Theorem 6. *The automorphisms of C which carry a given set of first category (with respect to C) into a set of measure zero (in the probability measure) form a residual set in the space of automorphisms of C.*

The proof runs parallel to that of Theorem 1 until the proof that E_{ik} is dense. Here we divide C into disjoint segments $\sigma_1, \ldots, \sigma_l$ of length less than δ. In each σ_i select a prefect subset lying outside hF_i. Since any two perfect subsets of C are homeomorphic, g can be chosen so as to leave each σ_j invariant and yet map s_j into a subset of σ_j having an arbitrarily large fraction of the measure of σ_i. Thus ghF_i can be made to have measure less than $1/k$. The proof then proceeds similarly as before.

Isotopic deformation of C is of course impossible, so that no analog of Theorem 5 is to be sought.

The possible extension of the results here obtained to locally compact groups with Haar measure is a question which the authors hope to discuss subsequently.

Society of Fellows, Harvard University.

The characteristic function of a sequence of sets and some of its applications [1].

By

Edward Szpilrajn (Warszawa).

The characteristic function of a sequence of sets, which is the subject of this paper, is a simple generalization of the well known notion of characteristic function of a single set due to de la Vallée-Poussin.

Since the time when the notion of characteristic function of a sequence of sets has been introduced [2]) it was applied in several publications by Knaster, Kuratowski, Sierpiński and the author of this paper [3]). In general, it may be stated that, for each sequence of sets which approximates a space in one sense or another, the application of its characteristic function leads to certain results. This happens e.g. in the case of 1^0 a basis of a separable space [3.8 (i)], 2^0 a basis consisting of closed-and-open subsets of a 0-dimensional separable space [3.8 (ii)], 3^0 the sequence of all closed-and-open subsets of a compact metrical space [4]), 4^0 a sequence approximating all the measurable sets from the point of view of the measure [5]).

---

[1]) Presented in part to the 3rd Polish Mathematical Congress in Warsaw, on September 30, 1937 (§§ 2 and 3) and to the Polish Mathematical Society, Warsaw section, on May 20, 1938 (§ 4).

[2]) Kuratowski [1], p. 124 defines in a certain proof a function which is the characteristic function of a basis of a 0-dimensional space. In a general way, the characteristic function of a sequence of sets has been defined in the paper: Szpilrajn [1], which contains several theorems on this notion and with its help solves some problems of Ulam.

[3]) Kuratowski [2], p. 56 and [3], pp. 244 and 245, Sierpiński [2], [3] and [4], Szpilrajn [1] and the preliminary reports [4a, b, c and d].

[4]) Kuratowski [3], p. 244.

[5]) Szpilrajn [4c].

A special field for applying the characteristic function are logical relations between certain questions of point sets and questions belonging to the General Theory of Sets. And so, with the help of the characteristic function, a series of known problems on sets of points, which have not as yet been solved, may be formulated in the language of the General Theory of Sets. Several such equivalences have been given by Sierpiński; in § 4 I show these equivalences in a somewhat simpler (but not essentially different) way (4.1, 4.4 and 4.5) and I give some analogous new theorems (4.2, 4.3, and 4.6).

It is worth noticing that equivalences of this type may be proved also without the aid of the characteristic function; namely, considering a sequence e of subsets of an abstract space, it is possible to form a topological space by treating as basis of the space all sets belonging to e (this remark is due to S. Eilenberg). One obtains so the topologization of problems concerning the sequence e. Nevertheless, it seems that the application of the characteristic function permits to obtain this topologization in a particularly simple way.

Many important properties of a sequence of sets may be formulated shortly with the help of its characteristic function. A part of this paper is just devoted to the study of the correspondence between different properties of a single sequence and those of its characteristic function (§ 3). Certain relations between two sequences of sets and corresponding relations between their characteristic functions will be treated in another paper [1]).

§ 1. Preliminaries.

1.1. Space. By the term *space* we understand in this paper a wholly arbitrary abstract set. The letters X, Y and Z, when no additional explications are given, will always denote arbitrary spaces.

A sequence b of open subsets of a metrical space X, such that each open subset of X is the sum of a certain subsequence of b, is called a *basis* of X.

1.2. Classes of sets and their atoms. $\boldsymbol{K}$ being a class of subsets of a space X, we denote by:

$\boldsymbol{K}_c$, $\boldsymbol{K}_s$, $\boldsymbol{K}_\sigma$, $\boldsymbol{K}_d$, $\boldsymbol{K}_\delta$ the class of all complements, finite sums, enumerable sums, finite products, enumerable products of sets belonging to $\boldsymbol{K}$;

[1]) See the preliminary reports: Szpilrajn [4b] and [4a], cf. also Szpilrajn [1].

$\boldsymbol{K}_o$, $\boldsymbol{K}_b$ the smallest class $\boldsymbol{Q}$ containing $\boldsymbol{K}$ and such that $\boldsymbol{Q}=\boldsymbol{Q}_c=\boldsymbol{Q}_s$, or $\boldsymbol{Q}=\boldsymbol{Q}_c=\boldsymbol{Q}_\sigma$ respectively;

$\boldsymbol{K}_a$ the class of all sets obtained by the operation (A) upon sets belonging to $\boldsymbol{K}$;

$\boldsymbol{K}_t$ the smallest totally additive and complementative class of sets containing the class $\boldsymbol{K}$.

The class $\boldsymbol{K}$ is called a *ring* if $\boldsymbol{K}=\boldsymbol{K}_o$ [1]).

$\{E_\xi\}$ denoting the (finite or transfinite) sequence consisting of all the sets which belong to $\boldsymbol{K}$ (where the indexes ξ build a set Ξ), each set of the form:

$$\prod_{\xi\in\Xi} Z_\xi \quad \text{where} \quad Z_\xi=E_\xi \quad \text{or} \quad Z_\xi=X-E_\xi \quad \text{for each} \quad \xi\in\Xi$$

is called *atom* of the class $\boldsymbol{K}$. It is easy to see that

(i) *Each two different atoms of a class of sets are disjoint and the whole space is the sum of all the atoms.*

(ii) *The class of all the sums of atoms of a class* $\boldsymbol{K}$ *is equal to* $\boldsymbol{K}_t$.

(iii) *If each one-element subset of* X *belongs to* $\boldsymbol{K}_t$, *then each atom of* $\boldsymbol{K}$ *contains at most a single point.*

1.3. Functions. Let $f(x)$ denote a function defined on X; further let be $f(X)=Y$. Consider two functions of a set: the operation of image: $f(E)$ (for $E\subset X$) and that of counterimage $\overset{-1}{f}(T)$ (for $T\subset Y$). It is well known that 1⁰ these functions are both totally additive, 2⁰ the function $\overset{-1}{f}(T)$ is complementative and 3⁰ in the general case, the function $f(E)$ is not complementative. Nevertheless, the function $f(E)$ considered for special sets E also becomes complementative. By applying 1⁰ and 2⁰ we obtain the following proposition:

Let $f(x)$ be a function such that $f(X)=Y$ and $\boldsymbol{K}$ *the class of all sets $\overset{-1}{f}(T)$ (for $T\subset Y$). Then the class* $\boldsymbol{K}$ *is totally additive and complementative and the functions of a set: $f(E)$ considered only for $E\in$* $\boldsymbol{K}$ *and $\overset{-1}{f}(T)$ (for $T\subset Y$) are totally additive and complementative and establish a biunivocal correspondence between* $\boldsymbol{K}$ *and the class of all subsets of Y.*

We may say that the functions $f(E)$ and $\overset{-1}{f}(T)$ establish a total isomorphism [2]) between the class of all sets of the form $\overset{-1}{f}(T)$ and the class of all subsets of Y.

[1]) Certain authors call such a class "ring with unit" and understand by "ring" each additive and substractive class.

[2]) For the notion of total isomorphism see Szpilrajn [4b].

f and g being two functions which transform a space X into a space Y and φ a function which transforms Y into Z, we denote by φf the function $\varphi[f(x)]$ and we write $f=g$ when $f(x)=g(x)$ for each $x \epsilon X$.

1.4. Cantor's discontinuum. We shall denote by C *Cantor's discontinuum*, i. e. the set of all numbers

(*) $t=2\cdot(0, i_1, i_2, i_3, \ldots)_3$ where $i_n=0$ or $i_n=1$ for $n=1,2,\ldots$

It is known that Cantor's discontinuum may be considered as identical (from the topological point of view) with the product $D\times D\times\ldots$ where D consists of 0 and 1. Namely the sequence $i_1, i_2, \ldots \epsilon D\times D\times\ldots$ can be identified with the number (*) [1]).

For each positive integer k and $i=0$ or $i=1$ we shall denote by C_k^i the set of all numbers (*) such that $i_k=i$. Evidently we have:

(i) $C=C_n^0+C_n^1$; $C_n^0\cdot C_n^1=0$ for $n=1,2,\ldots$

(ii) *The sets* $C_n^i (i=0,1; n=1,2,\ldots)$ *are closed and open in* C.

(iii) *Let two functions* g *and* h *assume only values belonging to* C. *If* $g^{-1}(C_n^1)=h^{-1}(C_n^1)$ *for* $n=1,2,\ldots$, *then* $g=h$.

For each finite system $i_1, i_2, \ldots, i_n$ consisting only of the numbers 0 and 1, we put

$$C_{i_1 i_2 \ldots i_n}=C_1^{i_1}\cdot C_2^{i_2}\cdot\ldots\cdot C_n^{i_n}.$$

Each set $C_{i_1 i_2 \ldots i_n}$ will be called *interval* of C. It is known that

(iv) *A subset of* C *is both open and closed in* C *if and only if it is the sum of a finite number of intervals of* C.

(v) *The class of all intervals of* C *is a basis of* C.

§ 2. Definition and fundamental properties of the characteristic function.

2.1. The characteristic function of a single set. For each set $E\subset X$ we denote by $c_E(x)$ *the characteristic function of* E, i. e. the function defined as follows:

$$c_E(x)=\begin{cases}0 & \text{for } x \epsilon X-E\\ 1 & \text{for } x \epsilon E.\end{cases}$$

[1]) Kuratowski [1], p. 79.

2.2. The characteristic function of a sequence of sets. Given a sequence $e=\{E_n\}$ of subsets of X, we define the characteristic function of e by the identity: $c_e(x)=\{c_{E_n}(x)\}$. Thus, the values of this function are sequences consisting of the numbers 0 and 1. Since all such sequences may be considered as points of Cantor's discontinuum (1.4), the characteristic function of a sequence $e=\{E_n\}$ may be written in the form:

$$c_e(x)=2\cdot(0, i_1 i_2 \ldots)_3, \text{ where } i_n=\begin{cases}0 & \text{for } x \epsilon X-E_n\\ 1 & \text{for } x \epsilon E_n\end{cases}$$

or, in other terms (Kuratowski),

$$c_e(x)=2\cdot\sum_{n=1}^{\infty}[c_{E_n}(x)/3^n].$$

2.3. Atoms and constituents. Denote by $e=\{E_n\}$ an arbitrary sequence of subsets of X and put for any sequence $\{i_n\}$ consisting only of the numbers 0 and 1:

$$E_{i_1 i_2 \ldots}=Z_1\cdot Z_2\cdot\ldots \quad \text{and} \quad E_{i_1 i_2 \ldots i_k}=Z_1\cdot Z_2\cdot\ldots\cdot Z_k,$$

where $Z_n=X-E_n$ if $i_n=0$ and $Z_n=E_n$ if $i_n=1$. The sets $E_{i_1 i_2 \ldots}$ are called *atoms* of the sequence e (cf. 1.2) and the sets $E_{i_1 i_2 \ldots i_k}$ *constituents* of the sequence e [1]).

Obviously we have:

(i) $c_e^{-1}(0)=\prod_{n=1}^{\infty}(X-E_n)=X-\sum_{n=1}^{\infty}E_n \qquad c_e^{-1}(1)=\prod_{n=1}^{\infty}E_n$

and more generally:

(ii) $c_e^{-1}[2\cdot(0, i_1 i_2 \ldots)_3]=E_{i_1, i_2, \ldots}$

Further:

(iii) $c_e^{-1}(C_n^0)=X-E_n \qquad c_e^{-1}(C_n^1)=E_n$

and more generally:

(iv) $c_e^{-1}(C_{i_1 i_2 \ldots i_n})=E_{i_1 i_2 \ldots i_n}$.

It follows directly from (ii) that

(v) *The power of the class of all non-void atoms of* e *is equal to the power of the set* $c_e(X)$.

[1]) Cf. Kuratowski-Posament [1], p. 283.

2.4. The correspondence between functions and sequences of sets. *Associating with each sequence e of subsets of X the function c_e we obtain a biunivocal correspondence between the class of all sequences of subsets of X and that of all functions defined on X and assuming values which belong to C* [1]).

Since the characteristic functions of two different sequences of sets obviously are different, it remains to prove that every function f defined on X and such that $f(X)\subset C$ is the characteristic function of a certain sequence of subsets of X. In fact, putting $E_n = f^{-1}(C_n^1)$ and $e=\{E_n\}$ and comparing this with 2.3 (iii) and 1.4 (iii), we obtain $f=c_e$.

2.5. Transformation of classes of sets by the characteristic function. Let $e=\{E_n\}$ denote a sequence of subsets of the space X and $\boldsymbol{E}$ — the class of all the sets E_n. Furthermore let us write $T=c_e(X)$, $T_n=C_n^1\cdot T$ and finally let us denote by $\boldsymbol{T}$ the class of all sets T_n. It follows from 2.3 (iii) that $E_n=c_e^{-1}(T_n)$ and consequently the operation $c_e^{-1}(U)$ establishes a biunivocal correspondence between the classes $\boldsymbol{T}_o$ and $\boldsymbol{E}_o$. Moreover we conclude from 1.4 (i) and (iv) that the class $\boldsymbol{T}_o$ is equal to that of the sets HT where H is both open and closed in C. Hence we obtain:

(i) *The operation $c_e^{-1}(U)$ (and also the operation $c_e(E)$) establishes a biunivocal correspondence between the class $\boldsymbol{E}_o$ and the class of all sets $H\cdot c_e(X)$, where H is both open and closed in C.*

Since the class $\boldsymbol{T}_o$ is a basis of the space $c_e(X)$, we derive from (i) and from the properties of the operation of counterimage (cf. 1.3) the following theorem which is important for applications:

(ii) *The operation $c_e^{-1}(U)$ [and also the operation $c_e(E)$] establishes a biunivocal correspondence between the classes*

$$\boldsymbol{E}_{o\sigma},\ \boldsymbol{E}_{o\delta},\ \boldsymbol{E}_{o\sigma\delta},\ \boldsymbol{E}_{o\delta\sigma},\ \ldots\ \boldsymbol{E}_b,\ \boldsymbol{E}_{oa}$$

and the classes of all sets: open, closed, G_δ, F_σ, ... borelian and analytical in $c_e(X)$ [2]).

[1]) Szpilrajn [1], p. 306.

[2]) Different particular cases of this theorem have been formulated in the following papers: Sierpiński [2], pp. 184-186, [3], p. 3, Szpilrajn [1], p. 307 and in the unpublished address *O przeliczalnych rodzinach zbiorów (On enumerable families of sets)* given by Sierpiński to the 3rd Polish Mathematical Congress in Warsaw, on September 30, 1937.

Remarks. 1. In connection with (i) notice that: (a) if $A\supset B$ then for every set H both open and closed in A the set BH is clearly both open and closed in B, but (b) the converse of this theorem would be false.

2. In many applications we have $\boldsymbol{E}=\boldsymbol{E}_o$; in that case we evidently can omit the index „o“ in Theorems (i) and (ii).

2.6. Relativization. Subsequences. Biunivocal transformations. We see at once that

(i) *If $Y\subset X$, $A_n\subset X$, $B_n\subset X$, $A_nY=B_nY$ (for $n=1,2,\ldots$), $a=\{A_n\}$, $b=\{B_n\}$, then $c_a(x)=c_b(x)$ for $x\,\epsilon\,Y$.*

$\varkappa=\{k_n\}$ denoting a sequence of positive integers, let us put for each number $t=2\cdot(0,i_1i_2\ldots)_3\,\epsilon\,C$ (where $i_n=0$ or 1) $\varphi_\varkappa(t)=2\cdot(0,i_{k_1},i_{k_2}\ldots)_3$. The function $\varphi_\varkappa$ is always continuous. If the sequence $\{k_n\}$ contains all positive integers and each of them only once, then the function $\varphi_\varkappa$ is a topological automorphism of C. It is easy to see that

(ii) *If $a=\{A_n\}$, $b=\{A_{k_n}\}$, $\varkappa=\{k_n\}$, then $c_b=\varphi_\varkappa c_a$* [1]).

Next, we shall prove the following theorem

(iii) *Let be $A_n\subset X$, $B_n\subset Y$, $a=\{A_n\}$, $b=\{B_n\}$ and φ a biunivocal transformation such that $\varphi(X)=Y$. Then the relations: $\varphi(A_n)=B_n$ $(n=1,2,\ldots)$ and $c_a=c_b\varphi$ are equivalent* [2]).

Put

$$(*)\qquad g=c_b\varphi.$$

The identity

$$\varphi(A_n)=B_n \qquad n=1,2,\ldots$$

is equivalent successively to the following identities:

$$\begin{aligned}
\varphi[c_a^{-1}(C_n^1)] &= c_b^{-1}(C_n^1) && \text{for } n=1,2,\ldots && \text{[according to 2.3 (iii)]}\\
\varphi^{-1}[c_b^{-1}(C_n^1)] &= c_a^{-1}(C_n^1) && \text{for } n=1,2,\ldots &&\\
g^{-1}(C_n^1) &= c_a^{-1}(C_n^1) && \text{for } n=1,2,\ldots && \text{[according to (*)]}\\
g &= c_a && && \text{[according to 1.4 (iii)]}\\
c_b\varphi &= c_a && && \text{[according to (*)].}
\end{aligned}$$

[1]) Cf. Szpilrajn [1], p. 308.

[2]) Szpilrajn [1], p. 309.

§ 3. Different properties of sequences of sets and the corresponding properties of their characteristic functions.

3.1. Identity. *All sets belonging to a sequence* $e=\{E_n\}$ *are identical if and only if the function* c_e *assumes only the values* 0 *and* 1. *Then we have* $c_e=c_{E_1}$.

If for each $x\epsilon X$ either $c_e(x)=0$ or $c_e(x)=1$, then we have either $x\epsilon E_n$ for $n=1,2,...$ or $x\epsilon X-E_n$ for $n=1,2,...$ and consequently $E_1=E_2=...$

Conversely, if $E_1=E_2=...$, then $c_e(x)=0,00...$ for each $x\epsilon X-E_1$ and $c_e(x)=2\cdot(0,11...)_3=1$ for each $x\epsilon E_1$. Hence, in our case, $c_e=c_{E_1}$.

3.2. Sets without common points. *No two sets belonging o a sequence of sets have common points if and only if its characteristic function assumes only the value* 0 *and values of the form* $2/3^k$.

In fact, $e=\{E_n\}$ denoting a sequence of sets, the point x belongs only to E_k if and only if $c_e(x)=2/3^k$.

3.3. Monotony. *A sequence of sets is ascending (descending) if and only if its characteristic function assumes only values of the form* $1/3^n$ [*or* $1-(1/3^n)$ *respectively*].

In fact, we have $E_1\subset E_2\subset ...$ if and only if all the values of c_e are of the form $2\cdot(0,00...0111...)_3=1/3^n$.

3.4. Convergence. It is easily seen that the sequence $e=\{E_n\}$ is convergent (in the sense of the General Theory of Sets) if and only if for each x either $x\epsilon E_n$ for $n>N$, or $x\bar{\epsilon}E_n$ for $n>N$. In other words: either $c_e(x)=2\cdot(0,i_1...i_N111...)_3$ or $c_e(x)=2\cdot(0,i_1i_2...i_N000...)_3$. Hence we obtain:

A sequence of sets is convergent if and only if its characteristic function assumes only values of the form $k/3^n$ *(i. e. values possessing finite ternary development).*

3.5. Independence. The sequence $e=\{E_n\}$ is called a *sequence of independent sets* if no constituent of e (cf. 2.3) is empty [1]). Similarly it is called a *sequence of* $\aleph_0$*-independent sets* if no atom of e (cf. 2.3) is empty. By Theorems 2.3 (iv), 1.4 (v) and 2.3 (ii) we obtain:

A sequence e *of subsets of the space* X *is a sequence of* (i) *independent,* (ii) $\aleph_0$*-independent sets if and only if* (i') $\overline{c_e(X)}=C$, (ii') $c_e(X)=C$.

[1]) Cf. Fichtenholz-Kantorovitch [1], p. 78 and Hausdorff [1], p. 18.

3.6. Distinguishable points. Two points are said to be *distinguishable by means of a sequence* $e=\{E_n\}$ (Sierpiński) if there exists a positive integer n such that the set E_n contains one and only one of these points. Each two points of the space are distinguishable by means of e if and only if each atom of e contains at most a single point. Consequently, in view of 2.3 (ii) we have:

Every two points of a space X *are distinguishable by means of a sequence* e *of subsets of* X *if and only if the characteristic function of* e *assumes each of its values only once.*

3.7. Measurability ($\boldsymbol{K}$). Let $\boldsymbol{K}$ denote a class of subsets of a space X. A function $f(x)$, which transforms X into a subset of a topological space Y, is called *measurable* ($\boldsymbol{K}$) whenever $f^{-1}(G)\epsilon\boldsymbol{K}$ for any open subset G of Y.

Now, let us denote by $e=\{E_n\}$ a sequence of sets contained in X. From the identities 2.3 (iii) it follows at once that

(i) *If the characteristic function of a sequence* $\{E_n\}$ *of sets contained in* X *is measurable* ($\boldsymbol{K}$), *then* $E_n\epsilon\boldsymbol{K}$ *and* $X-E_n\epsilon\boldsymbol{K}$ *for* $n=1,2,...$

On the other hand it follows from 2.5 (ii) that:

(ii) $\boldsymbol{E}$ *denoting the class of all sets belonging to a sequence* e, *the function* c_e *is measurable* ($\boldsymbol{E}_{\sigma\sigma}$).

Combining (i) with (ii) we obtain:

(iii) *If* $\boldsymbol{K}=\boldsymbol{K}_d=\boldsymbol{K}_\sigma$ *then, in order that the function* c_e *be measurable* ($\boldsymbol{K}$) *it is necessary and sufficient that all the sets* E_n *and* $X-E_n$ *belong to* $\boldsymbol{K}$ [1]).

The necessity of this condition is stated in (i). The sufficiency follows from (ii) and from the fact that the relations: $\boldsymbol{E}+\boldsymbol{E}_c\subset\boldsymbol{K}$ and $\boldsymbol{K}=\boldsymbol{K}_d=\boldsymbol{K}_\sigma$ imply the relation: $\boldsymbol{E}_{\sigma\sigma}\subset\boldsymbol{K}$.

It follows from (iii) that (in the case when the space X is topological or particularly if it is cartesian):

(iv) *The function* c_e *is* 1^0 *continuous,* 2^0 *of the first class,* 3^0 *measurable* (B), 4^0 *measurable* (L) *if and only if all the sets belonging to* e *are* 1^0 *both open and closed,* 2^0 *both* F_σ *and* G_δ, 3^0 *borelian,* 4^0 *measurable* (L).

[1]) Szpilrajn [1], p. 308.

3.8. Basis of a metrical space. With the help of the characteristic function of a basis of a metrical space we can easily obtain the following known theorems: (a) each metrical separable space is a continuous biunivocal image of a subset of Cantor's discontinuum [1]), (b) each separable and 0-dimensional space can be topologically imbedded in C [2]). In fact, these theorems follow directly from the following Theorems (i) and (ii).

(i) *Let $v=\{V_n\}$ denote a basis of a separable space X. Then the function c_v is a generalized homeomorphism of the class* (1,0) [3]).

Put $c_v(X)=T$. The sequence v being a basis of X, each two points of X are distinguishable by means of v and consequently c_v is a biunivocal transformation of X into T (cf. 3.6).

Since the sets $\{V_n\}$ are open, the function c_v is of the first class by 3.7 (iv, 2^0). On the other hand the function c_v^{-1} is continuous, because the sets $c_v(V_n)$ are open in T by 2.5 (ii) and accordingly, for any set G open in X, the set $c_v(G)$ is also open in T.

(ii) *Let $v=\{V_n\}$ denote a basis of sets both open and closed in a 0-dimensional, separable space. Then the function c_v is a homeomorphism.*

On account of (i), it suffices to prove that the function c_v is continuous. This follows directly from 3.7 (iv, 1^0).

3.9. A certain transformation of the interval. Now, by applying the preceding theorems, we shall obtain the following remark:

(i) *There exists a biunivocal transformation φ of the interval $I=\langle 0,1\rangle$ into a set $T\subset C$, which transforms the class of the Borel subsets of I into a proper subclass of the class of sets borelian in T.*

(In other words: the class of all Borel subsets of I is equivalent [4]) to a proper subclass of the class of all sets borelian in a certain subset of I).

---

[1]) Cf. e.g. Kuratowski [1], p. 226.

[2]) Cf. e.g. Kuratowski [1], p. 124. Our proof is briefer (as using general theorems on the characteristic function) but not essentially different from that of Kuratowski.

[3]) For the notion of homeomorphism of class (α,β) see Kuratowski [1], p. 221.

[4]) For the equivalence of classes of sets see e.g. Stone [1], p. 91, Szpilrajn [1] and [2].

Let $\{V_n\}$ denote a basis of I and P a non borelian subset of I. Let w be the sequence of sets: $P,V_1,V_2,\dots$ and $\boldsymbol{W}$ the class of all these sets. According to 3.6, the function c_w is a biunivocal transformation of I into a set $T\subset C$. By 2.5 (ii) the function c_w^{-1} transforms the class of all Borel sets in T into the class W_b to which belong all the Borel sets in I and the set P which is no Borel set.

§ 4. Logical equivalence of certain problems.

4.1. Set of Lusin. A linear set is called *Lusin set* if each of its nowhere dense subsets is at most enumerable. The existence of a non enumerable Lusin set is proved only with the help of the hypothesis of the continuum [1]). Next, it is known that the existence of a Lusin set of the power $\mathfrak{n}$ is equivalent to that of a set E of the power $\mathfrak{n}$ possessing the following weaker property (ν): each subset of E nowhere dense in E is at most enumerable [2]). Furthermore, it is easy to show that the property (ν) of a set E is equivalent to the following property (ν_0): each set closed in E is the sum of a set open in E and of a set at most enumerable [3]).

(i) Let $e=\{E_n\}$ be a sequence of subsets of X, $\boldsymbol{E}$ — the class of all sets E_n. If each set belonging to the class $\boldsymbol{E}_{\sigma\delta}$ is the sum of a set belonging to $\boldsymbol{E}_{\delta\sigma}$ and of a set at most enumerable, then the set $c_e(X)$ possesses the property (ν).

We shall prove that $c_e(X)$ have the property (ν_0). Let then F be a set closed in $c_e(X)$. Theorem 2.5 (ii) implies: $c_e^{-1}(F)\epsilon\boldsymbol{E}_{\sigma\delta}$ and consequently $c_e^{-1}(F)=H+D$ where $H\epsilon\boldsymbol{E}_{\delta\sigma}$ and D is at most enumerable. Accordingly, $F=c_e(H)+c_e(D)$ where $c_e(H)$ is open in $c_e(X)$ [also by 2.5 (ii)] and $c_e(D)$ is at most enumerable. Thus the set $c_e(X)$ has the property (ν_0), q. e. d.

(ii) *For the existence of a separable Lusin set of the power $\mathfrak{n}>\aleph_0$ it is necessary and sufficient that there exist an enumerable ring $\boldsymbol{K}$ of sets 1^0 possessing $\mathfrak{n}$ non-void atoms and 2^0 such that each set belonging to $\boldsymbol{K}_\delta$ is the sum of a set belonging to $\boldsymbol{K}_\sigma$ and of a set at most enumerable* [4]).

---

[1]) See e.g. Sierpiński [1], Chap. II.

[2]) See e.g. Kuratowski-Sierpiński [1], p. 137.

[3]) Szpilrajn [3], Th. 1.

[4]) It is a modification of a theorem of Sierpiński. See Sierpiński [2].

Necessity. Let L denote a separable Lusin set of the power $\mathfrak{n}$ and $\boldsymbol{B}$ a basis of L consisting of sets both open and closed in L [1]). It is easy to show that the ring $\boldsymbol{K}=\boldsymbol{B}_o$ satisfies the above formulated conditions. In fact 1^0 each point belonging to L is an atom of $\boldsymbol{B}_o$, and the power of L is equal to $\mathfrak{n}$, 2^0 since the set L possesses the property (ν_0), each set belonging to $\boldsymbol{K}_\delta$, as closed in L, is the sum of a set at most enumerable and of a set which is open in L and therefore belongs to $\boldsymbol{K}_\sigma$.

Sufficiency. It is a consequence of (i) and 2.3 (v).

Remarks. Theorem (ii) differs partially from Sierpiński's theorem cited above, but passing from one to the other presents no difficulties:

1. It is easy to prove that the condition 1^0 may be replaced by the following stronger condition considered by Sierpiński: there exists $\mathfrak{n}$ non-void disjoint sets which belong to the class $\boldsymbol{K}_\delta$.

2. The condition 2^0 may be replaced by the following weaker condition: for each set $H \epsilon \boldsymbol{K}_\delta$ there exists a set $K \epsilon \boldsymbol{K}_\sigma$ such that the set $(K-H)+(H-K)$ is at most enumerable. For that purpose it suffices to replace the property (ν_0) in the proof by another property (ν_1) and to apply a certain simple remark formulated in the paper: Szpilrajn [3], Th. 3.

3. In his paper, Sierpiński understands by *ring* every class of sets closed with respect to finite addition and substraction (and not with respect to complementation, as in this paper). Let us remark that (in the case $\mathfrak{n}=\mathfrak{c}$ considered by Sierpiński) if there exists an enumerable ring $\boldsymbol{K}$ in that sense, satisfying the condition 1^0 (for $\mathfrak{n}=\mathfrak{c}$) and 2^0, then there exists an enumerable ring $\boldsymbol{K}'$ in the sense of our paper which satisfies these conditions as well. In fact, there then exists a set $K_0 \epsilon \boldsymbol{K}$ which contains $\mathfrak{c}$ non-void atoms of $\boldsymbol{K}$. The sets KK_0 where $K \epsilon \boldsymbol{K}$ form the required ring $\boldsymbol{K}'$.

4.2. Besikovitch's concentrated sets. A metrical space X is called *concentrated in the neighbourhood of a set* $T \subset X$ if for every open set $G \supset T$ the set $X-G$ is at most enumerable (Besikovitch) [2]). A set concentrated in the neighbourhood of an enumerable set is briefly called *concentrated*. The existence of a non enumerable concentrated set is proved only with the help of the hypothesis of the continuum. Every separable set having the property (ν) (cf. 4.1) is concentrated (from the hypothesis of the continuum it follows that, on the other hand, the converse of this proposition is untrue). It is easy to show that a separable set E possesses the property (ν) if and only if it is concentrated in the neighbourhood of each enumerable set, dense in E.

[1]) Each separable Lusin set is 0-dimensional. See Kuratowski-Sierpiński [1], p. 138.

[2]) Besikovitch [1], p. 289.

(i) Let $e=\{E_n\}$ be a sequence of subsets of X, and $\boldsymbol{E}$ the class of all sets E_n. Suppose that there exists an enumerable set $D \subset X$ such that for any set $H \epsilon \boldsymbol{E}_{\sigma\sigma}$ containing D, the set $X-H$ is at most enumerable. Then the set $c_e(X)$ is concentrated [in the neighbourhood of the set $c_e(D)$].

Suppose that a set G is open in $c_e(X)$ and contains $c_e(D)$. Thus $c_e^{-1}(G) \epsilon \boldsymbol{E}_{\sigma\sigma}$ [by 2.5 (ii)] and $c_e^{-1}(G) \supset D$. Accordingly, the set $Z=X-c_e^{-1}(G)$ is at most enumerable by the hypothesis, and consequently so is the set $c_e(Z)$.

The relation $c_e(Z) \supset c_e(X)-G$ implies the enumerability of the set $c_e(X)-G$.

(ii) *For the existence of a linear concentrated set of the power* $\mathfrak{n}>\aleph_0$ *it is necessary and sufficient that there exist: an enumerable ring of sets* $\boldsymbol{R}$ *possessing* $\mathfrak{n}$ *non-void atoms and an enumerable set* D *contained in the sum* X *of the sets belonging to* $\boldsymbol{R}$ *such that if* $H \supset D$ *and* $H \epsilon \boldsymbol{R}_\sigma$ *then the set* $X-H$ *is at most enumerable.*

Necessity. Let X be a linear set of the power $\mathfrak{n}$ concentrated in the neighbourhood of an enumerable set D and $\boldsymbol{B}$ a basis of X consisting of sets both open and closed in X. The ring $\boldsymbol{R}=\boldsymbol{B}_o$ satisfies the conditions mentioned above.

Sufficiency. It is a consequence of (i) and 2.3 (v).

4.3. Property (λ). We say that a metrical space X has *the property* (λ) if each of its enumerable subsets is a G_δ-set in X (Kuratowski). Each space having the property (λ) is of the first category on every perfect set; but it follows from the hypothesis of the continuum that the converse of this theorem would be false (Lusin). It is known that there exists a space of the power $\aleph_1$ having the property (λ), but the problem of the existence of such a set of the power of the continuum is open [1]).

(i) Let $e=\{E_n\}$ be a sequence of subsets of a space X and $\boldsymbol{E}$ the class of all sets E_n. If each enumerable subset of X belongs to $\boldsymbol{E}_{\sigma\sigma\delta}$, then the set $c_e(X)$ has the property (λ).

Since each one-element subset of X belongs to $\boldsymbol{E}_{\sigma\sigma\delta}$, the function $c_e(X)$ assumes each of its values only once [by 1.2 (iii) and 3.6].

[1]) For the property (λ) see Kuratowski [1], pp. 269-271 and Sierpiński [1], pp. 94-98.

Let D denote an enumerable subset of $c_e(X)$. Thus the set $c_e^{-1}(D)$ is enumerable and consequently $c_e^{-1}(D)\,\epsilon\,\boldsymbol{E}_{\sigma\sigma\delta}$ by the hypothesis. Therefore, by 2.5 (ii), D is a G_δ-set in $c_e(X)$.

(ii) *For the existence of a linear set which has the property* (λ) *and the power* $\mathfrak{n}>\aleph_0$, *it is necessary and sufficient that there exists an enumerable class* $\boldsymbol{R}$ *of sets having* $\mathfrak{n}$ *non-void atoms and such that each enumerable set contained in the sum of the class* $\boldsymbol{R}$ *belongs to* $\boldsymbol{R}_{\sigma\delta}$.

Necessity. Any basis of a set having the property (λ) and the power $\mathfrak{n}$ is the required class $\boldsymbol{R}$.

Sufficiency. It suffices to put $\boldsymbol{E}=\boldsymbol{R}_o$ and to apply (i) and 2.3 (v).

4.4. Problems of Hausdorff and Sierpiński. It follows directly from the hypothesis of the continuum that the following propositions are false:

(H) E being a set of the power $\aleph_1$, there exists an enumerable class $\boldsymbol{R}$ of sets contained in E such that each subset of E belongs to $\boldsymbol{R}_{\sigma\delta}$.

(S) There exists a linear set U of the power $\aleph_1$, each subset of which is a G_δ-set in U.

The problem of proving the falseness of the proposition (H)[1]) and (S)[2]) without additional hypotheses is not solved.

(i) Let $e=\{E_n\}$ be a sequence of subsets of X and $\boldsymbol{E}$ the class of all sets E_n. If each subset of X belongs to $\boldsymbol{E}_{\sigma\sigma\delta}$, then every subset of $c_e(X)$ is a G_δ-set in $c_e(X)$.

Let be $T\subset c_e(X)$. By the hypothesis, $c_e^{-1}(T)\,\epsilon\,\boldsymbol{E}_{\sigma\delta\sigma}$, and, by 2.5 (ii), T is a G_δ-set in $c_e(X)$.

(ii) *The propositions* (H) *and* (S) *are equivalent*[3]).

(S)$\to$(H). Let φ denote a biunivocal transformation of the set U into an arbirary set E of the power $\aleph_1$. Any basis of U is transformed by φ into a class $\boldsymbol{R}$ of subsets of E which obviously satisfies the conditions of the proposition (H).

[1]) It is Sierpiński's modification of a problem raised by Hausdorff in Fund. Math. **20** (1933), p. 286 (Problème 58).

[2]) Problem of Sierpiński. See Sierpiński [1], p. 90.

[3]) The result shown in the paper: Sierpiński [3].

(H)$\to$(S). Let E denote an arbitrary set of the power $\aleph_1$ and $r=\{R_n\}$ the sequence of all sets belonging to the class $\boldsymbol{R}$. Since each one-element subset of E belongs to $\boldsymbol{R}_{\sigma\delta}$, it follows from 1.2 (iii) and 3.6 that the function c_r assumes each of its values only once. Therefore this function transforms the set X into a linear set U of the power $\aleph_1$. According to (i), the set U satisfies the condition contained in the proposition (S).

4.5. Problems of Mazurkiewicz and Kolmogoroff. For each class of sets put $\boldsymbol{B}_0(\boldsymbol{R})=\boldsymbol{R}$, next $\boldsymbol{B}_\xi(R)=[\boldsymbol{B}_{\xi-1}(\boldsymbol{R})]_\delta$ for each odd $\xi<\Omega$ and $\boldsymbol{B}_\xi(\boldsymbol{R})=[\sum_{\eta<\xi}\boldsymbol{B}_\eta(\boldsymbol{R})]_\sigma$ for each even $\xi<\Omega$. If $\boldsymbol{R}$ is a basis of a topological space, then $\boldsymbol{B}_1(\boldsymbol{R})$, $\boldsymbol{B}_2(\boldsymbol{R})$, $\boldsymbol{B}_3(\boldsymbol{R})$,... equal the classes of sets: open, $G_\delta, G_{\delta\sigma}$,... respectively.

Consider the following condition:

$$(*)\qquad \boldsymbol{B}_\alpha(\boldsymbol{K})\neq\boldsymbol{B}_{\alpha+1}(\boldsymbol{K})=\boldsymbol{B}_{\alpha+2}(\boldsymbol{K}).$$

The problem of existence for each $\alpha<\Omega$ of a metrical space X, the basis of which would satisfy this condition, has been raised by Mazurkiewicz[1]). An analogous problem for arbitrary classes $\boldsymbol{K}$ of sets has been set up by Kolmogoroff[2]). We shall show the following relation between these problems (theorem of Sierpiński[3])):

(i) *For the existence (for an ordinal number* $0<\alpha<\Omega$*) of a linear set* N *possessing a basis* $\boldsymbol{V}$ *which satisfies the condition* ($*$) (*for* $\boldsymbol{K}=\boldsymbol{V}$) *it is necessary and sufficient that there exists an enumerable ring* $\boldsymbol{R}$ *satisfying this condition (for* $\boldsymbol{K}=\boldsymbol{R}$*).*

Necessity. Let N be a linear set, a basis $\boldsymbol{V}$ of which satisfies the condition ($*$). The set N contains no interval and consequently there exists a basis $\boldsymbol{V}'$ being a ring which consists only of sets both open and closed in N. Since $\boldsymbol{B}_\alpha(\boldsymbol{V})=\boldsymbol{B}_\alpha(\boldsymbol{V}')$ for $0<\alpha<\Omega$, the condition ($*$) for $\boldsymbol{K}=\boldsymbol{V}$ is equivalent to that for $\boldsymbol{K}=\boldsymbol{V}'$.

Sufficiency. Take the sequence $k=\{K_n\}$ of all sets belonging to $\boldsymbol{K}$ and put $N=c_k(K_1+K_2+...)$. It follows from 2.5 (ii) that N is the required linear set.

[1]) See Poprougénko [1], p. 284.

[2]) Fund. Math. **25** (1935), p. 578, problème 65.

[3]) Sierpiński [4].

4.6. Problem of Ulam. Recently Ulam has raised the question whether the following proposition is true:

(U) There exists an enumerable class $\boldsymbol{D}$ of sets such that $\boldsymbol{D}_b$ contains the class of all the analytical subsets of the interval $I=\langle 0,1\rangle$. [1])

We shall prove that

(i) *The proposition* (U) *is equivalent to the following*:

(U′) There exists a biunivocal transformation φ of the interval I into a linear set transforming each set analytical in I into a Borel set in $\varphi(I)$.

(The proposition (U′) may be formulated also in the following manner: The class of analytical subsets of I is equivalent [2]) to a class of sets borelian in a certain linear set.)

(U)→(U′). Denote by $d=\{D_n\}$ the sequence of all the sets belonging to $\boldsymbol{D}$ and consider the function c_d. By the hypothesis, each one-element subset of I belong to $\boldsymbol{D}_b$ and hence the function c_d assumes each of its values only once [according to 1.2 (iii) and 3.6]. Next, on account of Theorem 2.5 (ii), the function c_d transforms each set belonging to $\boldsymbol{D}_b$, and in particular each set analytical in I, into a set borelian in $c_d(I)$.

(U′)→(U). Let φ satisfy the condition of the proposition (U′), let $\boldsymbol{V}=\{V_n\}$ be a basis of the set $\varphi(I)$ and let us put $\boldsymbol{W}=\{\varphi^{-1}(V_n)\}$. Since by the hypothesis the class $\boldsymbol{V}_b$ contains as elements all the sets $\varphi(A)$, where A is an analytical set in I, the class $\boldsymbol{W}_b$ contains as elements all the sets analytical in I, q. e. d.

References.

Besikovitch, A. S. [1] *Concentrated and rarified sets of points*. Acta Math. **62** (1934), pp. 289-300.

Fichtenholz, G. et Kantorovitch, L. [1] *Sur les opérations linéaires dans l'espace des fonctions bornées*. Studia Math. **5** (1934), pp. 69-98.

Hausdorff, F. [1] *Über zwei Sätze von G. Fichtenholz und L. Kantorovitch*. Studia Math. **6** (1936), pp. 18-19.

[1]) Fund. Math. **30** (1938), p. 365, problème 74. The proposition (U) is a modification of the original Ulam's proposition, which is concerned with the smallest class of sets containing $\boldsymbol{D}$ and closed with respect to enumerable addition and enumerable multiplication.

[2]) Cf. p. 216 [4]).

Kuratowski, C. [1] *Topologie I*. Monografie Matematyczne **3**, Warszawa-Lwów 1933.

— [2] *Sur les suites analytiques d'ensembles*. Fund. Math. **29** (1937), pp. 54-59.

— [3] *Sur la compactification des espaces à connexité n-dimensionnelle*. Fund. Math. **30** (1938), pp. 242-246.

Kuratowski, C. et Posament, T. [1] *Sur l'isomorphie algébro-logique et les ensembles relativement boreliens*. Fund. Math. **22** (1934), pp. 281-286.

Kuratowski, C. et Sierpiński, W. [1] *Sur les ensembles qui ne contiennent aucun sous-ensemble indénombrable non-dense*. Fund. Math. **26** (1936), pp. 137-142.

Poprougénko, G. [1] *Sur un problème de M. Mazurkiewicz*. Fund. Math. **15** (1930), pp. 284-286.

Sierpiński, W. [1] *Hypothèse du continu*. Monografie Matematyczne **4**, Warszawa-Lwów 1934.

— [2] *Le théorème de M. Lusin comme une proposition de la Théorie générale des ensembles*. Fund. Math. **29** (1937), pp. 182-190.

— [3] *Sur un problème de M. Hausdorff*. Fund. Math. **30** (1938), pp. 1-7.

— [4] *Sur l'équivalence des problèmes de M. Kolmogoroff et M. Mazurkiewicz*. Fund. Math. **30** (1938), pp. 65-67.

Stone, M. H. [1] *The theory of representations for Boolean algebras*. Trans. Amer. Math. Soc. **40** (1936), pp. 37-111.

Szpilrajn, E. [1] *Sur l'équivalence des suites d'ensembles et l'équivalence des fonctions*. Fund. Math. **26** (1936), pp. 302-326. *Correction*... Fund. Math. **27** (1936), p. 294.

— [2] *On the equivalence of some classes of sets*. Fund. Math. **30** (1938), pp. 235-241.

— [3] *Remarques sur l'ensemble de M. Lusin*. Mathematica, à paraître.

— [4] a) *Concerning convergent sequences of sets*. b) *On the isomorphism and the equivalence of classes and sequences of sets*. c) *On the space of measurable sets*. d) *Operations upon sequences of sets*. Annales Soc. Pol. Math. **17** (1938), pp. 115, 119, 120 and 123.