Theorem 6. The automorphisms of $C$ which carry a given set of first category (with respect to $C$) into a set of measure zero (in the probability measure) form a residual set in the space of automorphisms of $C$.

The proof runs parallel to that of Theorem 1 until the proof that $E_0$ is dense. Here we divide $C$ into disjoint segments $\sigma_1, \ldots, \sigma_j$ of length less than $\delta$. In each $\sigma_j$ select a prefect subset lying outside $kF_j$. Since any two perfect subsets of $C$ are homeomorphic, $g$ can be chosen so as to leave each $\sigma_j$ invariant and yet map $s_j$ into a subset of $\sigma_j$ having an arbitrarily large fraction of the measure of $\sigma_n$. Thus $gkF_j$ can be made to have measure less than $1/k$. The proof then proceeds similarly as before.

Isotopic deformation of $C$ is of course impossible, so that no analog of Theorem 5 is to be sought.

The possible extension of the results here obtained to locally compact groups with Haar measure is a question which the authors hope to discuss subsequently.

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The characteristic function of a sequence of sets and some of its applications 1).

By Edward Szpilrajn (Warszawa).

The characteristic function of a sequence of sets, which is the subject of this paper, is a simple generalization of the well known notion of characteristic function of a single set due to de la Vallée-Poussin.

Since the time when the notion of characteristic function of a sequence of sets has been introduced 2) it was applied in several publications by Knaster, Kuratowski, Sierpiński and the author of this paper 3). In general, it may be stated that, for each sequence of sets which approximates a space in one sense or another, the application of its characteristic function leads to certain results. This happens e.g. in the case of 10 a basis of a separable space [3.8 (i)], 20 a basis consisting of closed-and-open subsets of a 0-dimensional separable space [3.8 (ii)], 30 the sequence of all closed-and-open subsets of a compact metrical space 4), 40 a sequence approximating all the measurable sets from the point of view of the measure 5).

1) Presented in part to the 3rd Polish Mathematical Congress in Warsaw, on September 30, 1937 (§§ 2 and 3) and to the Polish Mathematical Society, Warsaw section, on May 20, 1938 (§ 4).

2) Kuratowski [1], p. 124 defines in a certain proof a function which is the characteristic function of a basis of a 0-dimensional space. In a general way, the characteristic function of a sequence of sets has been defined in the paper: Szpilrajn [1], which contains several theorems on this notion and with its help solves some problems of Ulam.

3) Kuratowski [2], p. 36 and [3], pp. 24 and 245, Sierpiński [2], [3] and [4], Szpilrajn [1] and the preliminary reports [4a, b, c and d].

4) Kuratowski [3], p. 244.

5) Szpilrajn [40].
A special field for applying the characteristic function are logical relations between certain questions of point sets and questions belonging to the General Theory of Sets. And so, with the help of the characteristic function, a series of known problems on sets of points, which have not as yet been solved, may be formulated in the language of the General Theory of Sets. Several such equivalences have been given by Sierpiński; in § 4 I show these equivalences in a somewhat simpler (but not essentially different) way (4.1, 4.4 and 4.5) and I give some analogous new theorems (4.2, 4.3, and 4.6).

It is worth noticing that equivalences of this type may be proved also without the aid of the characteristic function; namely, considering a sequence $\varepsilon$ of subsets of an abstract space, it is possible to form a topological space by treating as basis of the space all sets belonging to $\varepsilon$ (this remark is due to S. Eilenberg). One obtains so the topologization of problems concerning the sequence $\varepsilon$. Nevertheless, it seems that the application of the characteristic function permits to obtain this topologization in a particularly simple way.

Many important properties of a sequence of sets may be formulated shortly with the help of its characteristic function. A part of this paper is just devoted to the study of the correspondence between different properties of a single sequence and those of its characteristic function (§ 3). Certain relations between two sequences of sets and corresponding relations between their characteristic functions will be treated in another paper 1).

§ 1. Preliminaries.

1.1. Space. By the term space we understand in this paper a wholly arbitrary abstract set. The letters $X$, $Y$ and $Z$, when no additional explications are given, will always denote arbitrary spaces.

A sequence $b$ of open subsets of a metrical space $X$, such that each open subset of $X$ is the sum of a certain subsequence of $b$, is called a basis of $X$.

1.2. Classes of sets and their atoms. $K$ being a class of subsets of a space $X$, we denote by:

$K_\alpha, K_\beta, K_\gamma, K_\delta, K_\lambda$ the class of all complements, finite sums, enumerable sums, finite products, enumerable products of sets belonging to $K$;

1) See the preliminary reports: Szpirajn [4b] and [4a], cf. also Szpirajn [1].
2.2. The characteristic function of a sequence of sets.

Given a sequence $e = \{E_n\}$ of subsets of $X$, we define the characteristic function of $e$ by the identity: $c_e(x) = (\sigma_{E_n}(x))$. Thus, the values of this function are sequences consisting of the numbers 0 and 1. Since all such sequences may be considered as points of Cantor's discontinuum (1.4), the characteristic function of a sequence $e = \{E_n\}$ may be written in the form:

$$c_e(x) = 2 \cdot (0, i_1, i_2, \ldots)_3,$$

where $i_n = \begin{cases} 0 & \text{for } x \in X - E_n \\ 1 & \text{for } x \in E_n \end{cases}$

or, in other terms (Kuratowski),

$$c_e(x) = 2 \cdot \sum_{n=1}^{\infty} [c_{E_n}(x)/3^n].$$

2.3. Atoms and constituents. Denote by $e = \{E_n\}$ an arbitrary sequence of subsets of $X$ and put for any sequence $(i_n)$ consisting only of the numbers 0 and 1:

$$E_{i_1 i_2 \ldots i_n} = Z_{i_1} Z_{i_2} \ldots Z_{i_n},$$

where $Z_n = X - E_n$ if $i_n = 0$ and $Z_n = E_n$ if $i_n = 1$. The sets $E_{i_1 i_2 \ldots i_n}$ are called atoms of the sequence $e$ (cf. 1.2) and the sets $E_{i_1 i_2 \ldots i_n}$ constituents of the sequence $e$.

Obviously we have:

(i) $c_e^{-1}(0) = \bigcap_{n=1}^{\infty} (X - E_n) = X - \bigcup_{n=1}^{\infty} E_n$

and more generally:

(ii) $c_e^{-1}[2 \cdot (0, i_1, i_2, \ldots)_3] = E_{i_1 i_2 \ldots i_n}$

Further:

(iii) $c_e^{-1}(C_n) = X - E_n$

and more generally:

(iv) $c_e^{-1}(C_n) = E_{i_1 i_2 \ldots i_n}$

It follows directly from (ii) that

(v) The power of the class of all non-void atoms of $e$ is equal to the power of the set $c_e(X)$.

1) Kuratowski [1], p. 79.

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2.4. The correspondence between functions and sequences of sets. Associating with each sequence \( e \) of subsets of \( X \) the function \( e \) we obtain a biunivocal correspondence between the class of all sequences of subsets of \( X \) and that of all functions defined on \( X \) and assuming values which belong to \( C^1 \).

Since the characteristic functions of two different sequences of sets obviously are different, it remains to prove that every function \( f \) defined on \( X \) and such that \( f(X) \subseteq C^1 \) is the characteristic function of a certain sequence of subsets of \( X \). In fact, putting \( E_n = f^{-1}(G_n) \) and \( e = \{E_n\} \) and comparing this with 2.3 (iii) and 1.4 (iii), we obtain \( f = e \).

2.5. Transformation of classes of sets by the characteristic function. Let \( e = \{E_n\} \) denote a sequence of subsets of the space \( X \) and \( E \) — the class of all the sets \( E_n \). Furthermore let us write \( T = c_e(X) \), \( T = \mathcal{C}^1 \) and finally let us denote by \( T \) the class of all sets \( T_n \). It follows from 2.3 (iii) that \( E_n = c_{T_n}^{-1}(T_n) \) and consequently the operation \( c_{T_n}^{-1}(U) \) establishes a biunivocal correspondence between the classes \( T_n \) and \( E_n \). Moreover we conclude from 1.4 (i) and (iv) that the class \( T_n \) is equal to that of the sets \( HT \) where \( H \) is both open and closed in \( C \). Hence we obtain:

(i) The operation \( c_{T_n}^{-1}(U) \) (and also the operation \( c_{T_n}(E) \)) establishes a biunivocal correspondence between the class \( T_n \) and the class of all sets \( H \cdot c_e(X) \), where \( H \) is both open and closed in \( C \).

Since the class \( T_n \) is a basis of the space \( c_e(X) \), we derive from (i) and from the properties of the operation of counterimage (cf. 1.3) the following theorem which is important for applications:

(ii) The operation \( c_{T_n}^{-1}(U) \) (and also the operation \( c_{T_n}(E) \)) establishes a biunivocal correspondence between the classes

\[
\{E_0, E_1, E_2, E_3, \ldots \} \quad \text{and} \quad \{G_0, G_1, G_2, \ldots \}
\]

and the classes of all sets: open, closed, \( G \), \( F \), \( F \) borelian and analytical in \( c_e(X) \).

\[\]  

1) Szpilrajn [1], p. 306.
2) Different particular cases of this theorem have been formulated in the following papers: Sierpiński [2], pp. 184-186, [3], p. 3. Szpilrajn [1], p. 307 and in the unpublished address O problemach rodzinach zbiorów (On enumerable families of sets) given by Sierpiński to the 3rd Polish Mathematical Congress in Warsaw, on September 30, 1937.

\[\]  

2.6. Relativization. Subsequences. Biunivocal transformations. We see at once that

(i) If \( Y \subseteq X \), \( A_n \subseteq X \), \( B_n \subseteq X \), \( A_n \cdot B_n = A_n \cdot B_n \) (for \( n = 1, 2, \ldots \)), \( a = \{A_n\} \), \( b = \{B_n\} \), then \( c_a(x) = c_b(x) \) for \( x \cdot y \).

\( x = (k_n) \) denoting a sequence of positive integers, let us put for each number \( t = 2 : (0, t_1, t_2, \ldots) \cdot s \cdot C \) (where \( t_1 = 0 \) or 1)

\( \varphi_n(t) = 2 : (0, s_1, s_2, \ldots) \cdot s \cdot C \). The function \( \varphi_n \) is always continuous. If the sequence \( \{k_n\} \) contains all positive integers and each of them only once, then the function \( \varphi_n \) is a topological automorphism of \( C \). It is easy to see that

(ii) If \( a = \{A_n\} \), \( b = \{B_n\} \), \( x = \{k_n\} \), then \( c_a = c_b \cdot a \).

Next, we shall prove the following theorem

(iii) Let \( a = \{A_n\} \), \( b = \{B_n\} \) and \( \varphi \) a biunivocal transformation such that \( \varphi(X) = Y \). Then the relations: \( \varphi(A_n) = B_n \) (\( n = 1, 2, \ldots \)) and \( a = c_b \cdot \varphi \) are equivalent.

Put

\( g = c_b \cdot \varphi \).

The identity

\( \varphi(A_n) = B_n \)

is equivalent successively to the following identities:

\( \varphi^{-1}(C_n) = c_a^{-1}(C_n) \) for \( n = 1, 2, \ldots \) [according to 2.3 (iii)]

\( \varphi^{-1}(C_n) = c_a^{-1}(C_n) \) for \( n = 1, 2, \ldots \) [according to 2.3 (iii)]

\( g = c_a \) [according to 1.4 (iii)]

\( \varphi = c_a \) [according to (*)].

\[\]  

1) Cf. Szpilrajn [1], p. 308.
2) Szpilrajn [1], p. 309.
3. Different properties of sequences of sets and the corresponding properties of their characteristic functions.

3.1. Identity. All sets belonging to a sequence $e=\{E_n\}$ are identical if and only if the function $c_e$ assumes only the values 0 and 1. Then we have $c_e=c_{E_1}$.

If for each $x \in X$ either $c_e(x)=0$ or $c_e(x)=1$, then we have either $x \in E_n$ for $n=1, 2, \ldots$ or $x \notin E_n$ for $n=1, 2, \ldots$ and consequently $E_1=\ldots=\ldots$.

Conversely, if $E_1=E_2=\ldots$, then $c_e(x)=0, 00, \ldots$ for each $x \in X - E_1$ and $c_e(x)=2, 0, 01, \ldots=1$ for each $x \in E_1$. Hence, in our case, $c_e=c_{E_1}$.

3.2. Sets without common points. No two sets belonging to a sequence of sets have common points if and only if its characteristic function assumes only the value 0 and values of the form $2/3^k$.

In fact, $e=\{E_n\}$ denoting a sequence of sets, the point $x$ belongs only to $E_k$ if and only if $c_e(x)=2/3^k$.

3.3. Monotony. A sequence of sets is ascending (descending) if and only if its characteristic function assumes only values of the form $1/3^k$ [or $1-(1/3^k)$ respectively].

In fact, we have $E_1 \subseteq E_2 \subseteq \ldots$ if and only if all the values of $c_e$ are of the form $2, 0, 00, 011, \ldots=1/3^k$.

3.4. Convergence. It is easily seen that the sequence $e=\{E_n\}$ is convergent (in the sense of the General Theory of Sets) if and only if for each $x$ either $x \in E_n$ for $n>N$, or $x \notin E_n$ for $n>N$. In other words: either $c_e(x)=2, 0, 01, \ldots=1/3^k$ or $c_e(x)=2, 0, 01, \ldots=1/3^k$. Hence we obtain:

A sequence of sets is convergent if and only if its characteristic function assumes only values of the form $k/3^k$ (i.e., values possessing finite ternary development).

3.5. Independence. The sequence $e=\{E_n\}$ is called a sequence of independent sets if no constituent of $e$ (cf. 2.3) is empty $\dagger$). Similarly it is called a sequence of $\kappa$-independent sets if no atom of $e$ (cf. 2.3) is empty. By Theorems 2.3 (iv), 1.4 (v) and 2.3 (ii) we obtain:

A sequence $e$ of subsets of the space $X$ is a sequence of (i) independent, (ii) $\kappa$-independent sets if and only if (i') $c_e(X)=C$, (ii') $c_e(X)=C$.

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\dagger)\text{Fichtenholz-Kantorovitch} [1], p. 78 and Hausdorff [1], p. 18.

3.6. Distinguishable points. Two points are said to be distinguishable by means of a sequence $e=\{E_n\}$ (Sierpiński) if there exists a positive integer $n$ such that the set $E_n$ contains one and only one of these points. Each two points of the space are distinguishable by means of $e$ if and only if each atom of $e$ contains at most a single point. Consequently, in view of 2.3 (ii) we have:

Every two points of a space $X$ are distinguishable by means of a sequence $e$ of subsets of $X$ if and only if the characteristic function of $e$ assumes each of its values only once.

3.7. Measurability $(K)$. Let $K$ denote a class of subsets of a space $X$. A function $f(x)$, which transforms $X$ into a subset of a topological space $Y$, is called measurable $(K)$ whenever $f^{-1}(G) \in K$ for any open subset $G$ of $Y$.

Now, let us denote by $e=\{E_n\}$ a sequence of sets contained in $X$. From the identities 2.3 (iii) it follows at once that:

(i) If the characteristic function of a sequence $e=\{E_n\}$ of sets contained in $X$ is measurable $(K)$, then $E_n \in K$ and $X - E_n \notin K$ for $n=1, 2, \ldots$.

On the other hand it follows from 2.5 (ii) that:

(ii) $E$ denoting the class of all sets belonging to a sequence $e$, the function $c_e$ is measurable $(E_e)$.

Combining (i) with (ii) we obtain:

(iii) If $K=K_1=K_2$ then, in order that the function $c_e$ be measurable $(K)$ it is necessary and sufficient that all the sets $E_n$ and $X - E_n$ belong to $K$ $^1$.

The necessity of this condition is stated in (i). The sufficiency follows from (ii) and from the fact that the relations: $E_1 \subseteq E_2 \subseteq \ldots \subseteq K_1 \subseteq K_2 \subseteq \ldots$ imply the relation: $E_1 \subseteq E_2 \subseteq \ldots \subseteq K_1 \subseteq K_2 \subseteq \ldots$.

It follows from (iii) that (in the case when the space $X$ is topological or particularly if it is cartesian):

(iv) The function $c_e$ is $1^o$ continuous, $2^o$ of the first class, $3^o$ measurable $(B)$, $4^o$ measurable $(I)$ if and only if all the sets belonging to $e$ are $1^o$ both open and closed, $2^o$ both $F_\delta$ and $G_\delta$, $3^o$ borelian, $4^o$ measurable $(I)$.

\dagger) Szpilrajn [1], p. 308.
3.8. Basis of a metrical space. With the help of the characteristic function of a basis of a metrical space we can easily obtain the following known theorems: (a) each metrical separable space is a continuous biunivocal image of a subset of Cantor's discontinuum, (b) each separable and 0-dimensional space can be topologically imbedded in $C$. In fact, these theorems follow directly from the following Theorems (i) and (ii).

(i) Let $v = \{V_{\alpha}\}$ denote a basis of a separable space $X$. Then the function $c_v$ is a generalized homeomorphism of the class $(1,0)$.

Put $c_v(X) = T$. The sequence $v$ being a basis of $X$, each two points of $X$ are distinguishable by means of $v$ and consequently $c_v$ is a biunivocal transformation of $X$ into $T$ (cf. 3.6).

Since the sets $\{V_{\alpha}\}$ are open, the function $c_v$ is of the first class by 3.7 (iv, 28). On the other hand the function $c_v^{-1}$ is continuous, because the sets $c_v(V_{\alpha})$ are open in $T$ by 3.5 (ii) and accordingly, for any set $G$ open in $X$, the set $c_v(G)$ is also open in $T$.

(ii) Let $v = \{V_{\alpha}\}$ denote a basis of sets both open and closed in a 0-dimensional, separable space. Then the function $c_v$ is a homeomorphism.

On account of (i), it suffices to prove that the function $c_v$ is continuous. This follows directly from 3.7 (iv, 19).

3.9. A certain transformation of the interval. Now, by applying the preceding theorems, we shall obtain the following remark:

(i) There exists a biunivocal transformation $\varphi$ of the interval $I = (0,1)$ into a set $TCC$, which transforms the class of the Borel subsets of $I$ into a proper subclass of the class of sets borel in $T$.

(In other words: the class of all Borel subsets of $I$ is equivalent to a proper subclass of the class of all sets borel in a certain subset of $I$).

§ 4. Logical equivalence of certain problems.

4.1. Set of Lusin. A linear set is called Lusin set if each of its nowhere dense subsets is at most enumerable. The existence of a non enumerable Lusin set is proved only with the help of the hypothesis of the continuum. Next, it is known that the existence of a Lusin set of the power $\aleph_0$ is equivalent to that of a set $E$ of the power $\aleph_0$ possessing the following weaker property $(\tau)$: each subset of $E$ nowhere dense in $E$ is at most enumerable. Furthermore, it is easy to show that the property $(\tau)$ of a set $E$ is equivalent to the following property $(\tau_0)$: each set closed in $E$ is the sum of a set open in $E$ and of a set at most enumerable.

(i) Let $e = \{E_\alpha\}$ be a sequence of subsets of $X$, $E$ — the class of all sets $E_\alpha$. If each set belonging to the class $E_\alpha$ is the sum of a set belonging to $E_{\text{ord}}$ and of a set at most enumerable, then the set $c_e(X)$ possesses the property $(\tau)$.

We shall prove that $c_e(X)$ have the property $(\tau_0)$. Let then $F$ be a set closed in $c_e(X)$. Theorem 2.5 (ii) implies: $c_e^{-1}(F) \in E_{\text{ord}}$ and consequently $c_e^{-1}(F) = H + D$ where $H \subseteq E_{\text{ord}}$ and $D$ is at most enumerable. Accordingly, $F = c_e(H) + c_e(D)$ where $c_e(H)$ is open in $c_e(X)$ [also by 2.5 (ii)] and $c_e(D)$ is at most enumerable. Thus the set $c_e(X)$ has the property $(\tau_0)$, q. e. d.

(ii) For the existence of a separable Lusin set of the power $\aleph_0 > \aleph$, it is necessary and sufficient that there exist an enumerable ring $K$ of sets $1^a$ possessing $\aleph$ non-void atoms and $2^a$ such that each set belonging to $K_0$ is the sum of a set belonging to $K$ and of a set at most enumerable.

1) Cf. e.g. Kuratowski [1], p. 226.
2) Cf. e.g. Kuratowski [1], p. 124. Our proof is briefer (as using general theorems on the characteristic function) but not essentially different from that of Kuratowski.
3) For the notion of homeomorphism of class $(e,d)$ see Kuratowski [1], p. 221.
4) For the equivalence of classes of sets see e.g. Stone [1], p. 91, Szpilrajn [1] and [2].
The characteristic function

(i) Let \( e=\{E_n\} \) be a sequence of subsets of \( X \), and \( E \) the class of all sets \( E_n \). Suppose that there exists an enumerable set \( DCX \) such that for any set \( H \in E_{oc} \) containing \( D \), the set \( X-H \) is at most enumerable. Then the set \( e_1(X) \) is concentrated [in the neighbourhood of the set \( e_1(D) \)].

Suppose that a set \( G \) is open in \( e_1(X) \) and contains \( e_1(D) \). Thus \( e_1^{-1}(G) \in E_{oc} \) [by 2.5 (ii)] and \( e_1^{-1}(G) \supset D \). Accordingly, the set \( Z=X-e_1^{-1}(G) \) is at most enumerable by the hypothesis, and consequently so is the set \( e_1(Z) \).

The relation \( e_1(Z) \supset e_1(X) \supset G \) implies the enumerability of the set \( e_1(X) \supset G \).

(ii) For the existence of a linear concentrated set of the power \( \aleph_1 \), it is necessary and sufficient that there exist: an enumerable ring of sets \( R \) possessing \( n \) non-void atoms and an enumerable set \( D \) contained in the sum \( X \) of the sets belonging to \( R \) such that if \( H \supset D \) and \( H \in E_n \) then the set \( X-H \) is at most enumerable.

Necessity. Let \( X \) be a linear set of the power \( \aleph_1 \) concentrated in the neighbourhood of an enumerable set \( D \) and \( B \) a basis of \( X \) consisting of sets both open and closed in \( X \). The ring \( R=B \) satisfies the conditions mentioned above.

Sufficiency. It is a consequence of (i) and 2.3 (v).

4.3. Property (λ). We say that a metrical space \( X \) has the property (λ) if each of its enumerable subsets is a \( G_2 \)-set in \( X \) (Kuratowski). Each space having the property (λ) is of the first category on every perfect set; but it follows from the hypothesis of the continuum that the converse of this theorem would be false (Lusin). It is known that there exists a space of the power \( \aleph_1 \) having the property (λ), but the problem of the existence of such a set of the power of the continuum is open \(^1\).

(i) Let \( e=\{E_n\} \) be a sequence of subsets of a space \( X \) and \( E \) the class of all sets \( E_n \). If each enumerable subset of \( X \) belongs to \( E_{oc} \), then the set \( e_1(X) \) has the property (λ).

Since each one-element subset of \( X \) belongs to \( E_{oc} \), the function \( e_1(X) \) assumes each of its values only once [by 1.2 (iii) and 3.6].

\(^1\) For the property (λ) see Kuratowski [1], pp. 269-271 and Sierpinski [1], pp. 94-98.

4.2. Besikovitch’s concentrated sets. A metrical space \( X \) is called concentrated in the neighbourhood of a set \( T \subset X \) if for every open set \( G \supseteq T \) the set \( X-G \) is at most enumerable (Besikovitch) \(^2\). A set concentrated in the neighbourhood of an enumerable set is briefly called concentrated. The existence of a non enumerable concentrated set is proved only with the help of the hypothesis of the continuum. Every separable set having the property (ν) (cf. 4.1) is concentrated (from the hypothesis of the continuum it follows that, on the other hand, the converse of this proposition is untrue). It is easy to show that a separable set \( E \) possesses the property (ν) if and only if it is concentrated in the neighbourhood of each enumerable set, dense in \( E \).

\(^2\) Besikovitch [1], p. 289.
Let $D$ denote an enumerable subset of $c_\alpha(X)$. Thus the set $c_\alpha^{-1}(D)$ is enumerable and consequently $c_\alpha^{-1}(D) \in E_{n\alpha}$ by the hypothesis. Therefore, by 2.3 (ii), $D$ is a $G_\delta$-set in $c_\alpha(X)$.

(ii) For the existence of a linear set which has the property $(\lambda)$ and the power $\kappa = \kappa_\alpha$, it is necessary and sufficient that there exists an enumerable class $\mathcal{R}$ of sets having $\kappa$ non-empty atoms and such that each enumerable set contained in the sum of the class $\mathcal{R}$ belongs to $R_{\omega\alpha}$.

Necessity. Any basis of a set having the property $(\lambda)$ and the power $\kappa$ is the required class $\mathcal{R}$.

Sufficiency. It suffices to put $E = R_\alpha$ and to apply (i) and 2.3 (v).

4.4. Problems of Hausdorff and Sierpiński. It follows directly from the hypothesis of the continuum that the following propositions are false:

(H) $E$ being a set of the power $\kappa_\alpha$, there exists an enumerable class $\mathcal{R}$ of sets contained in $E$ such that each subset of $E$ belongs to $R_\alpha$.

(S) There exists a linear set $U$ of the power $\kappa_\alpha$, each subset of which is a $G_\delta$-set in $U$.

The problem of proving the falseness of the proposition (H) is 1) without additional hypotheses is not solved.

(i) Let $\mathcal{E} = (E_\alpha)$ be a sequence of subsets of $X$ and $E$ the class of all sets $E_\alpha$. If each subset of $X$ belongs to $E_{n\alpha}$, then every subset of $c_\alpha(X)$ is a $G_\delta$-set in $c_\alpha(X)$.

Let be $Tc_\alpha(X)$. By the hypothesis, $c_\alpha^{-1}(T) \in E_{n\alpha}$ and, by 2.5 (ii), $T$ is a $G_\delta$-set in $c_\alpha(X)$.

(ii) The propositions (H) and (S) are equivalent 2).

(S)$\rightarrow$(H). Let $\varphi$ denote a biunivocal transformation of the set $U$ into an arbitrary set $E$ of the power $\kappa_\alpha$. Any basis of $U$ is transformed by $\varphi$ into a class $\mathcal{R}$ of subsets of $E$ which obviously satisfies the conditions of the proposition (H).

Let $E$ denote an arbitrary set of the power $\kappa_\alpha$ and $\tau = (R_\alpha)$ the sequence of all sets belonging to the class $\mathcal{R}$. Since each one-element subset of $E$ belongs to $R_{\omega\alpha}$, it follows from 1.2 (iii) and 3.6 that the function $c_\alpha$ assumes each of its values only once. Therefore this function transforms the set $X$ into a linear set $U$ of the power $\kappa_\alpha$. According to (i), the set $U$ satisfies the condition contained in the proposition (S).

4.5. Problems of Mazurkiewicz and Kolmogoroff. For each class of sets put $B_\xi(R) = R_\xi$, next $R_\xi(R) = [B_{\xi^{-1}}(R)]_\xi$ for each odd $\xi < \Omega$ and $B_\xi(R) = [\sum B_\eta(R)]_\xi$ for each even $\xi < \Omega$.

If $R$ is a basis of a topological space, then $B_\xi(R)$, $B_\eta(R)$, $B_\eta(R)$, equal the classes of sets: open, $G_\delta$, $G_\delta$,... respectively.

Consider the following condition:

(*) $B_\alpha(K) + B_{\alpha + 1}(K) = B_{\alpha + 2}(K)$.\n
The problem of existence for each $\alpha < \Omega$ of a metrical space $X$, the basis of which would satisfy this condition, has been raised by Mazurkiewicz 3). An analogous problem for arbitrary classes $K$ of sets has been set up by Kolmogoroff 4). We shall show the following relation between these problems (theorem of Sierpiński 5):

(i) For the existence (for an ordinal number $0 < \alpha < \Omega$) of a linear set $\mathcal{N}$ possessing a basis $V$ which satisfies the condition (*) (for $K = V$) it is necessary and sufficient that there exists an enumerable ring $\mathcal{R}$ satisfying this condition (for $K = \mathcal{R}$).

Necessity. Let $N$ be a linear set, a basis $V$ of which satisfies the condition (*). The set $N$ contains no interval and consequently there exists a basis $V'$ being a ring which consists only of sets both open and closed in $N$. Since $B_\alpha(V) = B_\alpha(V')$ for $0 < \alpha < \Omega$, the condition (*) for $K = V$ is equivalent to that for $K = V'$.

Sufficiency. Take the sequence $k = [K_\alpha]$ of all sets belonging to $K$ and put $N = c_\alpha(K_1 + K_2 + ...).$ It follows from 2.5 (ii) that $N$ is the required linear set.

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1) It is Sierpiński’s modification of a problem raised by Hausdorff in Fund. Math. 20 (1933), p. 286 (Problème 55).
2) Problem of Sierpiński. See Sierpiński [1], p. 90.
3) The result shown in the paper: Sierpiński [3].
4.6. Problem of Ulam. Recently Ulam has raised the question whether the following proposition is true:

(U) There exists an enumerable class $D$ of sets such that $D_0$ contains the class of all the analytical subsets of the interval $I = (0, 1)$. 1)

We shall prove that

(i) The proposition (U) is equivalent to the following:

(U') There exists a bimivocal transformation $\varphi$ of the interval $I$ into a linear set transforming each set analytical in $I$ into a Borel set in $\varphi(I)$.

(The proposition (U') may be formulated also in the following manner: The class of analytical subsets of $I$ is equivalent 2) to a class of sets borelian in a certain linear set.)

(U')→(U). Denote by $d = (D_n)$ the sequence of all the sets belonging to $D$ and consider the function $c_d$. By the hypothesis, each one-element subset of $I$ belong to $D_n$ and hence the function $c_d$ assumes each of its values only once [according to 1.2 (iii) and 3.6]. Next, on account of Theorem 2.5 (ii), the function $c_d$ transforms each set belonging to $D_0$, and in particular each set analytical in $I$, into a set borelian in $\varphi(I)$.

(U')→(U'). Let $\varphi$ satisfy the condition of the proposition (U'), let $V = (V_n)$ be a basis of the set $\varphi(I)$ and let us put $W = (\varphi^{-1}(V_n))$. Since by the hypothesis the class $V_0$ contains as elements all the sets $\varphi(A)$, where $A$ is an analytical set in $I$, the class $W_0$ contains as elements all the sets analytical in $I$, q. e. d.

References.


1) Fund. Math. 30 (1938), p. 365, problème 74. The proposition (U) is a modification of the original Ulam's proposition, which is concerned with the smallest class of sets containing $D$ and closed with respect to enumerable addition and enumerable multiplication.

2) Cf. p. 2164.)