On the summability of double Fourier series.

By

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1. Let $f(x,y)$ be a function of period 2π with respect to each of the variables x,y. If $f(x,y)$ is integrable over the square

$$(Q) \qquad -\pi \leqslant x \leqslant \pi, \qquad -\pi \leqslant y \leqslant \pi,$$

then we may associate with the function its Fourier series

$$(1.1) \qquad \sum_{m,n=-\infty}^{+\infty} a_{mn}\, e^{i(mx+ny)},$$

where

$$a_{mn} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\, e^{-i(mx+ny)}\, dx\, dy \quad (m,n=0,\pm 1,\pm 2,\ldots).$$

By $\sigma_{m,n}(x,y)=\sigma_{m,n}(x,y;f)$ we shall mean the first arithmetic means of the series (1.1), that is

$$\sigma_{m,n}(x,y) = \sum_{\mu=-m}^{m} \sum_{\nu=-n}^{n} \left(1-\frac{|\mu|}{m+1}\right)\left(1-\frac{|\nu|}{n+1}\right) a_{\mu\nu}\, e^{i(\mu x+\nu y)}.$$

The following theorems have recently been established:

(A) *There is an integrable function $f(x,y)$ such that*

$$\limsup_{m,n\to+\infty} |\sigma_{m,n}(x,y;f)| = +\infty$$

at every point (x,y) [1].

[1] Cf. S. Saks, *Remark on the differentiability of the Lebesgue indefinite integral*, Fund. Math. **22** (1934), pp. 257-261.

(B) *If the function $f(x,y)\log^+|f(x,y)|$ is integrable over the square (Q), then*

$$\lim_{m,n\to+\infty} \sigma_{m,n}(x,y;f) = f(x,y)$$

at almost every point (x,y) [2].

The condition imposed on the function f in the last theorem cannot be relaxed. This is a consequence of the following result, which fills the gap between Theorems (A) and (B):

(C) *If $\varepsilon(u)$ is any bounded function defined for $u \geqslant 0$ and tending to 0 as $u\to+\infty$, then there is a function $f(x,y)$ of period 2π with respect to x,y, such that $f(x,y)\log^+|f(x,y)|\,\varepsilon(|f(x,y)|)$ is integrable, and yet*

$$(1.2) \qquad \limsup_{m,n\to+\infty} |\sigma_{m,n}(x,y;f)| = +\infty$$

at every point (x,y) [3].

The problem of the summability of double Fourier series is closely connected with that of the differentiability of double integrals. As a matter of fact, Theorems (B) and (C) are consequences of the following results concerning double integrals:

(B_1) *If the function $f(x,y)\log^+|f(x,y)|$ is integrable, then, at almost every point (x,y),*

$$(1.3) \qquad \lim_{h,k\to+0} \frac{1}{hk} \int_0^h\int_0^k f(x+u,y+v)\, du\, dv = f(x,y) \text{ [4]}.$$

[2] See Jessen, Marcinkiewicz and Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math. **25** (1935), pp. 217-234.

[3] See Jessen, Marcinkiewicz and Zygmund, *loc. cit.* The theorem is stated there in a slightly weaker form, viz. that there is a function $f(x,y)$ with $|f|\log^+|f|\,\varepsilon(|f|)\epsilon L$ and such that (1.2) holds at almost every point (x,y). Theorem (C) is, however, a simple consequence of Theorem (C_1) (see below).

[4] See Jessen, Marcinkiewicz and Zygmund, *loc. cit.*

This result is a special case of the following general theorem:

Let $f(x_1,x_2,\ldots,x_k)$ be a function defined in a k-dimensional cell Q, with sides parallel to the coordinate axes. If the function $|f|(\log^+|f|)^{k-r}$ is integrable over Q, where r is one of the numbers $1,2,\ldots,k$, then, at almost every point $(x_1,x_2,\ldots,x_k)$ of Q,

$$\lim_{h_1,h_2,\ldots,h_k\to 0} \frac{1}{h_1h_2\ldots h_k} \int_0^{h_1}\!\!\ldots\int_0^{h_k} f(x_1+u_1,\ldots,x_k+u_k)\, du_1\ldots du_k = f(x_1,\ldots,x_k),$$

provided that the ratios h_1/h_s and h_s/h_1 are bounded for $s=1,2,\ldots,r$.

In the case $r=1$ this theorem is proved in the paper just quoted. The general result can easily be deduced from Theorem 7 of that paper.

(C$_1$) *If $\varepsilon(u)$ is any bounded function defined for $u\geqslant 0$, and tending to 0 as u tends to $+\infty$, then there is a positive function $f(x,y)$ such that the function $f(x,y)\log^+|f(x,y)|\,\varepsilon(|f(x,y)|)$ is integrable, and that*

$$\limsup_{h,k\to+0}\frac{1}{hk}\int_0^h\int_0^k f(x+u,y+v)\,du\,dv=+\infty \tag{1.4}$$

at every point (x,y) [5].

In Theorems (B) and (C) the indices m,n tend to $+\infty$ independently of each other. The main object of this paper is to prove the following result, which completes those theorems:

Theorem 1. *Let $f(x,y)$ be a function of period 2π with respect to x and y, and integrable L over the square (Q). Let $\lambda\geqslant 1$ be any fixed number. Then, at almost every point (x,y),*

$$\sigma_{m,n}(x,y)\to f(x,y), \tag{1.5}$$

provided that m and n tend to $+\infty$ in such a way that

$$m/n\leqslant\lambda,\qquad n/m\leqslant\lambda \tag{1.6}$$ [6].

This theorem is an analogue for Fourier series of the classical result of Lebesgue asserting that at almost every point (x,y) we have (1.3), if only h and k tend to 0 in such a way that $h/k\leqslant\lambda$, $k/h\leqslant\lambda$. It must however be observed that, unlike in the previous cases, Theorem 1 is not a consequence of the corresponding result on the differentiability of double integrals (and the proof of Theorem 1 is not so easy as one might expect). This is understandable, since in order to estimate the integral defining $\sigma_{m,n}(x,y)$ we cannot apply integration by parts. An integration by parts would introduce integrals $\int_0^h\int_0^k f(x+u,y+v)\,du\,dv$ with ratios h/k and k/h unbounded.

[5]) S. Saks, *On the strong derivatives of functions of intervals*, Fund. Math. **25** (1935), 235–252. An example of a positive function f such that $|f|\log^+|f|\,\varepsilon(|f|)$ is integrable and (1.4) holds almost everywhere, was given independently and at the same time by Busemann and Feller; see the paper quoted in footnote [2]).

[6]) The summability of double Fourier series by spherical Cesàro means was investigated by S. Bochner. See his paper *Summation of multiple Fourier series by spherical means*, Transactions of the American Mathematical Society **40** (1936), 175–207. A comparison of Bochner's results with Theorems (B), (C), and 1 shows that the properties of the spherical Cesàro means of double Fourier series are quite different from those of ordinary (rectangular) Cesàro means.

Added in proof, 16. I. 1939: See also the recent paper of L. Fejér, *Zur Summabilitätstheorie der Fourierschen und Laplaceschen Reihe*. Proc. Cambridge Philos. Soc. **34** (1938), 503–509.

2. We begin by proving a number of lemmas.

Lemma 1. *Let a be any positive number, and let E be any plane set satisfying the condition*

$$0<|E|<\infty.$$

If to any point (x,y) belonging to E corresponds a rectangle $R=R_{x,y}$ with centre at (x,y), and with sides $\delta=\delta(R)$ and $a\delta$, parallel respectively to the axes x and y, then there is a finite number of non overlapping rectangles $R_{x_0,y_0},R_{x_1,y_1},\dots,R_{x_n,y_n}$, such that

$$\sum_{m=0}^{n}|R_m|\geqslant\frac{1}{26}|E|, \tag{2.1}$$

where $R_m=R_{x_m,y_m}$ [7]).

Proof. Let $\boldsymbol{K}_0$ denote the class of all the rectangles R, and let

$$\delta_0=\operatorname*{Max}_{R\in\boldsymbol{K}_0}\delta(R).$$

If $\delta_0=+\infty$, then there exist rectangles R with area as large as we please. If, for example, $|R_0|>|E|$, then (2.1) is satisfied with $n=0$. We may therefore confine our attention to the case when $\delta_0<+\infty$. Let R_0 be any rectangle belonging to $\boldsymbol{K}_0$ and such that $\delta(R_0)>\frac{1}{2}\delta_0$. Let $\boldsymbol{K}_1'$ denote the class of the rectangles $R\in\boldsymbol{K}_0$ overlapping with R_0, and let $\boldsymbol{K}_1$ denote the class of the remaining rectangles $R\in\boldsymbol{K}_0$. We write

$$\delta_1=\operatorname*{Max}_{R\in\boldsymbol{K}_1}\delta(R),$$

and denote by R_1 any rectangle belonging to $\boldsymbol{K}_1$ and such that $\delta(R_1)>\frac{1}{2}\delta_1$. Let $\boldsymbol{K}_2'$ and $\boldsymbol{K}_2$ denote respectively the classes of the rectangles $R\in\boldsymbol{K}_1$ overlapping and non overlapping with R_1, and let

$$\delta_2=\operatorname*{Max}_{R\in\boldsymbol{K}_2}\delta(R).$$

[7]) The proof which follows of Lemma 1 has certain features in common with the usual proof of Vitali's covering lemma. The point of Lemma 1 is that the numerical factor on the right hand side of (2.1) is independent of a. For the case $a=1$ cf. T. Radó, *Sur un problème relatif à un théorème de Vitali*, Fund. Math. **11** (1928), 228–229.

By R_2 we denote any rectangle of $\boldsymbol{K}_2$ such that $\delta(R_2)>\frac{1}{2}\delta_2$, and so on. Generally, having defined $\boldsymbol{K}'_i$, $\boldsymbol{K}_i$, δ_i, and R_i for $i=1,2,\dots,m-1$, we define $\boldsymbol{K}'_m$ and $\boldsymbol{K}_m$ as the classes of rectangles $R\in\boldsymbol{K}_{m-1}$ respectively overlapping or non overlapping with R_{m-1} (so that $\boldsymbol{K}_{m-1}=\boldsymbol{K}'_m+\boldsymbol{K}_m$). If

(2.2) $$\delta_m=\operatorname*{Max}_{R\in\boldsymbol{K}_m}\delta(R),$$

then by R_m we denote any rectangle $R\in\boldsymbol{K}_m$ such that

(2.3) $$\delta(R_m)>\tfrac{1}{2}\delta_m.$$

It may happen that the sequence $R_0,R_1,R_2,\dots$ is finite, that is that the class $\boldsymbol{K}_m$ is empty for a certain m. Let us assume for the moment that this is not the case. Since $\delta_0\geqslant\delta_1\geqslant\delta_2\dots$, there are two possibilities:

(i) the numbers δ_m are bounded below by a positive number;

(ii) the numbers δ_m tend to 0.

In the first case the inequality (2.1) is obvious, if only n is large enough. Passing to the second case, we write:

$$S=\sum_{R\in\boldsymbol{K}_0}R,\qquad S_m=\sum_{R\in\boldsymbol{K}'_m}R\qquad (m=1,2,\dots),$$

where S and the S_m denote point sets. It will now be shown that

(2.4) $$S=\sum_{m=1}^{\infty}S_m.$$

For let $\bar{R}$ be any rectangle of $\boldsymbol{K}_0$ and $\bar{\delta}=\delta(\bar{R})$. On account of (2.2) and of the relation $\delta_m\to 0$, the rectangle $\bar{R}$ cannot belong to all the classes $\boldsymbol{K}_m$. Let $\mu\geqslant 1$ be the lowest integer such that $\bar{R}$ does not belong to $\boldsymbol{K}_\mu$. Since $\bar{R}$ belongs to $\boldsymbol{K}_{\mu-1}$, and $\boldsymbol{K}_{\mu-1}=\boldsymbol{K}'_\mu+\boldsymbol{K}_\mu$, the rectangle $\bar{R}$ must belong to $\boldsymbol{K}'_\mu$, and so $S\subset\sum_{m=1}^{\infty}S_m$. The opposite relation being obvious, we obtain (2.4). Let us now observe that $S\supset E$. From this and from (2.4) it follows that

(2.5) $$\sum_{m=1}^{\infty}|S_m|\geqslant|E|.$$

We shall now prove that

(2.6) $$\sum_{m=0}^{\infty}|R_m|\geqslant\frac{1}{25}|E|.$$

For let $R\in\boldsymbol{K}'_m$, so that R belongs to $\boldsymbol{K}_{m-1}$ and overlaps with R_{m-1}. Hence

$$\delta(R)\leqslant\delta_{m-1},\qquad \delta(R_{m-1})>\tfrac{1}{2}\delta_{m-1},$$

which gives $\delta(R)<2\delta(R_{m-1})$. It follows that the set S_m is contained in a rectangle concentric with R_{m-1} and with sides five times those of R_{m-1}. Hence $|S_m|\leqslant 25|R_{m-1}|$. From this and (2.5) we obtain (2.6), which gives (2.1) for n big enough.

If the sequence $\{R_m\}$ is finite and consists of $n+1$ terms, the above argument gives (2.6) with ∞ replaced by n. The inequality (2.1) is then true *a fortiori*. The proof of Lemma 1 is thus complete.

Lemma 2. *Let $f(x,y)$ be an integrable function defined in the square (Q') $-2\pi\leqslant x\leqslant 2\pi$, $-2\pi\leqslant y\leqslant 2\pi$, and let α be any positive and fixed number. For (x,y) belonging to the square (Q) $-\pi\leqslant x\leqslant\pi$, $-\pi\leqslant y\leqslant\pi$, we write*

$$f^*_\alpha(x,y)=\operatorname*{Max}_{h}\frac{1}{4\alpha h^2}\int_{-\alpha h}^{\alpha h}dv\int_{-h}^{h}|f(x+u,y+v)|\,du,$$

where the number h is so small that the rectangle over which the integral is taken is contained in Q'. Let

$$\mathcal{E}_\alpha(\xi)=\mathop{E}_{(x,y)}\{f^*_\alpha(x,y)>\xi\}$$

for any $\xi>0$. Then

(2.7) $$|\mathcal{E}_\alpha(\xi)|\leqslant\frac{26}{\xi}\iint_{Q'}|f(x,y)|\,dx\,dy.$$

Proof. If the point (x,y) belongs to $\mathcal{E}_\alpha(\xi)$, then there is a rectangle $R\subset Q'$ with centre at (x,y), with sides parallel to the axes x and y equal respectively to δ, $\alpha\delta$, and such that

$$\frac{1}{|R|}\iint_{R}|f(x,y)|\,dx\,dy>\xi.$$

The class of these rectangles satisfies the conditions of Lemma 1. Hence we can find a finite number of these rectangles $R_0,R_1,\dots,R_n$, non overlapping and satisfying the inequality (2.1), with $E=\mathcal{E}_\alpha(\xi)$.

This gives

$$\iint_{Q'} |f(x,y)|\,dx\,dy \geqslant \sum_{m=0}^{n} \iint_{R_m} |f(x,y)|\,dx\,dy \geqslant \frac{\xi}{26}\,|\mathcal{E}_\alpha(\xi)|,$$

and (2.7) follows at once.

Lemma 3. *Let the function $f(x,y)$ satisfy the conditions of Lemma 2, and let*

$$f^*(x,y)=\operatorname*{Max}_s \{f^*_{2^s}(x,y)\,2^{-\frac{1}{2}|s|}\} \quad \text{for } s=0,\pm1,\pm2,\dots, \text{ and } (x,y)\,\epsilon\,Q.$$

We write

$$\mathcal{E}(\xi)=\underset{x,y}{E}\{f^*(x,y)>\xi\}$$

for any $\xi>0$. Then

$$|\mathcal{E}(\xi)| \leqslant \frac{160}{\xi}\iint_{Q'} |f(x,y)|\,dx\,dy.$$

Proof. A necessary and sufficient condition for the inequality $f^*(x,y)>\xi$ is that $f^*_{2^s}(x,y)>\xi 2^{\frac{1}{2}|s|}$ for some integer s. Hence

$$\mathcal{E}(\xi)\subset \sum_{s=-\infty}^{\infty} \mathcal{E}_{2^s}(\xi 2^{\frac{1}{2}|s|}),$$

and so

$$|\mathcal{E}(\xi)| \leqslant \sum_{s=-\infty}^{\infty} |\mathcal{E}_{2^s}(\xi 2^{\frac{1}{2}|s|})| \leqslant \frac{26}{\xi}\iint_{Q'} |f(x,y)| \left(\sum_{s=-\infty}^{\infty} 2^{-\frac{1}{2}|s|}\right) dx\,dy.$$

This completes the proof.

3. Using the results of § 2 we shall now prove certain inequalities for the sums $\sigma_{m,n}$. These sums are given by the formula

$$\text{(3.1)} \qquad \sigma_{m,n}(x,y;f)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)\,K_m(u)\,K_n(v)\,du\,dv,$$

where $K_m(u)$ denotes Fejér's kernel:

$$K_m(u)=\frac{1}{m+1}\,\frac{\sin^2\frac{1}{2}(m+1)\,u}{2\sin^2\frac{1}{2}u}.$$

The kernel $K_m(u)$ satisfies the following inequalities

$$\text{(3.2)} \qquad K_m(u)\leqslant Am, \qquad K_m(u)\leqslant A/mu^2 \qquad (m\geqslant 1,\ 0<u\leqslant\pi).$$

Here and in the sequel the letter A denotes an absolute constant. The constant need not be be the same at every occurence. Let

$$\text{(3.3)} \qquad \tilde\sigma_{m,n}(x,y;f)=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x+u,y+v)\,K_m(u)\,K_n(v)\,du\,dv.$$

Let k be any non negative integer, and let the integers m and n satisfy the inequalities:

$$\text{(3.4)} \qquad 2^k\leqslant m<2^{k+1}, \qquad m/n\leqslant\lambda, \qquad n/m\leqslant\lambda.$$

It is not difficult to see that then

$$|\tilde\sigma_{m,n}(x,y;f)|\leqslant A\lambda\int_0^{\pi 2^{-k}} du \int_{\pi 2^{-k}}^{\pi} \frac{|f(x+u,y+v)|}{v^2}\,dv + A\lambda\int_0^{\pi 2^{-k}} dv\int_{\pi 2^{-k}}^{\pi}\frac{|f(x+u,y+v)|}{u^2}\,du$$

$$+A\lambda 2^{-2k}\int_{\pi 2^{-k}}^{\pi}\int_{\pi 2^{-k}}^{\pi}\frac{|f(x+u,y+v)|}{u^2v^2}\,du\,dv + A\lambda 2^{2k}\int_0^{\pi 2^{-k}}\int_0^{\pi 2^{-k}}|f(x+u,y+v)|\,du\,dv$$

$$=A\lambda P_k(x,y)+A\lambda Q_k(x,y)+A\lambda R_k(x,y)+A\lambda S_k(x,y),$$

say. We write

$$P^*(x,y)=\operatorname*{Max}_{k=0,1,\dots} P_k(x,y)=\operatorname*{Max}_{k=0,1,\dots}\int_0^{\pi 2^{-k}} du\int_{\pi 2^{-k}}^{\pi}\frac{|f(x+u,y+v)|}{v^2}\,dv,$$

and similarly define the expressions $Q^*(x,y)$, $R^*(x,y)$, and $S^*(x,y)$.

Lemma 4. *Let $f(x,y)$ be a function of period 2π with respect to x and y, integrable over the square (Q), and let $\underset{x,y}{E}\{P^*>\xi\}$ denote the set of the points $(x,y)\,\epsilon\,Q$ at which $P^*(x,y)>\xi>0$. Then*

$$\text{(3.5)} \qquad |\underset{x,y}{E}\{P^*>\xi\}|\leqslant\frac{A}{\xi}\iint_Q |f(x,y)|\,dx\,dy,$$

and the same inequality holds for the functions $Q^*(x,y)$, $R^*(x,y)$, and $S^*(x,y)$.

Proof. In the formula

$$P_k(x,y)=\sum_{i=0}^{k-1}\int_0^{\pi 2^{-k}} du \int_{\pi 2^{-i-1}}^{\pi 2^{-i}} \frac{|f(x+u,y+v)|}{v^2}\, dv$$

the i-th term of the sum on the right does not exceed

$$(3.6)\qquad A2^{2i}\int_{-\pi 2^{-k}}^{\pi 2^{-k}} du \int_{-\pi 2^{-i}}^{\pi 2^{-i}} |f(x+u,y+v)|\, dv \leqslant A2^{-(k-i)} f^*_{2^{k-i}}(x,y)$$

(with the notation of Lemma 2). Hence (cf. Lemma 3):

$$P_k(x,y)\leqslant Af^*(x,y)\sum_{i=0}^{k-1} 2^{-\frac{1}{2}(k-i)} \leqslant Af^*(x,y), \qquad P^*(x,y)\leqslant Af^*(x,y).$$

That the last inequality holds with P^* replaced by Q^*, is obvious. Passing to the function R^*, we observe that

$$R_k(x,y)=2^{-2k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\int_{\pi 2^{-i-1}}^{\pi 2^{-i}} du \int_{\pi 2^{-j-1}}^{\pi 2^{-j}} \frac{|f(x+u,y+v)|}{u^2v^2}\, dv$$

$$\leqslant A2^{-2k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} 2^{2(i+j)}\int_{-\pi 2^{-i}}^{\pi 2^{-i}} du \int_{-\pi 2^{-j}}^{\pi 2^{-j}} |f(x+u,y+v)|\, dv \leqslant A2^{-2k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} 2^{i+j} f^*_{2^{i-j}}(x,y)$$

$$\leqslant A2^{-2k} f^*(x,y)\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} 2^{i+j+\frac{1}{2}|i-j|} \leqslant A\, f^*(x,y).$$

Hence $R^*(x,y)\leqslant Af^*(x,y)$. That $S^*(x,y)\leqslant Af^*(x,y)$, is obvious, since $S_k(x,y)$ does not exceed the expression (3.6) with $i=k$.

In order to complete the proof of Lemma 4 it is sufficient to apply Lemma 3.

Taking into account the inequalities (3.3), (3.4), Lemma 4, and observing that $\sigma_{m,n}(x,y)$ may be written as a sum of four integrals, all analogous to (3.3), we see that the function

$$(3.7)\qquad \sigma^*_\lambda(x,y;f)=\operatorname{Max}|\sigma_{m,n}(x,y;f)|, \quad \text{where } m,n\geqslant 1,\ m/n\leqslant\lambda,\ n/m\leqslant\lambda,$$

satisfies the inequality

$$\sigma^*_\lambda(x,y;f)\leqslant A\lambda f^*(x,y).$$

[8]) The restriction $m,n\geqslant 1$ was introduced for the sake of convenience only. The properties of the function hold if that condition is omitted.

From this and from Lemma 3, and taking into account the periodicity of f, we deduce the following

Lemma 5. *For every* $\xi>0$,

$$(3.8)\qquad \left|\underset{x,y}{E}\{\sigma^*_\lambda(x,y;f)>\xi\}\right|\leqslant\frac{A\lambda}{\xi}\iint_Q |f(x,y)|\, dx\, dy.$$

The last inequality implies in particular that the function $\sigma^*_\lambda(x,y;f)$ is finite almost everywhere.

Theorem 1 is a simple consequence of Lemma 5. It is sufficient to make a decomposition

$$(3.9)\qquad f=f_1+f_2,$$

where f_1 is a trigonometrical polynomial, and f_2 is such that

$$\left|\underset{x,y}{E}\{\sigma^*_\lambda(x,y;f_2)>\delta\}\right|<\delta,$$

where δ is a fixed positive number, as small as we please. Since $\sigma_{m,n}(x,y;f_1)$ tends (uniformly) to $f_1(x,y)$, and since $|\sigma_{m,n}(x,y;f_2)|$ does not exceed δ, except in a set of measure $<\delta$, we have (1.5) at almost every point (x,y). The proof of Theorem 1 is thus completed.

Theorem 2. *The function* $\sigma^*_\lambda(x,y;f)$ *defined by* (3.7) *satisfies, for every* $0<\varepsilon<1$, *the inequality*

$$(3.10)\qquad \left\{\iint_Q \left(\sigma^*_\lambda(x,y;f)\right)^{1-\varepsilon}\right\}^{1/(1-\varepsilon)}\leqslant\frac{A\lambda}{\varepsilon}\iint_Q |f(x,y)|\, dx\, dy.$$

Moreover, for m *and* n *satisfying* (1.6),

$$(3.11)\qquad \iint_Q |\sigma_{m,n}(x,y;f)-f(x,y)|^{1-\varepsilon}\, dx\, dy\to 0.$$

The inequality (3.10) is a simple consequence of (3.8). The relation (3.11) can be obtained from (3.10) by means of the decomposition (3.9), where again f_1 is a trigonometrical polynomial, and the integral $\iint_Q |f_2|\, dx\, dy$ is small.

4. We conclude the paper by a few remarks.

(a) It is plain that the number λ in Theorem 1 need not be a constant, but may be a function of the point (x,y). This slightly more general result is an immediate consequence of Theorem 1 in its previous form.

(b) Under the condition (1.6), the expression $\sigma_{m,n}^{\alpha,\beta}(x,y;f)$ tends almost everywhere to $f(x,y)$, where $\sigma_{m,n}^{\alpha,\beta}$ denotes the (C,α,β) means of the Fourier series of the function f, and the numbers α,β are positive. The proof follows the same line as that of Theorem 1. We must only observe that the (C,α) kernel $K_m^\alpha(u)$ satisfies for $0<\alpha<1$ the inequalities:

$$|K_m^\alpha(u)|\leqslant A(\alpha)\,m, \qquad |K_m^\alpha(u)|\leqslant A(\alpha)/m^\alpha u^{\alpha+1} \qquad (m=1,2,\ldots;\ 0<u\leqslant\pi)$$

analogous to (3.2). ($A(\alpha)$ depends on α only).

Similarly we prove that, *if $P(r,u)$ denotes Poisson's kernel, then*

$$(4.1)\qquad \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)\,P(r,u)\,P(\varrho,v)\,du\,dv\to f(x,y)$$

at almost every point (x,y), *provided that*

$$(4.2)\qquad r\to1,\quad \varrho\to1,\quad (1-\varrho)/(1-r)<\lambda,\quad (1-r)/(1-\varrho)<\lambda.$$

More generally, we have the following

Theorem 3. *At almost every point* (x,y),

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\xi+u,\eta+v)\,P(r,u)\,P(\varrho,v)\,du\,dv\to f(x,y),$$

provided that the conditions (4.2) *are satisfied, and that the points with polar coordinates* (r,ξ), (ϱ,η) *tend respectively to the points* $(1,x)$, $(1,y)$ *along non tangential paths.*

Similarly we may generalize Theorem 2 [9]).

(c) Theorems 1 and 2 are true for the Fourier series of functions of n variables. The proofs undergo no essential changes.

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[9]) If the function $|f|\log^+|f|$ is integrable over Q, we have (3.11), for m and n tending to $+\infty$ independently of each other. The function

$$\sigma^*(x,y;f)=\operatorname*{Max}_{m,n}|\sigma_{m,n}(x,y;f)|$$

satisfies the inequality

$$\left\{\iint_Q \left(\sigma^*(x,y;f)\right)^{1-\varepsilon} dx\,dy\right\}^{1/(1-\varepsilon)} \leqslant \frac{A}{\varepsilon}\iint_Q |f|\log^+|f|\,dx\,dy+\frac{A}{\varepsilon}.$$

These results are implicitly contained in the paper quoted in footnote [2]).

On the isomorphism and the equivalence of classes and sequences of sets [1]).

By

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Introduction. The notions which are the subject of this paper belong to the General Theory of Sets and particularly to its part which deals with classes and sequences of sets. It seems that one of the chief problems in this field is the investigation into such relations between classes (or sequences) of sets which are analogous to as important a relation between sets as the equality of powers.

The relationships in the domain of *classes of sets* are obviously more complicated than in the domain of *sets*; and so it is possible to define in a natural way many relations between classes of sets which are reflexive, symmetrical and transitive and which may be considered as analogous to the equality of powers. In this paper we examine four notions of this type: weak isomorphism, isomorphism ("isomorphie algébro-logique" in terms of Kuratowski-Posament [2])), total isomorphism, and equivalence ("double similarity" in terms of Whitehead-Russell [3])).

Let us denote by **K** and **L** two classes of sets. If **K** and **L**, considered as sets partially ordered by the relation of proper inclusion, are similar, then they are called *weakly isomorphic*. Further, we say that **K** and **L** are *isomorphic* when they have the same properties from the point of view of all finite operations upon sets (addition of two sets and complementation). Analogically, **K** and **L**

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[1]) Presented to the Polish Mathematical Society, Warsaw section, on May 6, 1938. Cf. the preliminary report Szpilrajn [6].

[2]) Kuratowski-Posament [1], p. 282.

[3]) Whitehead-Russell [1], p. 84. Cf. also Sierpiński [1], p. 80, Stone [1], p. 91, Szpilrajn [1], p. 306, [2] and [3].