), \( \alpha \neq 0 \), the equation \( a = w(\alpha) \) has only a finite set of solutions.

This implies:
(13.5) If \( \alpha_n(\mathcal{Y}) \neq 0 \), the group \( \pi_n(\mathcal{Y}) \) is finite,
(13.6) If \( \pi_n(\mathcal{Y}) = 0 \), the group \( \alpha_n(\mathcal{Y}) \) is finite,
(13.7) Theorem. If \( \mathcal{Y} \) is \( n \)-simple and \( \alpha_n(\mathcal{Y}) \neq 0 \) the group \( \pi_n(\mathcal{Y}) \) is finite.

Given a natural \( m \) let \( mB^n(\mathcal{Y}) \) be the subgroup of \( B^n(\mathcal{Y}) \) of all the cycles of the form \( m\gamma^n \).
(13.8) Theorem. If the group \( \pi_n(\mathcal{Y}) \) consists of \( m < \infty \) elements, the homomorphism \( h \) transforms \( \alpha_n(\mathcal{Y}) \) isomorphically into \( mB^n(\mathcal{Y}) \).

Proof. To each simplex in \( \mathcal{Y} \) there are in \( \mathcal{Y} \) exactly \( m \) simplexes \( A_1, A_2, \ldots, A_m \) such that \( u(A_i) = \alpha \) for \( i = 1, 2, \ldots, m \). Given \( \gamma^n \in mB^n(\mathcal{Y}) \), define \( A(\gamma^n) \) by taking each \( A_i \) with the coefficient of \( \alpha \) in \( \gamma^n \) divided by \( m \). Let \( I \) be the subgroup of \( B^n(\mathcal{Y}) \) composed of all the cycles \( \gamma^n \) such that \( \gamma^n = w(\gamma^n) \) for all \( \alpha \in \pi_n(\mathcal{Y}) \). We verify easily that \( A(mB^n(\mathcal{Y})) = I \) is a \((1-1)\)-isomorphism and that \( w(I) = mB^n(\mathcal{Y}) \) is the isomorphism inverse to \( A \). On the other hand \( h(\alpha) \) is a \((1-1)\)-isomorphism between \( \alpha_n(\mathcal{Y}) \) and \( I \), so that \( w[h(\alpha)] \) is a \((1-1)\)-isomorphism transforming \( \alpha_n(\mathcal{Y}) \) into \( mB^n(\mathcal{Y}) \). Using (12.2) we obtain (13.8).

14. Theorem. If \( \mathcal{Y} \) satisfies (12.1) and (13.1), then \( \mathcal{Y} \) is \( n \)-simple if and only if \( k \) transforms \( \pi_n(\mathcal{Y}) \) isomorphically into some subgroup of \( B^n(\mathcal{Y}) \).

Proof: The necessity of the condition follows from (13.7) if \( \pi_n(\mathcal{Y}) \) is infinite and from (11.2) and (13.8) if \( \pi_n(\mathcal{Y}) \) is finite. Sufficiency follows immediately from (12.4).

15. As an application let us consider the case when \( \mathcal{Y} = M^n \) is a \( n \)-dimensional simplicial manifold such that \( \mathcal{Y} \) is the \( n \)-dimensional sphere \( S^n(n > 1) \). Condition (13.2) being satisfied, (13.1) and obviously (12.1) are also satisfied. The group \( \pi_n(M^n) \) is finite. The group \( \pi_n(M^n) \), being isomorphic to \( B^n(S^n) \), is cyclic infinite.

Each homeomorphism \( \bar{w}(S^n) = S^n \) has a degree \( \epsilon_w = \pm 1 \). Since \( \bar{w}(\gamma^n) = \epsilon_w \gamma^n \) for each \( n \)-dimensional cycle \( \gamma^n \) in \( S^n \), it follows from (13.3) and (12.3) that
(15.1) \( w(\alpha) = \epsilon_w \alpha \).

We distinguish two cases:

1st \( M^n \) is orientable i.e. \( \epsilon_w = 1 \) for all \( w \in \pi_1(M^n) \). By (15.1) \( M^n \) is \( n \)-simple, \( \pi_n(M^n) = \pi_n(M^n) \) and \( \pi_n(M^n) = \epsilon_n(M^n) \). The \( n \)-dimensional real projective space for \( n = 2k + 1 \) and all the \( n \)-dimensional lens-species are contained in this case.

2nd \( M^n \) is not orientable, i.e. \( \epsilon_w = -1 \) for some \( w \in \pi_n(M^n) \). By (15.1) \( M^n \) is not \( n \)-simple. Since \( \pi_n(M^n) \) is cyclic infinite it follows from (11.2) and (10.6) that \( \epsilon_w(M^n) = 0 \). The group \( \epsilon_w(M^n) \) consists of all \( w \in \pi_1(M^n) \) such that \( \epsilon_w = 1 \). This case contains the \( n \)-dimensional real projective space for \( n = 2k \).